Multi-Fuzzy Sets

Sabu Sebastian
Department of Mathematical Sciences, Kannur University
Mangattuparamba, Kerala-670567, India
sabukannur@gmail.com

T. V. Ramakrishnan
Department of Mathematical Sciences, Kannur University
Mangattuparamba, Kerala-670567, India

Abstract

Multi-fuzzy set theory is an extension of fuzzy set theory, L-fuzzy set theory and Atanassov intuitionistic fuzzy set theory. In this paper we study the relation between Atanassov intuitionistic fuzzy and the proposed extension called Multi-fuzzy. Also we present the notion of Multi-fuzzy mappings and Atanassov Intuitionistic Fuzzy Sets Generating Maps.

Mathematics Subject Classification: MSC: 03E72; 47S40; 08A72

Keywords: Multi-fuzzy, Intuitionistic fuzzy, \( p \)-complement, Multi-fuzzy mapping, Atanassov intuitionistic fuzzy sets generating map

1 Introduction

We propose the theory of multi-fuzzy sets in terms of multi dimensional membership functions. Multi-fuzzy set theory is an extension of theories of fuzzy sets[5], L-fuzzy sets[2] and intuitionistic fuzzy sets[1]. Also we introduce the concept of multi-fuzzy mappings and conduct a study on multi-fuzzy mappings which produces intuitionistic fuzzy sets from multi-fuzzy sets called Atanassov Intuitionistic Fuzzy Sets Generating Maps. Our other papers [3,4] dealt with \( T \), \( S \) operations and their \( p \)-conjugates of Multi-fuzzy sets, the \( p \)-complements of multi-fuzzy sets and their relations with \( p \)-conjugates of multi-fuzzy sets. In this paper an intuitionistic fuzzy set means Atanassov intuitionistic fuzzy set.
2 Preliminary

Throughout this paper, we will use the following notations. $P$, $X$, $I$ and $I^X$ stand for the set of all positive integers, the universal set, the unit interval $[0, 1]$ and the set of all functions from $X$ to $I$ respectively.

**Definition 2.1** An Atanassov Intuitionistic Fuzzy Set on $X$ is a set $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where $\mu_A(x) \in I$ denotes the membership degree and $\nu_A(x) \in I$ the non-membership degree of $x$ in $A$ and $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

**Definition 2.2** Let $X$ be a nonempty set and let $\{L_i : i \in P\}$ be a family of complete lattices. A multi-fuzzy set $A$ in $X$ is a set of ordered sequences:

$$A = \{(x, \mu_1(x), \mu_2(x), ..., \mu_i(x), ...) : x \in X\},$$

where $\mu_i \in L_i^X$, for $i \in P$.

**Remark 2.3** If the sequences of the membership functions have only $k$-terms (finite number of terms), $k$ is called the dimension of $A$. If $L_i = [0, 1]$ (for $i = 1, 2, ..., k$), then the set of all multi-fuzzy sets in $X$ of dimension $k$ is denoted by $\mathbb{M}^k\text{FS}(X)$. The multi-membership function $\mu_A$ is a function from $X$ to $I^k$ such that for all $x$ in $X$, $\mu_A(x) = (\mu_1(x), \mu_2(x), ..., \mu_k(x))$. For the sake of simplicity we denote the multi fuzzy set $A = \{(x, \mu_1(x), \mu_2(x), ..., \mu_k(x)) : x \in X\}$ as $A=(\mu_1, \mu_2, ..., \mu_k)$. In this paper $L_i = [0, 1]$ (for $i = 1, 2, ..., k$) and we will study some properties of Multi-fuzzy sets of dimension $k$.

**Definition 2.4** Let $k$ be a positive integer and let $\mu$ and $\nu$ in $\mathbb{M}^k\text{FS}(X)$, that is $\mu = (\mu_1, ..., \mu_k) = \{(x, \mu_1(x), ..., \mu_k(x)) : x \in X\}$ and $\nu = (\nu_1, ..., \nu_k) = \{(x, \nu_1(x), ..., \nu_k(x)) : x \in X\}$, then we have the following relations and operations:

1. $\mu \subseteq \nu$ if and only if $\mu_i \leq \nu_i$, for all $i = 1, 2, ..., k$;
2. $\mu = \nu$ if and only if $\mu_i = \nu_i$, for all $i = 1, 2, ..., k$;
3. $\mu \sqcap \nu = (\mu_1 \sqcap \nu_1, ..., \mu_k \sqcap \nu_k) = \{(x, \max(\mu_1(x), \nu_1(x)), ..., \max(\mu_k(x), \nu_k(x))) : x \in X\}$;
4. $\mu \sqcup \nu = (\mu_1 \sqcup \nu_1, ..., \mu_k \sqcup \nu_k) = \{(x, \min(\mu_1(x), \nu_1(x)), ..., \min(\mu_k(x), \nu_k(x))) : x \in X\}$;
5. $\mu + \nu = (\mu_1 + \nu_1, ..., \mu_k + \nu_k) = \{(x, \mu_1(x) + \nu_1(x) - \mu_1(x) \cdot \nu_1(x), ..., \mu_k(x) + \nu_k(x) - \mu_k(x) \cdot \nu_k(x)) : x \in X\}$. 

Multi-fuzzy sets

Definition 2.5 Let \( \mu = (\mu_1, ..., \mu_k) \) be a multi-fuzzy set and let \( \mu'_i \) be the classical fuzzy complement of the ordinary fuzzy set \( \mu_i \), for \( i = 1, 2, ..., k \). The Classical Multi-fuzzy Complement of the multi-fuzzy set \( \mu \) is a multi-fuzzy set \( (\mu'_1, ..., \mu'_k) \) and it is denoted by \( C(\mu) \).

That is \( C(\mu) = \{ (x, c(\mu_1(x)), ..., c(\mu_k(x))) : x \in X \} = \{ (x, 1 - \mu_1(x), ..., 1 - \mu_k(x)) : x \in X \} \), where \( c \) is the classical fuzzy complement operation.

3 Multi-fuzzy Mappings

In this section we introduce the concept of mappings from \( M^k FS(X) \) into \( M^n FS(X) \), for every positive integers \( k \) and \( n \).

Definition 3.1 Let \( \mu = (\mu_1, ..., \mu_k) \), \( \nu = (\nu_1, ..., \nu_n) \) be multi-fuzzy sets in \( X \) of dimension \( k \) and \( n \) respectively. A multi-fuzzy mapping is a mapping \( F \) from \( M^k FS(X) \) into \( M^n FS(X) \), which maps each \( \mu \in M^k FS(X) \) into a unique multi-fuzzy set \( \nu \in M^n FS(X) \).

Example 3.2 The mapping \( F : M^k FS(X) \rightarrow M^n FS(X) \) defined by \( F(\mu_1, \mu_2, ..., \mu_k) = (\frac{1}{k} \sum_{i=1}^{k} \mu_i, \frac{1}{k} \sum_{i=1}^{k} \mu_i^2, ..., \frac{1}{k} \sum_{i=1}^{k} \mu_i^n) \) is a multi-fuzzy mapping, where \( \mu_i^m(x) = (\mu_i(x))^m, \forall x \in X \) and for \( m = 1, 2, ..., n \).

Example 3.3 The following mappings are multi-fuzzy mappings from \( M^k FS(X) \) into \( M^k FS(X) \). Let \( \mu = (\mu_1, \mu_2, ..., \mu_k) \) be multi-fuzzy set of dimension \( k \) and let \( p \in \{1, 2, ..., k\} \).

1. \( U_p(\mu) = (\mu_1, \mu_2, ..., \mu_{p-1}, 1, \mu_{p+1}, ..., \mu_k) \).
2. \( C_p(\mu) = (\mu_1, \mu_2, ..., \mu_{p-1}, \mu'_p, \mu_{p+1}, ..., \mu_k) \), where \( \mu'_p \) is the classical fuzzy complement of \( \mu_p \).
3. The identity mapping \( I(\mu) = (\mu_1, \mu_2, ..., \mu_k) \).
4. The classical multi-fuzzy complement operation \( C(\mu) = (\mu'_1, \mu'_2, ..., \mu'_k) \).
5. \( F(\mu)(x) = (t(\alpha, \mu_1(x)), t(\alpha, \mu_2(x)), ..., t(\alpha, \mu_k(x))) \), \( \forall x \in X \), where \( t \) is a \( t \)-norm and \( \alpha \) is a constant in \( [0, 1] \).

Remark 3.4 Let \( F \) and \( G \) be multi-fuzzy mappings from \( M^k FS(X) \) into \( M^n FS(X) \). For the sake of simplicity we use the notations \( FG(\mu) \) and \( (F + G)(\mu) \) instead of \( F(G(\mu)) \) and \( F(\mu) + G(\mu) \) respectively.
Note 3.5 Let $\mu = (\mu_1, \mu_2)$ be multi-fuzzy set of dimension 2 and let $\mu_1(x)$, $\mu_2(x)$ are the grade membership and grade nonmembership values of $x$ in $\mu$ respectively. If $\mu_1(x) + \mu_2(x) \leq 1$, then $\mu$ is an intuitionistic fuzzy set. Therefore every intuitionistic fuzzy set in $X$ is a multi-fuzzy set in $X$ of dimension 2 and every intuitionistic fuzzy operation is a multi-fuzzy mapping on multi-fuzzy sets. But multi-fuzzy set need not be an intuitionistic fuzzy set, for example the multi-fuzzy set $\mu = \{(x, \mu_1(x), \mu_2(x)) : \mu_1(x) = .9, \mu_2(x) = .8, x \in X\}$ is not an intuitionistic fuzzy set.

Proposition 3.6 Let ‘$\subseteq$’ and ‘$\subseteq$’ be the subset symbols in multi-fuzzy sets and intuitionistic fuzzy sets respectively. If $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ are intuitionistic fuzzy sets in $X$, then

1. $C_2(\mu) \subseteq C_2(\nu) \iff \mu \subseteq \nu$;
2. $C_2(\mu \cup C_2(\nu)) = \mu \cup \nu$;
3. $C_2(\mu \cap C_2(\nu)) = \mu \cap \nu$,

where $C_2(\mu) = (\mu_1, \mu_2)$.

Definition 3.7 A mapping $F : M^kFS(X) \rightarrow M^2FS(X)$ is said to be an Atanassov Intuitionistic Fuzzy Sets Generating Map (AIFSGM), if $F(\mu)$ is an intuitionistic fuzzy set in $M^2FS(X)$, for every $\mu \in M^kFS(X)$.

Example 3.8 Let $m, k$ be positive integers and let $\mu^m_i(x) = (\mu_i(x))^m$ for all $x$ in $X$. The mapping $F : M^kFS(X) \rightarrow M^2FS(X)$ is an AIFSGM, if $F(\mu_1, \mu_2, ..., \mu_k) = \left(\frac{1}{k} \sum_{i=1}^{k} \mu^m_i(x), \frac{1}{k} \sum_{i=1}^{k} (1 - \mu_i)^m\right)$. We know that for every $x \in X$, $\frac{1}{k} \sum_{i=1}^{k} (\mu_i(x))^m + \frac{1}{k} \sum_{i=1}^{k} (1 - \mu_i(x))^m \leq \frac{1}{k} \sum_{i=1}^{k} (\mu_i(x)) + \frac{1}{k} \sum_{i=1}^{k} (1 - \mu_i(x)) = \frac{1}{k} \sum_{i=1}^{k} [\mu_i(x) + (1 - \mu_i(x))] = 1$, since $\mu_i(x), 1 - \mu_i(x) \in [0, 1]$ and $m \geq 1$.

Example 3.9 If $\mu$ is an arbitrary multi-fuzzy set of dimension 2, then $G(\mu) = \{(x, \alpha \cdot \mu_1(x), (1 - \alpha) \cdot \mu_2(x)) : x \in X\}, \alpha \in [0, 1]$ is an intuitionistic fuzzy set.

Theorem 3.10 If $F$ and $G$ are Atanassov Intuitionistic Fuzzy Sets Generating Maps (AIFSGM) from $M^kFS(X)$ into $M^2FS(X)$ then

1. $F \wedge G$ is an AIFSGM on $M^kFS(X)$ and
2. $F \vee G$ need not be an AIFSGM on $M^kFS(X)$, where $(F \wedge G)(\mu) = F(\mu) \cap G(\mu)$ and $(F \vee G)(\mu) = F(\mu) \cup G(\mu)$. 
Proof (1). For any $\mu \in M^{k}\text{FS}(X)$, we have $(F \wedge G)(\mu) = F(\mu) \cap G(\mu) = (\nu_1, \nu_2) \cap (\lambda_1, \lambda_2) = \{(x, \min(\nu_1(x), \lambda_1(x)), \min(\nu_2(x), \lambda_2(x))) : x \in X\}$, where $F(\mu) = (\nu_1, \nu_2)$ and $G(\mu) = (\lambda_1, \lambda_2)$. By the definition of intuitionistic fuzzy sets; $\forall x \in X$, we have $0 \leq \nu_1(x) + \nu_2(x) \leq 1$ and $0 \leq \lambda_1(x) + \lambda_2(x) \leq 1$, that implies $0 \leq \min(\nu_1(x), \lambda_1(x)) + \min(\nu_2(x), \lambda_2(x)) \leq 1, \forall x \in X$. Therefore $F \wedge G$ is an AIFSGM.

(2). We prove this part by a counter example. Let $F(\mu) = (\nu_1, \nu_2)$, $G(\mu) = (\lambda_1, \lambda_2)$, where $\nu_1(x) = \frac{5}{7}, \nu_2(x) = \frac{1}{7}$, $\lambda_1(x) = \frac{2}{7}$ and $\lambda_2(x) = \frac{4}{7}, \forall x \in X$. $(F \lor G)(\mu) = F(\mu) \cup G(\mu) = \{(x, \frac{5}{7}, \frac{1}{7}) : x \in X\} \cup \{(x, \frac{2}{7}, \frac{4}{7}) : x \in X\} = \{(x, \max(\frac{5}{7}, \frac{2}{7}), \min(\frac{1}{7}, \frac{4}{7})) : x \in X\} = \{(x, \frac{5}{7}, \frac{4}{7}) : x \in X\}$; $F \lor G$ is not an AIFSGM, since $\frac{5}{7} + \frac{4}{7} > 1$.

Theorem 3.11 If $F$ and $G$ are AIFSGM from $M^{2}\text{FS}(X)$ into $M^{2}\text{FS}(X)$ then $F \circ G$ is an AIFSGM on $M^{2}\text{FS}(X)$

Proof. Let $\mu(x) = (\mu_1(x), \mu_2(x))$ and $G(\mu) = G(\mu_1, \mu_2) = (\nu_1, \nu_2)$. That is $G(\mu_1, \mu_2)(x) = (\nu_1(x), \nu_2(x))$ and $(F \circ G)(\mu) = F(G(\mu_1, \mu_2))(x) = F(\nu_1, \nu_2)(x)$. Since $F$ is an AIFSGM, there exists a $\gamma \in M^{2}\text{FS}(X)$ such that $F(\nu_1, \nu_2)(x) = \gamma(x) = (\gamma_1(x), \gamma_2(x))$ with $0 \leq \gamma_1(x) + \gamma_2(x) \leq 1, \forall x \in X$. Therefore $(F \circ G)(\mu) = \gamma$ is an intuitionistic fuzzy set.

Note 3.12 Atanassov Intuitionistic Fuzzy Sets Generating Maps (AIFSGM) and Atanassov Intuitionistic Fuzzy operations are different concepts.

Theorem 3.13 If $F : M^{2}\text{FS}(X) \rightarrow M^{2}\text{FS}(X)$ is a mapping defined by $F(\mu_1, \mu_2) = (r_1, r_2, r_2, \mu_2)$, where $r_1, r_2 \in [0, \frac{1}{2}]$, then $F$ is an AIFSGM on $M^{2}\text{FS}(X)$. Moreover if $F^n(\mu_1, \mu_2) = F \circ F^{n-1}(\mu_1, \mu_2)$, any integer $n \geq 2$, then $\lim_{n \to \infty} F^n(\mu_1, \mu_2) = (0, 0)$

Proof. $F(\mu_1, \mu_2)$ is an intuitionistic fuzzy set, since $F(\mu_1, \mu_2)(x) = (r_1, \mu_1(x), r_2, \mu_2(x))$ and $0 \leq r_1, \mu_1(x) + r_2, \mu_2(x) \leq r_1 + r_2, \leq 1, \forall x \in X$. By the mathematical induction $F^n(\mu_1, \mu_2)(x) = (r^n_1, \mu_1(x), r^n_2, \mu_2(x)), \forall x \in X$. Since $r^n_1$ and $r^n_2$ tend to 0 as $n$ tends to 0, $\lim_{n \to \infty} F^n(\mu_1, \mu_2)(x) = (0, 0), \forall x \in X$. Hence $\lim_{n \to \infty} F^n(\mu_1, \mu_2) = (0, 0)$.

Theorem 3.14 If $F$ is a mapping on $M^{2}\text{FS}(X)$ defined by $F(\mu)(x) = F(\mu_1, \mu_2)(x) = (t(\mu_1(x), \mu_2(x)), t(1 - \mu_1(x), 1 - \mu_2(x)))$, where $t$ is a $t$-norm, then $F$ is an AIFSGM on $M^{2}\text{FS}(X)$.

Proof.

$t(\mu_1(x), \mu_2(x)) + t(1 - \mu_1(x), 1 - \mu_2(x)) = t(\mu_1(x), \mu_2(x)) + (1 - s(\mu_1(x), \mu_2(x)))$
=1 - [s(\mu_1(x), \mu_2(x)) - t(\mu_1(x), \mu_2(x))], where s is the dual norm of t. We obtain the inequality 0 \leq 1 - [s(\mu_1(x), \mu_2(x)) - t(\mu_1(x), \mu_2(x))] \leq 1, since both t(\mu_1(x), \mu_2(x)) and t(1 - \mu_1(x), 1 - \mu_2(x)) \in [0, 1] and t(\mu_1(x), \mu_2(x)) \leq s(\mu_1(x), \mu_2(x)) for all x in X. Therefore 0 \leq t(\mu_1(x), \mu_2(x)) + t(1 - \mu_1(x), 1 - \mu_2(x)) \leq 1 and F(\mu) is an intuitionistic fuzzy set.

**Example 3.15** Let F be a mapping on \(M^2\text{FS}(X)\) defined by \(F(\mu_1, \mu_2)(x) = (\mu_1(x), \mu_2(x), (1-\mu_1(x)), (1-\mu_2(x)))\), then F is an AIFSGM on \(M^2\text{FS}(X)\).

### 4 Concluding Remarks

In this paper we have proposed the concept of multi-fuzzy mappings and Atanassov Intuitionistic Fuzzy Sets Generating Maps, which have the fundamental role in the study of multi-fuzzy set theory. Multi-fuzzy set theory is a direct extension of fuzzy set theory.

**ACKNOWLEDGEMENTS.**

The authors would like to thank Prof.T.Thrivikraman and the reviewers for their valuable comments and helpful suggestions for improvement of the original manuscript.

**References**


**Received:** February, 2010