Some Multipliers Results on Compact Hypergroups

Norbert Youmbi

Department of Mathematics
Saint Francis University
117 Evergreen Dr, Sullivan 114
Loretto, PA 15931, USA
nyoumbi@francis.edu

Abstract

A hypergroup is roughly speaking a locally compact Hausdorff space which has enough structure so that a convolution on the corresponding vector space of Radon measures makes it a Banach algebra. Hypergroups generalize in many ways topological groups. In this paper we extend to compact not necessarily commutative hypergroups some basic techniques on multipliers set forth for compact groups in Hewitt and Ross [3]. Our main result is a proof of an extended version of Wendel’s theorem for compact not necessarily commutative hypergroups.

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1 Introduction

Let $H$ be an infinite compact hypergroup with dual object $\hat{H}$, that is, the set of continuous unitary representations $U$ of $H$. Suppose $U \in \hat{H}$ and $\{\tau_j\}_{j=1}^{d_U}$ is an orthonormal basis for $H_U$ (the Hilbert space associated with $U$ with dimension $d_U$). We define coordinate functions for $U$ as in [10] by

$$u_{jk}(x) = < u_x \tau_k, \tau_j >$$

where $1 \leq j, k \leq d_U$. Trig$_U(H)$ is the linear span of coordinate functions of $U$ and Trig($H$) = $\bigcup\{\text{Trig}_U(H) : U \in \hat{H}\}$. For more details about representations on compact hypergroups, see [10].
The $*$-algebra $\prod_{U \in \hat{H}} B(H_U)$ will be denoted by $\mathcal{E}(\hat{H})$; scalar multiplication, addition, multiplication and adjoint of an element are defined coordinate wise. Let $E = (E_U)$ be an element of $\mathcal{E}(\hat{H})$. For $1 \leq p < \infty$ we define

$$\|E\|_p = \left(\sum_{U \in \hat{H}} k_U \|E_U\|_{\varphi_p}^p\right)^{\frac{1}{p}}$$

and

$$\|E\|_\infty = \sup\{\|E_U\|_{\varphi_\infty}\}$$

The norms $\|\cdot\|_{\varphi_p}$ are the operator norms of [[3] D.37, D.36(e)] and the notations $\mathcal{E}_p(\hat{H})$, $\mathcal{E}_{00}(\hat{H})$, and $\mathcal{E}_0(\hat{H})$ are as in [[3] 28.24].

We denote by $C(H)$ the Banach space of all continuous complex valued functions on $H$ with uniform norm. Let $L_p(H)$, $1 \leq p < \infty$, has its usual meaning.

Let $U = C(H)$ or $L_p(H)$. Given $A \subset U$, we define by $\hat{A}$, the set of all Fourier transforms $\hat{f}$ of $f \in A$. A complex valued mapping $\varphi$ on the dual space $\hat{H}$ of $H$ is called an $(A, B)$-multiplier if and only if $\varphi \hat{f} \in \hat{B}$ for each $f \in A$ where $A, B$ are subsets of $U$. An $(A, A)$-multiplier will be described simply an $A$-multiplier.

In this paper we generalize to compact hypergroups, two important results of Hewitt and Ross [3] on multipliers on compact Groups. The first result puts into one place all those closed graph theorem argument used in the theory of multipliers. The second result, describes the duality properties of multipliers, it also provides useful shortcuts in computations involving multipliers.

Our main result is an extended version of Wendel theorem for compact not necessarily commutative hypergroup. There is often more than one equally valid definition of multipliers. The appropriate choice of definition will often depend on the context in which we are considering multipliers. Equivalent definitions of multipliers are put together in the extended version of Wendel’s theorem. This theorem recapitulates the characterizations of bounded linear operators on $L_1(H)$. It was stated and proved for locally compact commutative hypergroups by Lasser [9]. We will prove that this result is true for compact not necessarily commutative hypergroups.

If $\mathcal{U}$ and $\mathcal{B}$ are given their usual norm, then $T$ is a bounded linear transformation from $\mathcal{U}$ to $\mathcal{B}$.

2 Preliminaries

Let $M(H)$ be the set of finite regular Borel measures on $H$; $M_1(H)$ be the space of probability measures. If $\mu \in M(H)$ then $\text{Supp}(\mu) = \{x \in H : \mu(V) > 0\}$.
any open set containing \( x \) then \( \mu(V) > 0 \). If \( x \in H \), \( \delta_x \) is the point mass at \( x \). An unspecified topology on \( M_+(H) \) is the cone topology. In this paper we are using the following definition of a hypergroup.

### 2.1 Definition

A nonempty locally compact Hausdorff space \( H \) will be called a hypergroup if the following conditions are satisfied:

\((H_1)\) \( (M(H), +, \ast) \) is a Banach algebra.

\((H_2)\) For all \( x, y \in H \), \( \delta_x \ast \delta_y \) is a probability measure with compact support.

\((H_3)\) The mapping \( (x, y) \mapsto \delta_x \ast \delta_y \) of \( H \times H \) into \( M_1(H) \) is continuous.

\((H_4)\) The mapping \( (x, y) \mapsto \text{Supp}(\delta_x \ast \delta_y) \) of \( H \times H \) into \( C^0(H) \) is continuous where \( C^0(H) \) is the space of compact subsets of \( H \) endowed with the Michael topology, that is the topology generated by the subbasis of all \( \text{C}_U(V) = \{ C \in C(S) : C \cap U \neq \emptyset \text{ and } C \subset V \} \) where \( U \) and \( V \) are open subsets of \( S \).

\( H_5 \) There exists \( e \in H \) such that \( \delta_x \ast \delta_e = \delta_e \ast \delta_x = \delta_x \forall x \in H \).

\( H_6 \) There exists a topological involution \( - \) (a homeomorphism) from \( H \) onto \( H \) such that \( (x^-)^- = x \forall x \in H \), with \( (\delta_x \ast \delta_y)^- = \delta_y^- \ast \delta_x^- \) and \( e \in \text{Supp}(\delta_x \ast \delta_y) \) if and only if \( x = y^- \) where for any Borel set \( B \), \( \mu^-(B) = \mu(\{ x^- : x \in B \}) \).

Remarks

(i) If \( \delta_x \ast \delta_y = \delta_y \ast \delta_x \) for all \( x, y \in H \) we say that \( (H, \ast) \) is a commutative hypergroup.

(ii) The convolution \( \ast \) on \( M(H) \) is defined by

\[
\mu \ast \nu(f) = \int_H f \, d\mu \ast \nu = \int_H \mu(dx) \int_H \nu(dy) \int_H f \, d\delta_x \ast \delta_y
\]

for all \( f \in C_b(H) \).

### 2.2 Examples

1. If \( (G, \cdot) \) is a locally compact Hausdorff group, then with convolution defined by \( \delta_x \ast \delta_y = \delta_{xy} \), \( (G, \ast) \) is a hypergroup. Also if a hypergroup is such that the convolution of two point masses is a point mass then it is a topological group.
2. Consider the segment $[0,1]$ with convolution defined by

$$\delta_r \ast \delta_s = \frac{1}{2} \delta_{|r-s|} + \frac{1}{2} \delta_{1-|1-r-s|}$$

for all $r, s \in [0,1]$, then $([0,1], \ast)$ is a compact commutative hypergroup.

For a detailed discussion on hypergroups see Bloom and Heyer [1], Jewett [6], Dunkl [2], Spector [11].

### 2.3 Definition

Let $(H, \ast)$ be a hypergroup. If $f$ is a Borel function on $H$ and $x, y \in H$ then we define

$$f(x \ast y) = f_x(y) = f^y(x) = \int_H f d(\delta_x \ast \delta_y)$$

if this integral exists, even when it is not finite. $f_x$ is called the left translation of $f$ by $x$, and $f^y$ is called the right translation of $f$ by $y$.

### 2.4 Definition

Let $H$ be a locally compact hypergroup. A measure $m$ not necessarily finite, will be called left subinvariant (invariant) if $\delta_x \ast m$ is defined and $\delta_x \ast m \leq m$ ($\delta_x \ast m = m$) for all $x \in H$. (Right invariant measures are defined in the same way).

### 2.5 Example

If a system of orthogonal functions with respect to a measure $m$, has a product formula which defines a hypergroup $H$ then the measure $m$ is the invariant measure of the hypergroup $H$.

### 2.6 Remarks


2. Unlike in the group case, $\hat{H}$ is not always a hypergroup even in the commutative case see Jewett [6]. From now on, we will assume that $\hat{H}$ is a hypergroup with invariant measure $\pi$ such that $supp(\pi) = \hat{H}$.
3 Proofs of Results

3.1 Proposition
Let $H$ be a compact hypergroup. Let $\mathcal{U}$ and $\mathcal{B}$ be any of the spaces

i. $\mathcal{E}_p(\hat{H})$, $(1 \leq p \leq \infty)$, $\mathcal{E}_0(\hat{H})$

ii. $L_p(H)$, $(1 \leq p \leq \infty)$, $C_b(H)$, $M(H)$

where $\hat{H}$ denote the dual object of $H$.

Let $E$ be a ($\mathcal{U}, \mathcal{B}$)-multiplier. Define the mapping

$$T : \mathcal{U} \rightarrow \mathcal{B}$$

by the following rules

iii. $T(g) = Eg$ for $g \in \mathcal{U}$ if $\mathcal{U}$ and $\mathcal{B}$ are chosen from (i)

iv. $\hat{T}(g) = Eg$ for $g \in \mathcal{U}$, $\hat{\mathcal{U}}$ chosen from (i) and $\mathcal{B}$ from (ii)

v. $T(f) = E\hat{f}$ for $f \in \mathcal{U}$ [or $T(\mu) = E\hat{\mu}$ for $\mu \in M(H)$] if $\mathcal{U}$ is chosen from (ii) and $\mathcal{B}$ from (i)

vi. $\hat{T}(f) = E\hat{f}$ for $f \in \mathcal{U}$ [or $\hat{T}(\mu) = E\hat{\mu}$ for $\mu \in M(H)$] if $\mathcal{U}$ and $\mathcal{B}$ are chosen from (ii).

Proof:
The proof is adapted from [[3] 35.2] for the group case.

First, we need to show that $T$ is well-defined in (iii),(iv),(v),(vi).

For (iii), since $\mathcal{U}$ and $\mathcal{B}$ are chosen from (i), by the definition of a ($\mathcal{U}, \mathcal{B}$)-multiplier, for all $g \in \mathcal{U}$, $Eg \in \mathcal{B}$ uniquely so that $T$ is well defined.

In (v) $\mathcal{U}$ is from (ii) so that $\hat{\mathcal{U}}$ is a subset of a set in (i) and $E$ is a ($\mathcal{U}, \mathcal{B}$)-multiplier, therefore $T$ is well defined as $\hat{f}$ is uniquely defined.

For (iv) $\mathcal{U}$ is chosen from (i) and $\mathcal{B}$ from (ii) so $E$ is a ($\mathcal{U}, \hat{\mathcal{B}}$)-multiplier so for all $f \in \mathcal{U}$, $\hat{f} \in \hat{\mathcal{U}}$ and $E\hat{f} \in \mathcal{B}$ uniquely, therefore $T$ is well defined as $\hat{f}$ is uniquely defined.

For (vi) $\mathcal{U}$ is chosen from (i) and $\mathcal{B}$ from (ii) so $E$ is a ($\hat{\mathcal{U}}, \hat{\mathcal{B}}$)-multiplier so $\forall g \in \mathcal{U}$, $Eg \in \mathcal{B}$. Now if $\hat{T}(g) = Eg$, by the uniqueness of the Fourier transform [[6] 7.3E], $T(g)$ is well defined. Similarly in (vi) for ($\mathcal{U}, \hat{\mathcal{B}}$) from (ii) $E$ is a ($\hat{\mathcal{U}}, \hat{\mathcal{B}}$)-multiplier that is $\forall f \in \mathcal{U}$, $\hat{f} \in \hat{\mathcal{U}}$ and $E\hat{f} \in \mathcal{B}$ so if $\hat{T}(f) = E\hat{f}$ then by the uniqueness of the Fourier Stieltjes transform $T(f)$ is unique, $T$ is then well defined.

Now if $\mathcal{U}$ is chosen from (i) then we have $\mathcal{U} \subset \mathcal{E}_\infty(\hat{H})$ and [[3] 28.32(iv)] shows that for all $g \in \mathcal{U}$

$$\|g\|_\infty \leq \|g\|_{\mathcal{U}}$$

(1)
If $\mathcal{U}$ is chosen from (ii) $\mathcal{U} \subset M(H)$ and as $M(H)$ is isomorphic with $\mathfrak{C}_\infty(\hat{H})$ [[10] 3.2] and since the isomorphism is norm-decreasing we have

$$\|\mu\| \leq \|\mu\|_\mathcal{U}$$

For all $g \in \hat{\mathcal{U}}$ (This is obtained by writing $\|\hat{\mu}\|_\mathcal{U}$ for $\|\mu\|_\mathcal{U}$ in the previous inequality), we have

$$\|g\|_\infty \leq \|g\|_\mathcal{U}$$

Relations (1) and (2) shows that in all cases $\mathcal{U}$ can be regarded as a subspace of $\mathfrak{C}_\infty(\hat{H})$ for which (1) holds. The same remark evidently holds for $\mathcal{B}$. Thus we may consider $\mathcal{U}$ and $\mathcal{B}$ as linear subspaces of $\mathfrak{C}_\infty(\hat{H})$ with complete norm $\|\cdot\|_\mathcal{U}$ and $\|\cdot\|_\mathcal{B}$ satisfying the inequality (1). So we will just prove that the mapping $T$ defined from $\mathcal{U}$ to $\mathcal{B}$ by $T(g) = E g$ is a bounded linear transformation carrying $\mathcal{U}$ to $\mathcal{B}$ for all subspaces of $\mathfrak{C}_\infty(\hat{H})$ having complete norms that satisfy (1). Since $E, g \in \mathfrak{C}_\infty(\hat{H})$ we have for $g_1, g_2 \in \mathcal{U}, g_1, g_2 \in \mathfrak{C}_\infty(\hat{H})$ and $E(g_1 + g_2) = E g_1 + E g_2$ so $T(g) = E g$ is a linear transformation. Now let $g \in \mathcal{U}$ and $\{g^{(n)}\}_{n=1}^\infty$ be a sequence in $\mathcal{U}$ such that

$$\lim_{n \to \infty} \|g^{(n)} - g\|_\mathcal{U} = 0 \quad (3)$$

Suppose that $g'$ is a limit point of $\mathcal{B}$ such that

$$\lim_{n \to \infty} \|T(g^{(n)}) - g'\|_\mathcal{B} = 0 \quad (4)$$

Then from (1) applied to $\mathcal{B}$ and (4)

$$\lim_{n \to \infty} \|E g^{(n)} - g'\|_\infty = 0 \quad (5)$$

For each $U \in \hat{H}$, (5) shows that

$$\lim_{n \to \infty} \|E_U g^{(n)} - g'_U\|_{\varphi_\infty} = 0 \quad (6)$$

and from [3] D.52i and (1)

$$\|E_U g^{(n)} - E_U g_U\|_{\varphi_\infty} \leq \|E_U\|_{\varphi_\infty} \|g^{(n)} - g_U\|_{\varphi_\infty}$$

From (3) we have

$$\lim_{n \to \infty} \|E_U g^{(n)} - E_U g_U\|_{\varphi_\infty} = 0 \quad (7)$$

For each $U \in \hat{H}$. The inequalities (6) and (7) imply that $E g = g'$, that is, $T$ has a closed graph in $\mathcal{U} \times \mathcal{B}$. and from the closed graph theorem $T$ is continuous.

Our next result describes the duality properties of multipliers. It provides useful shortcuts in computations involving multipliers.
3.2 Proposition

Suppose that $\mathcal{U}$ and $\mathcal{B}$ and also their conjugate spaces $\mathcal{U}^*$, $\mathcal{B}^*$ are among the spaces listed in Proposition (3.1) [(i),(ii)] [Thus neither $\mathcal{U}$ nor $\mathcal{B}$ can be $\mathcal{C}_\infty(\hat{H})$, $L_\infty(H)$ or $M(H)$] Then we have

$$\mathcal{M}(\mathcal{U}, \mathcal{B}^*) = \mathcal{M}(\mathcal{B}, \mathcal{U}^*)^\sim$$

Proof:

It is enough to show that $E \in \mathcal{M}(\mathcal{U}, \mathcal{B}^*)$ implies $E^\sim \in \mathcal{M}(\mathcal{B}, \mathcal{U}^*)$

We will examine four cases according to whether $\mathcal{U}$ and $\mathcal{B}$ are chosen from Proposition (3.1)(i) or Proposition (3.1)(ii). Throughout this prove, let $T$ be as defined in Proposition (3.1) for the element $E \in \mathcal{M}(\mathcal{U}, \mathcal{B}^*)$.

First suppose that both $\mathcal{U}$ and $\mathcal{B}$ are chosen from Proposition (3.1)(i), consider a fixed $B \in \mathcal{B}$. For each $A \in \mathcal{U}$ define

$$\phi(A) = \langle T(A), B \rangle$$

where $\langle, \rangle$ is defined as in [[3] 28.28i]. Holder’s inequality [ [3] 28.28ii] and the boundedness of $T$ show that

$$|\phi(A)| = |\langle T(A), B \rangle| = \|T(A)\|_{\mathcal{B}^*} \|B\|_B \leq \|\|T\||\|A\|_U \|B\|_B$$

For all $A \in \mathcal{U}$. The space $\mathcal{C}_{00}(\hat{H})$ is contained in $\mathcal{U}$ and for $A \in \mathcal{C}_\infty(\hat{H})$ we have

$$\langle A, C \rangle = \langle T(A), B \rangle = \langle EA, B \rangle = \langle A, E^\sim B \rangle$$

It follows that $E^\sim B = C$ and consequently that $E^\sim \in \mathcal{M}(\mathcal{B}, \mathcal{U}^*)^\sim$

Next suppose that $\mathcal{U}$ is chosen from Proposition (3.1)(i) and $\mathcal{B}$ from Proposition (3.1)(ii). Consider a fixed but arbitrary $f \in \mathcal{B}$. For each $A \in \mathcal{U}$, $T(A)$ belongs to $M(H)$ [In case $T(A)$ is a function $C \in L_p(H)$ we mean by this that $T(A)$ is a measure such that $dT(A) = gdm$], we define $\phi$ on $\mathcal{U}$ by

$$\phi(A) = \int \tilde{f} dT(A)$$

for all $A \in \mathcal{U}$ then

$$\|\phi(A)\| \leq \|\tilde{f}\|_B \|T(A)\| \leq \|\tilde{f}\|_B \|T\| \|A\|$$

so $\phi$ is bounded because $T$ is bounded. $\phi$ being a bounded linear functional on $\mathcal{U}$, there is a $C \in \mathcal{U}^*$ for which

$$\phi(A) = \langle A, C \rangle$$

for all $A \in \mathcal{U}$. 
Since \( \mathfrak{E}_{00}(\hat{H}) \subset \mathcal{U} \), \( T(A_0) \) is defined for \( A_0 \subset \mathfrak{E}_{00}(\hat{H}) \). The element \( EA_0 \) is in \( \mathfrak{E}_{00}(\hat{H}) \) and the definition Proposition (3.1)iv of \( T \) shows that \( dT(A_0) = gdm \) where \( g \in \text{Trig}(H) \) and \( \hat{g} = EA_0 \). Applying (8), (9) and [[3] 34.33] we obtain

\[
< A_0, C > = \int \bar{f} dT(A_0) = \int \bar{f} gdm = < \hat{g}, \hat{f} > = < EA_0, \hat{f} > = < A_0, E^{-} \hat{f} >. \tag{10}
\]

Since \( f \) is arbitrary in \( \mathcal{B} \) and \( A_0 \) is arbitrary in \( \mathfrak{E}_{00}(\hat{H}) \), (10) shows that \( E^{-} \) carries \( \hat{\mathcal{B}} \) into \( \mathcal{U}^* \); that is \( E^{-} \) is in \( \mathcal{M}(\mathcal{B}, \mathcal{U}^*) \).

Third suppose that \( \mathcal{U} \) is chosen from Proposition (3.1)(ii) and \( \mathcal{B} \) from Proposition (3.1)(i), for a fixed but arbitrary \( B \in \mathcal{B} \) define \( \phi \) on \( \mathcal{U} \) by

\[
\phi(f) = < T(f), B >
\]

for all \( \mathcal{U} \). As defined before \( \phi \) is a bounded linear functional, applying [[3] 14.10], if \( \mathcal{U} = C(H) \) and [[3] 12.18] if \( \mathcal{U} = L_p(H) \), we define a measure \( \mu \) (which has the form \( gdm \) if \( \mathcal{U} = L_p(H) \)) such that

\[
\phi(f) = \int f d\mu
\]

for all \( f \in \mathcal{U} \).

for each \( f \in \text{Trig}(H) \subset \mathcal{U} \) we have

\[
< \hat{f}, \hat{\mu} > = < \hat{\mu}, \hat{f} > = \int_H \bar{f} d\mu =
\]

\[
\phi(f) = < T(f), B > = < E\hat{f}, B > = < \hat{f}, E^{-} B >
\]

and hence \( E^{-} B = \hat{\mu} \). Thus again \( E^{-} \) belongs to \( \mathcal{M}(\mathcal{B}, \mathcal{U}^*) \). Suppose finally that \( \mathcal{U} \) and \( \mathcal{B} \) are chosen from Proposition (3.1)(ii), and consider \( g \in \mathcal{B} \). For \( f \in \mathcal{U} \) define \( \phi(f) = \int \bar{g} dT(f) \) as in the previous case, there is a \( \mu \in \mathcal{U}^* \) such that \( \phi(f) = \int f d\mu \) for \( f \in \mathcal{U} \). For \( f \in \text{Trig}(H) \) we have \( dT(f) = hdm \) where \( h \in \text{Trig}(H) \) and \( \hat{h} = E\hat{f} \). Hence we can write

\[
< \hat{f}, \hat{\mu} > = \phi(f) = \int \bar{g} dT(f) = \int \bar{g} hdm =
\]

\[
< \hat{h}, \hat{g} > = < E\hat{f}, \hat{g} > = < \hat{f}, E^{-} \hat{g} >
\]

once again \( E^{-} \hat{g} = \hat{\mu} \) and \( E^{-} \) is in \( \mathcal{M}(\mathcal{B}, \mathcal{U}^*) \).

We now give a proof of our main result.
3.3 Wendel’s Theorem

Let $H$ be a compact (not necessarily commutative) hypergroup. Suppose $T : L_1(H) \to L_1(H)$ is a bounded linear transformation. Then the following statements are equivalent:

i. $T$ commutes with right translation operators that is $T(f^s) = T(f)^s$ for all $s \in H$

ii. $T(f \ast g) = T(f) \ast g$ for each $f, g \in L_1(H)$

iii. There exists a unique transformation $\varphi$ on $\hat{H}$ such that $\hat{T}(f) = \varphi \hat{f}$ for each $f \in L_1(H)$.

iv. There exists a unique measure $\mu \in M(H)$ such that $\hat{T}(f) = \mu \hat{f}$ for each $f \in L_1(H)$

v. There exists a unique measure $\mu \in M(H)$ such that $T(f) = f \ast \mu$ for each $f \in L_1(H)$

Proof (i) implies (ii)

Suppose $T$ commute with right translation, let $k \in L_\infty(H)$ then the mapping defined on $L_1(H)$ by

$$ f \mapsto \int T(f)(t)k(t^-)dm(t) $$

is a linear functional on $L_1(H)$ moreover

$$ \| \int T(f)(t)k(t^-)dm(t) \| \leq \|k\|_\infty \|T(f)\|_1 \leq \|k\|_\infty \|T\| \|f\|_1 $$

where $\|T\|$ denotes the usual operator norm of $T$. Consequently there exists a function $h \in L_\infty(H)$ such that

$$ \int T(f)(t)k(t^-)dm(t) = \int f(t)h(t^-)dm(t) \quad (11) $$

by virtue of ([4] 20.20). If $f, g \in L_1(H)$ we have

$$ \int [T(f) \ast g](t)k(t^-)dm(t) = \int \int T(f)(t \ast s)g(s^-)dm(s)k(t^-)dm(t) = $$

$$ \int [T(f)^s(t)g(s^-)dm(s)]k(t^-)dm(t) = $$

$$ \int \int T(f^s)(t)g(s^-)dm(s)k(t^-)dm(t) $$
and from Fubini’s theorem

\[ \int g(s^-) \int T(f^*)(t)k(t^-)dm(t)dm(s) \]

and from (11)

\[ \int g(s^-) \int f^*(t)h(t^-)dm(t)dm(s) = \int h(t^-) \int f^*(t)g(s^-)dm(t)dm(s) = \int (f * g)(t)h(t^-)dm(t) = \int T(f * g)(t)k(t^-)dm(t). \]

And since \( k \) was arbitrarily chosen in \( L_\infty(H) \) it follows that \( T(f) * g = T(f * g) \) for all \( f, g \in L_1(H) \). At this point commutativity is not assume and will be assume now

(ii) implies (iii) Let \( \hat{H} \) be the dual object of \( H \), suppose \( \hat{U} \in \hat{H} \) and \( \{\tau_j\}_{j=1}^{d_U} \) is an orthonormal basis for \( H_U \) (The Hilbert space associated with \( U \) with dimension \( d_U \)). With coordinate functions defined for \( U \) by

\[ u_{jk}(x) = \langle u_{x \tau_k}, \tau_j \rangle \]

where \( 1 \leq j, k \leq d_U \) then if \( U, V \in \hat{H} \), there exists a constant \( k_U \) with \( k_U \geq d_U \) such that

\[ \int u_{jk}(v^r)dm = \left\{ \begin{array}{ll} k^{-1}_u & \text{ when } U = V, j = r, k = s, \\ 0 & \text{ otherwise.} \end{array} \right. \]

moreover if \( H \) is a compact hypergroup then \( k_U = d_U \) \[10\] theorem 2.6.

Now let \( \chi_U = k^{-1}_U I_U \) where \( k_U \geq d_U \). Then \( \chi_U \) is in the center \( Z(L_1(H)) \) of \( L_1(H) \) that is \( f * \chi_U = \chi_U * f \) because

\[ (\hat{f} \chi_U) \hat{f} = \hat{f} k^{-1}_U I_U = k^{-1}_U I_U \hat{f} \\
(k^{-1}_U I_U) \hat{f} = [(k^{-1}_U I_U) * \hat{f}] = (\chi_U * \hat{f}) \]

So \( f * \chi_U = \chi_U * f \)

We can now define \( \varphi(U) = \widehat{T(k_U \chi_U)}(U) \)

\[ \widehat{Tf}(U) = \widehat{Tf}(U) \hat{I}_U(U) = \widehat{Tf}(U)(k_U \chi_U)(U) = ((Tf) * k_U \chi_U)(U) = [T(f * k_U \chi_U)](U) = [T(k_U \chi_U) \hat{f}](U) = T(k_U \chi_U) \hat{f}(U) = \varphi(U) \hat{f}(U) \]
that is \((T \hat{f})(U) = \varphi(U) \hat{f}(U) = (\varphi \hat{f})(U)\) which implies \(T f = \varphi \hat{f}\).

(iii) implies (iv)

Suppose that \(\hat{T(f)} = \varphi \hat{f}\) for all \(f \in L_1(H)\). That is \(\varphi \hat{f} \in L_1(H)\). It follows that \(\varphi \hat{f}\) is a Fourier transform (of \(T(f)\)) and since \(\varphi \in C(\hat{H})\) \([\varphi \hat{f}\) is continuous] \(\varphi\) is a Fourier Stieltjes transform [[9] Theorem 2.1.3], that is, there exists \(\mu \in M(H)\) such that \(\varphi = \hat{\mu}\) so \(\hat{T(f)} = \hat{\mu} \hat{f}\)

(iv) implies (v)

\(\hat{T(f)} = \hat{\mu} \hat{f} = \mu \ast \hat{f}\) Now \((T(f) - \mu \ast f) = 0\) implies \(T(f) = \mu \ast f\)

Finally (v) implies (i)

Since \(f^* \in L_1(H)\) there exists \(\mu \in M(H)\) such that

\[T(f^*) = \mu \ast f^* = \mu \ast (f^* \delta_s) = \mu \ast f \ast \delta_s\]

but \((\mu \ast \nu) \ast f = \mu \ast (\nu \ast f)\) so that

\[T(f^*) = (\mu \ast f) \ast \delta_s = (\mu \ast f)^* = T(f)^*\]

So \(T(f^*) = T(f)^*\).

References


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