Coincidence and Fixed Points of Weakly Contractive Mappings in Intutionistic Fuzzy Metric Spaces

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Abstract
In this paper we define a new weak contractive condition for a pair of self mappings in Intutionistic fuzzy metric space and prove the existence of coincidence and common fixed points

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1 Introduction

P. Vijayaraju and Z. M. I. Sajath introduced the notion of cauchy sequences and proved the well known fixed point theorems of Banach in the context of intuitionistic fuzzy metric space. In this paper we introduce weak contractivity condition for one mapping with respect to another in Intuitionistic fuzzy metric space and prove the existence of coincidence and common fixed points in the intutionistic fuzzy metric space. We begin with some definitions and preliminary concepts.

**Definition 1.1.** A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a continuous \( t \)-norm if \(([0,1],*)\) is an abelian topological monoid with unit 1 such that \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0, 1] \).

Examples of \( t \)-norm are \( a * b = ab \) and \( a * b = \min\{a, b\} \).

**Definition 1.2.** A binary operation \( \odot : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called continuous \( t \)–conorm if it satisfies the following conditions:

(a) \( \odot \) is commutative and associative;
(b) \( \odot \) is continuous;
(c) \( a \odot 0 = a \) for all \( a \in [0, 1] \);
(d) \( a \odot b \leq c \odot d \) whenever \( a \leq c \) and \( b \leq d \) for each \( a, b, c, d \in [0, 1] \).

Example of \( t \)–conorm is \( a \odot b = \min\{1, a + b\} \) where \( a, b \in [0, 1] \).

**Definition 1.3.** A 5-tuple \((X, M, N, *, \odot)\) is said to be an intuitionistic fuzzy metric space if \( X \) is an arbitrary set, \(* \) is a continuous \( t \)-norm, \( \odot \) is a continuous \( t \)–conorm and \( M, N \) are a fuzzy sets on \( X^2 \times [0, \infty) \) satisfying the following conditions: for all \( x, y, z \in X \) and all \( s, t > 0 \),

1. \( M(x, y, t) + N(x, y, t) \leq 1 \);
2. \( M(x, y, 0) = 0 \);
3. \( M(x, y, t) = 1 \) for all \( t > 0 \) if and only if \( x = y \);
4. \( M(x, y, t) = M(y, x, t) \);
5. \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \);
6. \( M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1] \) is left continuous;
7. \( \lim_{t \to \infty} M(x, y, t) = 1 \);
8. \( N(x, y, 0) = 1 \);
(9) $N(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$;

(10) $N(x, y, t) = N(y, x, t)$;

(11) $N(x, y, t) \circ N(y, z, s) \geq N(x, z, t + s)$;

(12) $N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is right continuous;

(13) $\lim_{t \to \infty} N(x, y, t) = 0$.

$(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and non-nearness between $x$ and $y$ with respect to $t$ respectively.

**Definition 1.4.** Let $(X, M, N, \ast, \cdot)$ be an intuitionistic fuzzy metric space

A sequence $\{x_n\}$ in $X$ is called Cauchy sequence if and only if

$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0$ for each $p > 0$, $t > 0$.

A sequence $\{x_n\}$ in $X$ is converging to $x$ in $X$ if and only if

$\lim_{n \to \infty} M(x_n, x, t) = 1$ and $\lim_{n \to \infty} N(x_n, x, t) = 0$ for each $t > 0$.

An intuitionistic fuzzy metric space $(X, M, N, \ast, \circ)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent in $X$.

**Definition 1.5.** Two mappings $f$ and $g$ are compatible if and only if

$\lim_{n \to \infty} M(fg(x_n), gf(x_n), t) = 1$ and $\lim_{n \to \infty} N(fg(x_n), gf(x_n), t) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = x_0 \in X$.

**Definition 1.6.** Two mappings $f$ and $g$ are compatible if $f(x_0) = g(x_0)$ for some $x_0 \in X$, then $fg(x_0) = gf(x_0)$

**Lemma 1.7.** Let $f, g$ be two compatible mappings on $X$. If $f(x) = g(x)$ for some $x$ in $X$, then $fg(x) = gf(x)$.

**Definition 1.8.** A point in $X$ is a coincidence point (fixed point) of $f$ and $T$ if $f(x) = T(x)(T(x) = f(x) = x)$.

## 2 Main Results

We begin with the following definition.

**Definition 2.1.** Let $(X, M, N, \ast, \circ)$ be an intuitionistic fuzzy metric space. A mapping $T : X \rightarrow X$ is called weakly contractive with respect to $f : X \rightarrow X$ if for each $x, y$ in $X$, $M(Tx, Ty, t) \geq M(fx, fy, t) + \phi(M(fx, fy, t))$ and $N(Tx, Ty, t) \leq N(fx, fy, t) - \psi(N(fx, fy, t))$ for all $t > 0$, where $\phi : [0, 1] \rightarrow$
[0, 1] is continuous and nonincreasing such that \( \phi(\alpha) \leq 1 - \alpha \), \( \phi \) is positive in \((0, 1)\), \( \phi(0) = 1 \) and \( \phi(1) = 0 \) and \( \psi : [0, 1] \to [0, 1] \) is continuous and nondecreasing such that \( \psi(\alpha) \leq \alpha \), \( \psi \) is positive in \((0, 1)\), \( \psi(0) = 0 \) and \( \psi(1) = 1 \).

**Theorem 2.2.** Let \((X, M, N, *, \circ)\) be an intutionistic fuzzy metric space and let \(T : X \to X\) be a weakly contractive mapping with respect to \(f : X \to X\). If the range of \(f\) contains the range of \(T\) and \(f(X)\) is a complete subspace of \(X\), then \(f\) and \(T\) have a coincidence point in \(X\).

**Proof.** Let \(x_0\) be an arbitrary point in \(X\). Choose a point \(x_1 \in X\) such that \(f(x_1) = T(x_0)\). This is possible since the range of \(f\) contains the range of \(T\). Continuing this process and having chosen \(x_n\) in \(X\), we obtain \(x_{n+1}\) in \(X\) such that \(f(x_{n+1}) = T(x_n)\). Now consider

\[
M(f(x_{n+1}), f(x_{n+2}), t) = M(T(x_n), T(x_{n+1}), t) \\
\geq M(f(x_n), f(x_{n+1}), t) + \phi(M(f(x_n), f(x_{n+1}), t)) \\
\geq M(f(x_n), f(x_{n+1}), t). \quad (1)
\]

This implies that \(\{M(f(x_{n+1}), f(x_{n+2}), t)\}\) is a nondecreasing sequence of positive real numbers with lower bound 0 and upper bound 1. Hence it converges to a limit \(0 < l \leq 1\). If \(l \neq 1\), then we have

\[
M(f(x_{n+1}), f(x_{n+2}), t) \geq M(f(x_n), f(x_{n+1}), t) + \phi(l) \\
M(f(x_{n+2}), f(x_{n+3}), t) \geq M(f(x_{n+1}), f(x_{n+2}), t) + \phi(l) \\
\geq M(f(x_n), f(x_{n+1}), t) + 2\phi(l)
\]

Thus, \(M(f(x_{n+N}), f(x_{n+N+1}), t) \geq M(f(x_n), f(x_{n+1}), t) + N\phi(l)\)......................(2)

which is a contradiction for sufficiently large \(N\).

Therefore, \(\lim_{n \to \infty} M(f(x_n), f(x_{n+1}), t) = 1\) for all \(t > 0\)......................(3)

Further,from (2)we have

\[
M(f(x_{n+N}), f(x_{n+N+1}), t) \geq M(f(x_n), f(x_{n+1}), t) \quad \text{..........................(4)}
\]

Now using (4) and the property (iv) in Definition(1.3), we obtain

\[
M(f(x_{n+p}), f(x_n), t) \geq M(f(x_{n+1}), f(x_n), \frac{t}{2}) \circ M(f(x_{n+1}), f(x_n), \frac{t}{2^2}) \circ \cdots \circ M(f(x_{n+1}), f(x_n), \frac{t}{2^p})
\]

where \(t > 0\).

Taking limit as \(n \to \infty\) and using (3), we get

\[
\lim_{n \to \infty} M(f(x_{n+p}), f(x_n), t) = 1.
\]
Thus \( \{f(x_n)\} \) is a Cauchy sequence. Since \( f(X) \) is complete, there exists \( p \in X \) such that \( \lim_{n \to \infty} f(x_n) = f(p) \). Let \( f(p) = q \). Now,

\[
M(f(x_{n+1}), T(p), t) = M(T(x_n), T(p), t) \\
\geq M(f(x_n), f(p), t) + \phi(M(f(x_n), f(p), t))
\]

Taking limit as \( n \to \infty \), we obtain

\[
M(q, T(p), t) \geq 1 + \phi(1).
\]

That is \( M(q, T(p), t) = 1 \).

Now consider

\[
N(f(x_{n+1}), f(x_{n+2}), t) = N(T(x_n), T(x_{n+1}), t) \\
\leq N(f(x_n), f(x_{n+1}), t) - \psi(N(f(x_n), f(x_{n+1}), t)) \\
\leq N(f(x_n), f(x_{n+1}), t).
\tag{2}
\]

This implies that \( \{N(f(x_{n+1}), f(x_{n+2}), t)\} \) is a nonincreasing sequence of positive real numbers with lower bound 0. Hence it converges to a limit \( c \geq 0 \).

We show that \( c = 0 \). If it is not so, then

\[
N(f(x_{n+1}), f(x_{n+2}), t) \leq N(f(x_n), f(x_{n+1}), t) - \psi(c) \\
N(f(x_{n+2}), f(x_{n+3}), t) \leq N(f(x_{n+1}), f(x_{n+2}), t) - \psi(c) \\
\leq N(f(x_n), f(x_{n+1}), t) - 2\psi(c)
\]

Thus \( N(f(x_{n+K}), f(x_{n+K+1}), t) \leq N(f(x_n), f(x_{n+1}), t) - K\psi(c) \),............(2)

which is a contradiction for sufficiently large \( K \).

\[
N(f(x_{n+p}), f(x_n), t) \leq N(f(x_{n+1}), f(x_n), \frac{t}{2}) \circ N(f(x_{n+1}), f(x_n), \frac{t}{2^2}) \\
\circ \cdots \cdots \cdots \circ N(f(x_{n+1}), f(x_n), \frac{t}{2^{p-1}})
\]

where \( t > 0 \).

Taking limit as \( n \to \infty \) and using (3), we get

\[
\lim_{n \to \infty} N(f(x_{n+p}), f(x_n), t) = 0.
\]

Thus \( \{f(x_n)\} \) is a Cauchy sequence. Since \( f(X) \) is complete, there exists \( p \in X \) such that \( \lim_{n \to \infty} f(x_n) = f(p) \). Let \( f(p) = q \). Now,

\[
N(f(x_{n+1}), T(p), t) = N(T(x_n), T(p), t) \\
\leq N(f(x_n), f(p), t) - \psi(N(f(x_n), f(p), t))
\]

Taking limit as \( n \to \infty \), we obtain

\[
N(q, T(p), t) \leq 0 + \psi(0).
\]

That is \( N(q, T(p), t) = 0 \).

Therefore, \( q = T(p) = f(p) \).
Theorem 2.3. Let \((X, M, *, \diamond)\) be an intuitionistic fuzzy metric space which is complete and let \(T : X \rightarrow X\). If \(M(Tx, Ty, t) \geq M(x, y, t) + \phi(M(x, y, t))\) and \(N(Tx, Ty, t) \leq N(x, y, t) - \psi(N(x, y, t))\) for each \(x, y \in X\) and for all \(t > 0\), where
\[
\begin{align*}
(i) & \phi : [0, 1] \rightarrow [0, 1] \text{ is continuous and nonincreasing such that } \phi(\alpha) \leq 1 - \alpha, \\
& \phi \text{ is positive in } (0, 1), \phi(0) = 1 \text{ and } \phi(1) = 0 \\
(ii) & \psi : [0, 1] \rightarrow [0, 1] \text{ is continuous and nondecreasing such that, } \psi(\alpha) \geq \alpha \\
& \psi \text{ is positive in } (0, 1), \psi(0) = 0 \text{ and } \psi(1) = 1,
\end{align*}
\]
then \(T\) has a fixed point in \(X\).

**Proof.** If \(f = id_x\) (the identity map of \(X\)) in Theorem(2.2), then \(p = T(p)\) is the fixed point of \(T\).

Theorem 2.4. Let \((X, M, *, \diamond)\) be an intuitionistic fuzzy metric space and let \(T\) be a weakly contractive mapping with respect to \(f\). If \(T\) and \(f\) are weakly compatible and \(T(X) \subset f(X)\) and \(f(X)\) is complete subspace of \(X\), then \(T\) and \(f\) have a common fixed point in \(X\).

**Proof.** By Theorem(2.2), we get a point \(p \in X\) such that \(f(p) = T(p) = q\) (say). Then by Lemma(1.7), \(Tf(p) = fT(p)\). Further \(ff(p) = fT(p)\). This implies \(ff(p) = Tf(p)\). Therefore \(f(q) = T(q)\). Now we show that \(f(q) = q\). If it is not so, consider
\[
\begin{align*}
M(f(q), q, t) & = M(T(q), T(p), t) \\
& \geq M(f(q), f(p), t) + \phi(M(f(q), f(p), t)) \\
& \geq M(f(q), q, t) + \phi(M(f(q), q, t)) \\
& > M(f(q), q, t).
\end{align*}
\]
\[
\begin{align*}
N(f(q), q, t) & = N(T(q), T(p), t) \\
& \leq N(f(q), f(p), t) - \psi(N(f(q), f(p), t)) \\
& \leq N(f(q), q, t) - \psi(N(f(q), q, t)) \\
& < N(f(q), q, t).
\end{align*}
\]
This contradiction leads to the result.

**References**


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