Optimal Sequential Procedures
with Bayes Decision Rules

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Abstract

In this article, a general problem of sequential statistical inference for general discrete-time stochastic processes is considered. Let $X_1, X_2, \ldots$ be a discrete-time stochastic process, whose distribution depends on an unknown parameter $\theta$, $\theta \in \Theta$. We consider a problem of optimal sequential decision-making in the following framework. Let $w_n(\theta, d; x_1, \ldots, x_n)$, $\theta \in \Theta$, $d \in D$, be a loss function representing losses from making a decision $d$ at stage $n$ of a statistical experiment, when the true parameter value is $\theta$, and the data observed up to this stage are $x_1, \ldots, x_n$. Let $K^\theta_n(x_1, \ldots, x_n)$ be the cost of the observations when $\theta$ is the true value of the parameter. The decision is supposed to be taken through a sequential decision-making procedure $(\tau, \delta)$, where $\tau$ is a stopping time with respect to the sequence of $\sigma$-algebras $F_n = \sigma(X_1, X_2, \ldots, X_n)$, $n = 1, 2, \ldots$, and $\delta$ is an $F_\tau$-measurable decision function with values in $D$. For any sequential decision procedure $(\tau, \delta)$ let us define the average loss due to incorrect decision

$$W(\theta; \tau, \delta) = \mathbb{E}_\theta w_{\tau}(\theta, \delta; X_1, \ldots, X_{\tau}),$$

and the average cost of observations as

$$C(\theta; \tau) = \mathbb{E}_\theta K_{\theta}^\tau(X_1, \ldots, X_{\tau}).$$

Let, finally, the “risk function” be defined as

$$R(\tau, \delta) = \int_{\Theta} W(\theta; \tau, \delta) d\pi_1(\theta) + \int_{\Theta} C(\theta; \tau) d\pi_2(\theta),$$

where $\pi_1$ and $\pi_2$ are some probability measures on $\Theta$. The main goal of this article is to give conditions of existence of sequential decision procedures which minimize $R(\tau, \delta)$ (optimal decision procedures), and
characterize their structure. In particular, when $\pi_1 = \pi_2 = \pi$ is an 
a priori distribution of the parameter, we give a characterization of 
opimal (Bayesian) sequential decision procedures minimizing $R(\tau, \delta)$ 
among all sequential decision procedures $(\tau, \delta)$.

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process, optimal decision rule, optimal stopping rule, randomized stopping 
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## 1 Introduction

Let $X_1, X_2, \ldots, X_n, \ldots$ be a discrete-time stochastic process, whose distribution depends on an unknown "parameter" $\theta$, $\theta \in \Theta$. In this article, we consider a general problem of sequential statistical decision making based on the observations of this process.

Let us define a **sequential statistical procedure** as a pair $(\psi, \delta)$, being $\psi$ a (randomized) **stopping rule**, $\psi = (\psi_1, \psi_2, \ldots, \psi_n, \ldots)$, and $\delta$ a (terminal) **decision function**, $\delta = (\delta_1, \delta_2, \ldots, \delta_n, \ldots)$, supposing that $\psi_n = \psi_n(x_1, x_2, \ldots, x_n)$ and $\delta_n = \delta_n(x_1, x_2, \ldots, x_n)$ are measurable functions, and $\psi_n(x_1, \ldots, x_n) \in [0, 1]$, $\delta_n(x_1, \ldots, x_n) \in D$ for any observations vector $(x_1, \ldots, x_n)$, for any $n = 1, 2, \ldots$ (see, for example, [10], [1], [3]).

For any stage number $n \geq 1$, $\psi_n(x_1, \ldots, x_n)$ is interpreted as the conditional probability to stop and proceed to decision making, given that we did not stop before and that the observations up to this stage were $(x_1, \ldots, x_n)$, and $\delta_n(x_1, \ldots, x_n)$ as the decision to take when stopping occurs, $n = 1, 2, \ldots$.

The stopping rule $\psi$ generates a random variable $\tau_\psi$ (**stopping time**) whose distribution is given by

$$P_\theta(\tau_\psi = n) = E_\theta(1 - \psi_1)(1 - \psi_2)\ldots(1 - \psi_{n-1})\psi_n, \quad n = 1, 2, \ldots \quad (1)$$

(here, and in what follows, we interchangeably use $\psi_n$ both for $\psi_n(x_1, x_2, \ldots, x_n)$ and for $\psi_n(X_1, X_1, \ldots, X_n)$: it $\psi_n$ is under the expectation or probability sign, then it is $\psi_n(X_1, \ldots, X_n)$, otherwise it is $\psi_n(x_1, \ldots, x_n)$).

Let $w_n(\theta, d; x_1, \ldots, x_n)$ be a non-negative loss function, $n = 1, 2, \ldots$ (we suppose that $w_n$ is a measurable function of all its arguments for any $n \geq 1$). Let $\pi_1$ be any probability measure. We define the average loss of the sequential statistical procedure $(\psi, \delta)$ due to wrong decision as

$$W(\psi, \delta) = \sum_{n=1}^{\infty} \int [E_\theta(1 - \psi_1)\ldots(1 - \psi_{n-1})\psi_n w_n(\theta, \delta_n; X_1, \ldots, X_n)] d\pi_1(\theta). \quad (2)$$
Let also $K^n_\theta = K^n_\theta(x_1, \ldots, x_n)$ be a non-negative (and measurable with respect to $(\theta, x_1, \ldots, x_n)$) cost function, $n \geq 1$, such that $K^n_\theta(x_1, \ldots, x_n) \leq K^{n+1}_\theta(x_1, \ldots, x_{n+1})$ for any observation sequence $x_1, x_2, \ldots, x_{n+1}$, $n \geq 1$, $\theta \in \Theta$.

Let us define the average cost of the sequential decision procedure $(\tau, \delta)$ as

$$C(\theta; \psi) = E_\theta K^{\tau_\psi}_\theta(X_1, \ldots, X_{\tau_\psi})$$

(we suppose that $C(\theta; \psi) = \infty$ if $\sum_{n=1}^{\infty} P_\theta(\tau_\psi = n) < 1$, see (1)).

Let us also define a “weighted” value of the average cost

$$C(\psi) = \int C(\theta; \psi) d\pi_2(\theta),$$

(3)

where $\pi_2$ is some probability measure giving “weights” to particular values of $\theta$.

Our main goal is minimizing the “weighted risk”

$$R(\psi, \delta) = C(\psi) + W(\psi, \delta),$$

(4)

supposing that $\pi_1$ in (2) and $\pi_2$ in (3) are, generally speaking, two different probability measures. If $\pi_1 = \pi_2 = \pi$, $R(\psi, \delta)$ is called Bayesian risk of $(\psi, \delta)$ corresponding to the a priori distribution $\pi$ (see, for example, [11], [10], [1], [9], [3], among many others).

To guarantee that $\inf R(\psi, \delta)$ is finite we suppose that $\inf_\delta R(\psi^1, \delta) < \infty$ with $\psi^1 = (1, \ldots)$.

We use essentially the same method as in [4], where the case of $K^n_\theta \equiv n$ and $w_n(\theta, d; x_1, \ldots, x_n) \equiv w(\theta, d)$ for any $\theta \in \Theta$, $d \in D$, and for any $(x_1, \ldots, x_n)$, $n \geq 1$, was considered. In turn, the method of [4] is an extension of the results of [6]. In [6], there is a number of applications of the results of this nature to hypothesis testing problems starting from classical problems of Wald and Wolfowitz [11] and of Kiefer-Weiss (see [12]) to Bayesian hypothesis testing problems for stochastic processes considered in [2]. The same method is used in [7] for multiple hypothesis testing problems. An extension of this method for statistical problems with control variables can be found in [5] and in [8].

2 Main results

Throughout the paper we suppose that for any $n = 1, 2, \ldots$, the vector $(X_1, X_2, \ldots, X_n)$ has a probability “density” function

$$f^n_\theta = f^n_\theta(x_1, x_2, \ldots, x_n)$$
(Radon-Nikodym derivative of its distribution) with respect to a product-measure

\[ \mu^n = \underbrace{\mu \otimes \mu \otimes \cdots \otimes \mu}_n \]

with some \( \sigma \)-finite measure \( \mu \) on the respective space. As usual in the Bayesian context, we suppose that \( f^n_\theta(x_1, x_2, \ldots, x_n) \) is measurable with respect to \((\theta, x_1, \ldots, x_n)\), for any \( n = 1, 2, \ldots \).

Let us suppose that for any \( n \geq 1 \) there exists a measurable \( \delta^B_n = \delta^B_n(x_1, \ldots, x_n) \) such that for any \( d \in D \)

\[
\int w_n(\theta, d; x_1, \ldots, x_n) f^n_\theta(x_1, \ldots, x_n) d\pi_1(\theta)
\geq \int w_n(\theta, \delta^B_n; x_1, \ldots, x_n) f^n_\theta(x_1, \ldots, x_n) d\pi_1(\theta)
\] (5)

for all data sequences \((x_1, \ldots, x_n)\). Let \( \delta^B = (\delta^B_1, \delta^B_2, \ldots, \delta^B_n, \ldots) \). It is easy to see that in this case for any decision function \( \delta_n = \delta_n(x_1, \ldots, x_n) \)

\[
\int_{\Theta} E_{\theta} w_n(\theta, \delta_n; X_1, \ldots, X_n) d\pi_1(\theta) \geq \int_{\Theta} E_{\theta} w_n(\theta, \delta^B_n; X_1, \ldots, X_n) d\pi_1(\theta),
\]

i.e. \( \delta^B_n \) is a Bayesian decision function (corresponding to the “a priori” distribution \( \pi_1 \)) based on \( n \) observations.

Let us denote \( l_n = l_n(x_1, \ldots, x_n) \) the right-hand side of (5). From this time on, we suppose that \( \int l_n d\mu_n < \infty \) for any \( n = 1, 2, \ldots \).

In the same way as in [4] we easily get

**Theorem 2.1**  For any sequential decision procedure \((\psi, \delta)\)

\[
W(\psi, \delta) \geq W(\psi, \delta^B) = \sum_{n=1}^{\infty} \int (1 - \psi_1) \cdots (1 - \psi_{n-1}) \psi_n l_n d\mu_n.
\]

It follows from Theorem 2.1 that \( \inf_\delta W(\psi, \delta) = W(\psi, \delta^B) \), and the aim of what follows is to minimize

\[
L(\psi) = C(\psi) + W(\psi, \delta^B)
\]

over all stopping rules \( \psi \) (see (4)).

It is easy to see that, by definition of \( C(\psi) \),

\[
L(\psi) = \sum_{n=1}^{\infty} \int (1 - \psi_1) \cdots (1 - \psi_{n-1}) \psi_n \left( \int K^n_\theta f^n_\theta d\pi_2(\theta) + l_n \right) d\mu^n
\] (6)
if \( P_\theta(\tau_\psi < \infty)d\pi_2(\theta) = 1 \), and \( L(\psi) = \infty \) otherwise.

Let us denote

\[
k_n = k_n(x_1, \ldots, x_n) = \int K^n_\theta(x_1, \ldots, x_n)f^n_\theta(x_1, \ldots, x_n)d\pi_2(\theta)
\]

(see (6)), and let for any \( \pi = \pi_1 \) or \( \pi = \pi_2 \)

\[
P_\pi(A) = \int P_\theta(A)d\pi(\theta) \quad \text{for any event } A.
\]

Let also

\[
s^n_\psi = s^n_\psi(x_1, \ldots, x_n) = (1 - \psi_1(x_1)) \ldots (1 - \psi_{n-1}(x_1, \ldots, x_{n-1}))\psi_n(x_1, \ldots, x_n)
\]

for any \( n = 1, 2, \ldots \) and for any stopping rule \( \psi \).

Thus, by (6),

\[
L(\psi) = \sum_{n=1}^{\infty} \int s^n_\psi(k_n + l_n) \, d\mu^n
\]

if \( P^{\pi_2}(\tau_\psi < \infty) = 1 \), and \( L(\psi) = \infty \) otherwise.

First, let us solve the problem of minimization of \( L(\psi) \) in the class \( F^N \) of truncated stopping rules, that is such that

\[
\psi = (\psi_1, \psi_2, \ldots, \psi_{N-1}, 1, \ldots), \quad N = 1, 2, \ldots \quad \text{(see also [4]).}
\]

For any \( \psi \in F^N \) let

\[
L_N(\psi) = \sum_{n=1}^{N} \int s^n_\psi(k_n + l_n) \, d\mu^n = \sum_{n=1}^{N-1} \int s^n_\psi(k_n + l_n) \, d\mu^n + \int t^\psi_N(k_N + l_N) \, d\mu^N,
\]

where, for any \( n \geq 1 \) and for any stopping rule \( \psi \)

\[
t^\psi_n = t^\psi_n(x_1, \ldots, x_n) = (1 - \psi_1(x_1)) \ldots (1 - \psi_{n-1}(x_1, \ldots, x_{n-1})) \quad t_1 \equiv 1.
\]

**Theorem 2.2** Let \( \psi \in F^N \) be any (truncated) stopping rule, \( N \geq 2 \). Then for any \( 1 \leq r \leq N - 1 \) the following inequalities hold true

\[
L_N(\psi) \geq \sum_{n=1}^{r} \int s^n_\psi(k_n + l_n) \, d\mu^n + \int t^\psi_r(k_{r+1} + V^N_{r+1}) \, d\mu^{r+1}
\]

\[
\geq \sum_{n=1}^{r-1} \int s^n_\psi(k_n + l_n) \, d\mu^n + \int t^\psi_r(k_r + V^N_r) \, d\mu^r,
\]

where \( V^N_N \equiv l_N \), and recursively for \( m = N - 1, N - 2, \ldots, 1 \)

\[
V^N_m = \min \{l_m, Q^N_m\},
\]

\[
(7), \quad (8), \quad (9)
\]
where

\[ Q_m^N = \int \left( k_{m+1} + V_{m+1}^N \right) d\mu(x_{m+1}) - k_m \] (10)

(it should be remembered that the function under the integral sign on the right-hand side of (10) is a function of \((x_1, \ldots, x_{m+1})\), and, because of this, \(Q_m^N = Q_m^N(x_1, \ldots, x_m)\)).

The lower bound in (8) is attained if and only if

\[ I_{\{l_m < Q_m^N\}} \leq \psi_m \leq I_{\{l_m \leq Q_m^N\}} \] (11)

\(\mu^m\)-almost everywhere on

\[ T_m^\psi = \{(x_1, \ldots, x_m) : t_m^\psi(x_1, \ldots, x_m) > 0\}, \]

for all \(m = r, r+1, \ldots, N-1\).

In particular, \((\psi_1, \psi_2, \ldots, \psi_{N-1}, 1, \ldots)\) is an optimal truncated stopping rule in \(F_N\), if and only if (11) is satisfied \(\mu^m\)-almost everywhere on \(T_m^\psi\) for all \(m = 1, \ldots, N-1\). In addition,

\[ \inf_{\psi \in F^N} L(\psi) = Q_0^N, \] (12)

where

\[ Q_0^N = \int (k_1(x) + V_1^N(x)) \, d\mu(x). \]

**Proof.** The proof can be implemented by induction as in the proof of Theorem 3 in [4] using instead of Lemma 2 [4] the following extension of it.

**Lemma 2.3** Let \(r \geq 1\) be any natural number, and let \(v_{r+1} = v_{r+1}(x_1, x_2, \ldots, x_{r+1})\) be any non-negative measurable function, such that \(\int v_{r+1} \, d\mu^{r+1} < \infty\). Then

\[ \int s_r^\psi(k_r + l_r) \, d\mu^r + \int t_{r+1}^\psi(k_{r+1} + v_{r+1}) \, d\mu^{r+1} \geq \int t_r^\psi(k_r + v_r) \, d\mu^r, \] (13)

where \(v_r = \min\{l_r, Q_r\}\), with \(Q_r = Q_r(x_1, \ldots, x_r)\) defined as

\[ Q_r(x_1, \ldots, x_r) = \int (k_{r+1}(x_1, \ldots, x_{r+1}) + v_{r+1}(x_1, \ldots, x_{r+1})) \, d\mu(x_{r+1}) \]

\[-k_r(x_1, \ldots, x_r).\]

There is an equality in (13) if and only if \(I_{\{l_r < Q_r\}} \leq \psi_r \leq I_{\{l_r \leq Q_r\}} \mu^r\)-almost everywhere on \(T_r^\psi\).
**Proof** of Lemma 2.3 can be implemented following the steps of the proof of Lemma 2 in [4] and is omitted here. ■

Starting with the class of non-truncated stopping rule, let us define for any \( \psi \)

\[
L_N(\psi) = \sum_{n=1}^{N-1} \int s_n^\psi (k_n + l_n) d\mu_n + \int t_n^\psi (k_N + l_N) d\mu_N.
\]

The idea of construction of optimal stopping rules is to pass to the limit, as \( N \to \infty \), in (7), (8), (9) and (10).

Let \( F \) be a class of stopping rules such that for every \( \psi \in F \)

\[
\pi_2(\tau_\psi < \infty) = 1 \quad \text{and} \quad \lim_{N \to \infty} L_N(\psi) = L(\psi).
\]

In a very similar manner as in [4] it can be shown that for any \( m = 1, 2, \ldots \) any \( N \geq m \) \( V_m^N(x_1, \ldots, x_m) \geq V_m^{N+1}(x_1, \ldots, x_m) \) for any \( (x_1, \ldots, x_m) \), so there exists

\[
V_m = V_m(x_1, \ldots, x_m) \underset{N \to \infty}{\to} V_m^N(x_1, \ldots, x_m).
\]

Thus, passing to the limit, for any \( \psi \in F \), in (7), (8), (9) and (10) is justified by the Lebesgue’s dominated convergence theorem. In particular, let

\[
Q_m = Q_m(x_1, \ldots, x_m) \underset{N \to \infty}{\to} Q_m^N(x_1, \ldots, x_m), \quad m = 0, 1, 2, \ldots
\]

In the same way as in [4] it can be shown that (cf. (12))

\[
\inf_{\psi \in F} L(\psi) = Q_0 = \int (k_1(x) + V_1(x)) d\mu(x).
\]

Combining all these ideas, we immediately have

**Theorem 2.4** If there exists \( \psi \in F \) such that

\[
L(\psi) = \inf_{\psi' \in F} L(\psi')
\]

then

\[
I_{\{l_m < Q_m\}} \leq \psi_m \leq I_{\{l_m \leq Q_m\}} \quad (15)
\]

\( \mu^m \)-almost everywhere on \( T_m^\psi \), for all \( m = 1, 2, \ldots \).

On the other hand, if \( \psi \) satisfies (15) \( \mu^m \)-almost everywhere on \( T_m^\psi \), for any \( m = 1, 2, \ldots \), and \( \psi \in F \), then it satisfies (14) as well.

**Proof.** The proof can be conducted following the steps of the proof of Theorem 4 in [4], using Lemma 2.3 instead of Lemma 2 of [4]. ■

Very much like in [4], we can give some conditions, under which the structure of (15) is necessary and sufficient for optimality in the class of all stopping rules.

Let us call the problem of minimizing \( L(\psi) \) **truncatable** if for any \( \psi \) such that \( \pi_2(\tau_\psi < \infty) = 1 \) it holds \( L_N(\psi) \to L(\psi) \), as \( N \to \infty \).
**Theorem 2.5** Let the problem of minimizing $L(\psi)$ be truncatable, and let for any $c > 0$
\[ \int P_\theta(K_\theta^n(X_1, \ldots, X_n) < c) d\pi_2(\theta) \to 0 \quad \text{as} \quad n \to \infty. \quad (16) \]
Then
\[ L(\psi) = \inf_{\psi'} L(\psi') \]
if and only if
\[ I\{l_\text{m}<Q_\text{m}\} \leq \psi_\text{m} \leq I\{l_\text{m} \leq Q_\text{m}\} \quad (17) \]
$\mu^m$-almost everywhere on $T_\psi^m$, for all $m = 1, 2, \ldots$.

**Proof.** The “if”-part can be proved analogously to the proof of Theorem 4 in [4], using Lemma 2.3 instead of Lemma 2 in [4].

To prove the “only if”-part we suppose that $\psi$ satisfies (15) $\mu^m$-almost everywhere on $T_\psi^m$, for any $m = 1, 2, \ldots$. It follows from Lemma 2.3 that for any fixed $m = 1, 2, \ldots$
\[ \sum_{n=1}^{m-1} \int s_n^\psi(k_n + l_n) d\mu^n + \int t_m^\psi(k_m + V_m) d\mu^m = \int (k_1(x) + V_1(x)) d\mu(x) = I < \infty. \quad (18) \]
In particular, this implies that $\int t_m^\psi k_m d\mu^m \leq I$, or
\[ \int E_\theta t_m^\psi K_\theta^m d\pi_2(\theta) \leq I, \quad (19) \]
where $t_m^\psi = t_m^\psi(X_1, \ldots, X_m)$ and $K_\theta^m = K_\theta^m(X_1, \ldots, X_m)$.

Let $C$ be any positive constant. Then (19) implies
\[ C \int E_\theta t_m^\psi I_{\{K_\theta^m > C\}} d\pi_2(\theta) < I, \quad m = 1, 2, \ldots. \quad (20) \]
Because
\[ \int E_\theta t_m^\psi d\pi_2(\theta) = \int E_\theta t_m^\psi I_{\{K_\theta^m > C\}} d\pi_2(\theta) + \int E_\theta t_m^\psi I_{\{K_\theta^m \leq C\}} d\pi_2(\theta) \quad (21) \]
and the second summand by virtue of (16) tends to 0, as $m \to \infty$, we have that the difference between the first summand on the right-hand side of (21)
and the left-hand side of it, goes to 0 as $m \to \infty$. Thus, from (20), we have that

$$
\lim_{m \to \infty} \int E_{\theta} t_{m}^{\psi} d\pi_{2}(\theta) = \lim_{m \to \infty} \int P_{\theta}(\tau_{\psi} \geq m) d\pi_{2}(\theta) = \int P_{\theta}(\tau_{\psi} = \infty) d\pi_{2}(\theta) < I/C,
$$

and, because of arbitrariness of $C$, $P^{{\pi}_{2}}(\tau = \infty) = 0$, or

$$
P^{{\pi}_{2}}(\tau < \infty) = 1. \tag{22}
$$

Now, from (18) we get that

$$
\lim_{m \to \infty} \sum_{n=1}^{m-1} \int s^{\psi}_{n}(k_{n} + l_{n}) d\mu^{n} = L(\psi) \leq I. \tag{23}
$$

Because the problem is truncatable, it follows from (22) that $L_{N}(\psi) \to L(\psi)$, as $N \to \infty$. Now, passing to the limit in (12), we get $L(\psi) \geq I$. From this and (23) it follows that $L(\psi) = I = \inf_{\psi'} L(\psi')$. ■

Very much like in [4] (see Corollary 1 therein), there are simple conditions which guarantee that the problem is truncatable.

**Theorem 2.6** The problem of minimization of $L(\psi)$ is truncatable if any of the following two conditions is fulfilled.

(i) There is $M$, $0 < M < \infty$, such that $w_{n}(\theta, d; x_{1}, \ldots, x_{n}) \leq M$ for any $\theta, d, x_{1}, \ldots, x_{n}$, and for any $n \geq 1$, and from $L(\psi) < \infty$ it follows that

$$
P^{{\pi}_{1}}(\tau_{\psi} < \infty) = 1.
$$

(ii)

$$
\int l_{n} d\mu^{n} \to 0, \quad \text{as} \quad n \to \infty.
$$

Theorem 2.6 can be proved in the same way as Corollary 1 in [4].

Combining Theorem 2.1 with Theorem 2.2 or Theorem 2.4 or Theorem 2.5, we have, under respective conditions, sequential decision procedures $(\psi, \delta^{B})$ minimizing $R(\psi, \delta)$ in the corresponding class of sequential decision procedures, and the respective necessary conditions under which the minimum is attained, for example, using Theorem 2.5 we get:

**Theorem 2.7** Under the conditions of Theorem 2.5

$$
\inf_{(\psi, \delta)} R(\psi, \delta) = Q_{0}.
$$
For every \( \psi \) satisfying (17) (\( \mu^m \)-almost everywhere on \( T^\psi_m \)) for all \( m = 1, 2, \ldots \), it holds
\[
R(\psi, \delta^B) = Q_0.
\]
If for a sequential decision procedure \( (\psi, \delta) \) \( R(\psi, \delta) = Q_0 \), then \( \psi \) satisfies (17) \( \mu^m \)-almost everywhere on \( T^\psi_m \) for all \( m = 1, 2, \ldots \).

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