Isometry of a Sequence Space Generated by a Difference Operator

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Abstract

The difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$ were studied by H. Kizmaz [3]. These difference sequence spaces were then generalized by Colak and Et [2]. In this study, we consider the difference sequence space $\ell_p(\Delta^m)$. Using a recursive method, we will show that if $\ell_p(\Delta^m)$ is equipped with an appropriate norm then this sequence space is linearly isometric to the usual sequence space $\ell_p$.

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1 Introduction

All throughout this paper we let $p \in [1, \infty)$. By $\omega$, we shall denote the space of all sequences $x = (x_k)$, where $x_k \in \mathbb{C}$ for all $k \in \mathbb{N}$. Given $x \in \omega$, define

$$\|x\|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

and let

$$\ell_p = \{x = (x_k) : \|x\|_p < \infty\}.$$

We define the linear difference operator $\Delta : \omega \rightarrow \omega$ which maps a sequence $x \in \omega$ into a sequence $\Delta x = (\Delta x_k) \in \omega$ having components

$$\Delta x_k = x_k - x_{k+1}.$$
For $m \geq 2$, the linear operator $\Delta^m : \omega \rightarrow \omega$ is defined recursively as the composition $\Delta^m = \Delta \circ \Delta^{m-1}$. One can easily check that for $m \geq 1$ and $x \in \omega$ we have the following Binomial representation

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v},$$

for all $k \in \mathbb{N}$.

Given $m \in \mathbb{N}$ we define the sequence space

$$\ell_p(\Delta^m) = \{x = (x_k) : \Delta^m x \in \ell_p\}$$

and for $x \in \ell_p(\Delta^m)$ we let

$$\|x\|_{p,\Delta^m} = \left(\sum_{i=1}^{m} |x_i|^p + \|\Delta^m x\|_p^p\right)^{1/p}. \quad (1)$$

It can be easily seen that the pair $(\ell_p(\Delta^m), \| \cdot \|_{p,\Delta^m})$ is a normed space.

If $\ell_\infty, c,$ and $c_0$ are the linear spaces of bounded, convergent, and null sequences $x = (x_k)$ having complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, then we similarly define the difference sequence spaces

$$\ell_\infty(\Delta^m) = \{x = (x_k) : \Delta^m x \in \ell_\infty\},$$

$$c(\Delta^m) = \{x = (x_k) : \Delta^m x \in c\},$$

$$c_0(\Delta^m) = \{x = (x_k) : \Delta^m x \in c_0\}.$$

We have the following inclusions $\ell_p(\Delta^m) \subset \ell_\infty(\Delta^m)$ and $\ell_p(\Delta^m) \subset c(\Delta^m) \subset c(\Delta^m)$. The difference sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$, and $c_0(\Delta^m)$ have been considered by Colak and Et [2] and they showed that these are Banach spaces with norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty.$$

For Euler difference sequence spaces and sequence spaces generated by a sequence of Orlicz functions, the reader may consult Qamaruddin and Saifi [4] and Altay and Polat [1], respectively. In this paper, we will show that $\ell_p(\Delta^m)$ with norm $\| \cdot \|_{p,\Delta^m}$ is a Banach space linearly isometric to the ordinary sequence space $\ell_p$. Furthermore, a sufficient condition for the inclusion $\ell_p(\Delta^m) \subset \ell_p(\mathcal{M}, \Delta^m)$, where $\mathcal{M}$ is a family of Orlicz functions satisfying the $\Delta_2$-condition, shall be given.
2 Main Results

Theorem 2.1. The sequence space \( \ell_p(\Delta^m) \) equipped with the norm \( \| \cdot \|_{p, \Delta^m} \) is a Banach space.

Proof. Let \((x^{(n)}) = ((x^{(n)}_k))\) be a Cauchy sequence in \( \ell_p(\Delta^m) \). Then given \( \epsilon > 0 \) we can find a positive integer \( N \) such that \( \|x^{(n)} - x^{(r)}\|_{p, \Delta^m} < \epsilon \) whenever \( n, r \geq N \), that is,

\[
\left( \sum_{i=1}^{m} |x_i^{(n)} - x_i^{(r)}|^p + \|\Delta^m x^{(n)} - \Delta^m x^{(r)}\|_p \right)^{1/p} < \epsilon,
\]

for \( n, r \geq N \). Since

\[
|x_i^{(n)} - x_i^{(r)}| \leq \|x^{(n)} - x^{(r)}\|_{p, \Delta^m}
\]

for all \( i = 1, 2, \ldots, m \) and

\[
\|\Delta^m x^{(n)} - \Delta^m x^{(r)}\|_p \leq \|x^{(n)} - x^{(r)}\|_{p, \Delta^m},
\]

it follows that \((x^{(n)}_i)\) and \((\Delta^m x^{(n)})\) are Cauchy sequence in \( \mathbb{C} \) and \( \ell_p \), respectively. The completeness of the spaces \( \mathbb{C} \) and \( \ell_p \) imply the existence of elements \( y_i \in \mathbb{C}, i = 1, 2, \ldots, m \), and \( z = (z_k) \in \ell_p \) such that

\[
\lim_{n \to \infty} |x_i^{(n)} - y_i| = 0 \tag{2}
\]

for all \( i = 1, 2, \ldots, m \) and

\[
\lim_{n \to \infty} \|\Delta^m x^{(n)} - z\|_p = 0. \tag{3}
\]

Furthermore, since \( |\Delta^m x^{(n)}_k - z_k| \leq \|\Delta^m x^{(n)} - z\|_p \) it follows from Equation (3) that \( |\Delta^m x^{(n)}_k - z_k| \to 0 \) as \( n \to \infty \) for all \( k \in \mathbb{N} \).

Next, we will find a recursive formula for the limit of \( x^{(n)}_{m+i}, i \geq 1 \), as \( n \to \infty \). Notice that

\[
(-1)^m x^{(n)}_{m+1} = \Delta^m x^{(n)}_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} x^{(n)}_{v+1}
\]

and so

\[
w_{m+1} := \lim_{n \to \infty} x^{(n)}_{m+1} = (-1)^m \left[ z_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} \right].
\]
Suppose that \( w_{m+1}, \ldots, w_{m+k-1}, 1 < k \leq m \), have been constructed where
\[
w_{m+i} := \lim_{n \to \infty} x_{m+i}^{(n)}, \quad i = 1, 2, \ldots, k-1.
\]
Using these we have, for \( 1 < k \leq m \),
\[
w_{m+k} := \lim_{n \to \infty} x_{m+k}^{(n)} = (-1)^m \left[ z_k - \sum_{v=0}^{m-k} (-1)^v \binom{m}{v} y_{v+k}
- \sum_{v=1}^{k-1} (-1)^{m-k+v} \binom{m}{m-k+v} w_{m+v} \right].
\]
On the other hand, for \( k > m \) we have
\[
(-1)^{x_{m+k}^{(n)}} = \Delta^m x_k^{(n)} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} x_{v+k}^{(n)}
\]
so that
\[
w_{m+k} = \lim_{n \to \infty} x_{m+k}^{(n)} = (-1)^m \left[ z_k - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} w_{k+v} \right].
\]
Let \( w = (y_1, \ldots, y_m, w_{m+1}, w_{m+2}, \ldots) \). We claim that \( w \in \ell_p(\Delta^m) \), that is, \( \Delta^m w \in \ell_p \). First, observe that
\[
(\Delta^m w)_1 = \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} + (-1)^m w_{m+1}
= \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} + \left[ z_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} \right]
= z_1.
\]
Also, for \( k = 2, \ldots, m \), we have
\[
(\Delta^m w)_k = \sum_{v=0}^{m-k} (-1)^v \binom{m}{v} y_{v+k} + \sum_{v=m-k+1}^{m-1} (-1)^v \binom{m}{v} w_{v+k} + (-1)^m w_{m+k}
= z_k.
\]
Similarly, for \( k > m \) we obtain
\[
(\Delta^m w)_k = \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} w_{v+k} + (-1)^m w_{m+k}
= z_k.
\]
Therefore we have shown that $\triangle^m w = z \in \ell_p$. Finally, it remains to show that $\|x^{(n)} - w\|_{p,\triangle^m} \to 0$ as $n \to \infty$. This follows directly from Equations (2) and (3) and the following computations

$$
\lim_{n \to \infty} \|x^{(n)} - w\|_{p,\triangle^m} = \lim_{n \to \infty} \left( \sum_{k=1}^{m} |x_k^{(n)} - y_k|^p + \|\triangle^m x^{(n)} - \triangle^m w\|^p_{p} \right) = \sum_{k=1}^{m} \lim_{n \to \infty} |x_k^{(n)} - y_k|^p + \lim_{n \to \infty} \|\triangle^m x^{(n)} - z\|^p_{p} = 0.
$$

This completes the proof of the theorem.

**Theorem 2.2.** The sequence spaces $(\ell_p(\triangle^m), \cdot \cdot \cdot, \|\cdot\|_{p,\triangle^m})$ and $(\ell_p, \cdot \cdot \cdot, \|\cdot\|_{p})$ are linearly isometric.

**Proof.** Consider the map $T : \ell_p(\triangle^m) \to \ell_p$ defined by $Ty = x$, where $y = (y_k) \in \ell_p(\triangle^m)$ and $x = (x_k)$ with

$$x_k = \begin{cases} y_k, & \text{if } 1 \leq k \leq m; \\ \triangle^m y_{k-m}, & \text{if } k > m. \end{cases}$$

The linearity of the difference operator $\triangle$ implies the linearity of $T$. If $y \in \ell_p(\triangle^m)$ and $Ty = x$ then

$$\|Ty\|^p_p = \|x\|^p_p = \sum_{k=1}^{m} |y_k|^p + \sum_{k=m+1}^{\infty} |\triangle^m y_{k-m}|^p = \sum_{k=1}^{m} |y_k|^p + \sum_{k=1}^{\infty} |\triangle^m y_k|^p = \|y\|^p_{p,\triangle^m} < \infty.$$

This shows that $T$ is well-defined and it is also norm preserving. Now, we are going to show that $T$ is one-to-one and onto. Assume that $Ty = 0$. Then it follows that

$$\triangle^m y_k = 0 \text{ for all } k \geq 1, \quad (4)$$
$$y_1 = y_2 = \cdots = y_m = 0. \quad (5)$$

We note that the difference equation (4) with initial conditions (5) implies that $y_k = 0$ for all $k \geq 1$, that is, $y = (0, 0, \ldots)$. Hence $T$ is one-to-one.

Suppose that $x = (x_k) \in \ell_p$. Define the sequence $y = (y_k)$ as follows. For $k = 1, 2, \ldots, m$, let $y_k = x_k$. The succeeding terms of the sequence $y$ is then
defined recursively by
\[
\begin{align*}
y_{m+1} &= (-1)^m \left[ x_{m+1} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} x_{v+k} \right] \\
y_{m+k} &= (-1)^m \left[ x_{m+k} - \sum_{v=0}^{m-k} (-1)^v \binom{m}{v} x_{v+k} - \sum_{v=1}^{k-1} (-1)^{m-k+v} \binom{m}{m-k+v} y_{m+v} \right], & 1 < k \leq m,
\end{align*}
\]
and
\[
\begin{align*}
y_{m+k} &= (-1)^m \left[ x_{m+k} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+k} \right], & k > m.
\end{align*}
\]

Using a similar argument as in the proof of the previous theorem, we can show that
\[
\Delta^m y_k = x_{k+m}
\]
for \( k \in \mathbb{N} \). Hence it follows that \( Ty = x \). Furthermore,
\[
\|\Delta^m y\|^p_p = \sum_{k=1}^{\infty} |\Delta^m y_k|^p_p = \sum_{k=1}^{\infty} |x_{k+m}|^p \leq \|x\|^p_p < \infty,
\]
so that \( y \in \ell_p(\Delta^m) \). Hence, \( T \) is onto. Therefore \( \ell_p(\Delta^m) \) and \( \ell_p \) are linearly isometric.

An **Orlicz function** \( M : [0, \infty) \to [0, \infty) \) is a continuous, convex, non-decreasing function with \( M(u) = 0 \) if and only if \( u = 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). An Orlicz function is said to satisfy the \( \Delta_2 \)-condition if there exists a positive constant \( K \) such that \( M(2u) \leq KM(u) \) for all \( u \geq 0 \). Let \( \mathcal{M} = (M_k) \) be a sequence of Orlicz functions satisfying the \( \Delta_2 \)-condition. Define the sequence spaces
\[
\ell_p(\mathcal{M}) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |M_k(|x_k|/\rho)|^p < \infty \right\}
\]
and
\[
\ell_p(\mathcal{M}, \Delta^m) = \{ x = (x_k) : \Delta^m x \in \ell_p(\mathcal{M}) \}.
\]
Theorem 2.3. Assume that $\mathcal{M} = (M_k)$ is a sequence of Orlicz functions satisfying the $\Delta_2$-condition. If
\[ \sum_{k=1}^{\infty} |M_k(t/\rho)|^p < \infty \tag{6} \]
for all $t, \rho > 0$ then $\ell_p(\Delta^m) \subset \ell_p(\mathcal{M}, \Delta^m)$.

Proof. Suppose that condition (6) holds and let $x = (x_k) \in \ell_p(\Delta^m)$. Then it follows that
\[ \sum_{k=1}^{\infty} |\Delta^m x_k|^p < \infty. \]
The convergence of the above series implies that
\[ \lim_{k \to \infty} |\Delta^m x_k| = 0. \]
Then we can find a positive integer $N$ such that $|\Delta^m x_k| \leq 1$ for all $k \geq N$. Let
\[ M = \max(|\Delta^m x_1|, \ldots, |\Delta^m x_{N-1}|, 1). \]
Then $|\Delta^m x_k| \leq M$ for all $k \in \mathbb{N}$. For $\rho > 0$, using the monotonicity of $M_k$, we have $M_k(|\Delta^m x_k|/\rho) \leq M_k(M/\rho)$ for all $k \in \mathbb{N}$. This inequality implies that
\[ \sum_{k=1}^{\infty} |M_k(|\Delta^m x_k|/\rho)|^p \leq \sum_{k=1}^{\infty} |M_k(M/\rho)|^p \]
and from equation (6) this estimate implies that $\Delta^m x \in \ell_p(\mathcal{M})$, that is, $x \in \ell_p(\mathcal{M}, \Delta^m)$. Therefore the inclusion $\ell_p(\Delta^m) \subset \ell_p(\mathcal{M}, \Delta^m)$ holds.

References


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