A Note on Approximation Problems of Neural Network

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Abstract

In this paper, the proofs of approximation on neural network given by T.Chen and Ch.Jiang was revised.

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1 introduction

In 1994 and 1995, T.Chen[1−3] investigated the approximation property of neural network and proved that an activation function belongs to $L_{loc}^p(R^1) \cap S'(R^1)$ can uniformly approximation any integrable function on a compact set if and only if the activate function is not a polynomial. 1998, followed Chen’s method, Ch.Jiang[4] get almost similar result in radial basis function (RBF) neural networks.

These results are interesting and important. However, the proofs of theorems in Chen[1] and Jiang[4] is not mathematically accurate. Since these theorems play an important role in neural network approximation theorems, and the idea or way of Chen[1] and Jiang[4] are wonderful, only we try in this note is to revise the proofs of these theorems. Some symbols and notations is as follows: $S(R^n)$ is all infinitely differentiable functions, which are rapidly decreasing at
infinity. \( S'(R^n) \) is all linear continuous functionals defined on \( S(R^n) \), which is also named tempered distribution.

The following lemmas and definitions are given by Rudin\(^5\):

**Lemma 1.1.** A distribution is the Fourier transform of a polynomial if and only if its support is the origin (or empty set).

**Lemma 1.2.** Suppose \( f \in S(R^n) \), \( g \in S'(R^n) \). Then \( g * f = \hat{g} \cdot \hat{f} \).

**Lemma 1.3.** If \( g, f, h \in D'(R^1) \) and two of them have compact supports, then \( f * g = g * f \), \( f * (g * h) = (g * f) * h \).

**Lemma 1.4.** If \( g \in S'(R^n) \), then \( \hat{g}(t) = g(-t) \).

**Lemma 1.5.** If \( f \in L^1(R^n) \), then \( \hat{f} \in C_0(R^n) \), and \( ||\hat{f}||_{\infty} \leq ||f||_1 \).

**Definition 1.6.** If \( \phi \in S(R^n) \), then its Fourier transform \( \hat{\phi}(t) = \int f(x)e^{it \cdot x}dx , \) \( t \in R^n \). If \( u \in S'(R^n) \) and \( \phi \in S(R^n) \), then \( \hat{u}(\phi) = u(\hat{\phi}) \), where \( \hat{u} \) denotes the Fourier transform of \( u \).

T. Chen:

**Theorem 1.7**\(^1\). Suppose \( g \in L^p_{loc}(R^1) \cap S'(R^1) \), then \( \{ \sum_{i=1}^{N} c_i g(\lambda_i x + \theta_i) \} \) is dense in \( L^p[a, b] \) if and only if \( g \) is not a polynomial.

**Theorem 1.8**\(^2\). Suppose \( g \in C(R^1) \cap S'(R^1) \), then \( \{ \sum_{i=1}^{N} c_i g(\lambda_i ||x - \theta_i||) \} \) is dense in \( C(K) \) if and only if \( g \) is not an even polynomial \( (K \) is a compact set of \( R^1 \)).

**Theorem 1.9**\(^3\). Suppose \( g \in C(R^1) \cap S'(R^1) \), then \( \{ \sum_{i=1}^{N} c_i g(\lambda_i x + \theta_i) \} \) is dense in \( C(K) \) if and only if \( g \) is not a polynomial \( K \) is a compact set of \( R^1 \).

Ch. Jiang:

**Theorem 1.10**\(^4\). Suppose \( g \in L^p_{loc}(R^n) \cap S'(R^n) \), then \( \{ \sum_{i=1}^{N} c_i g(\lambda_i \rho_i x + \theta_i) \} \) is dense in \( L^p(K) \) if and only if \( g \) is not an even polynomial. (Here, \( c_i, \lambda_i \in R^1, \rho_i \) is a rotation, \( b_i \in R^n \) \( K \) is a compact set of \( R^n \). There are some defects in the proofs of Chen\(^1-3\) and Jiang\(^4\). Here we only discuss Theorem 1.7 and Theorem 1.10, the other problems are similarity.

**Remark 1.11.** In Chen\(^1\), the key points to the proof of theorem 1.7 are the statements that “we have \( \int_{R^n} g(u)du \int_{R^n} W(u - \lambda x)h(x)dx = 0 \) and \( < g(t), W(t)*h(\lambda t) >= 0^+ \)”. However “which is equivalent to “< \( \hat{g}(t), \hat{W}(t)*\hat{h}(\lambda t) >= 0^+ \)” is no reasonable according to the definition of Fourier transform of \( S'(R^n) \) if \( u \in S'(R^n) \) and \( \phi \in S(R^n) \), then \( \hat{u}(\phi) = u(\hat{\phi}) \). This means we can not do Fourier transform to \( g(t) \) and \( W(t)*h(\lambda t) \) simultaneous.

**Remark 1.12** In Jiang\(^4\), the point to proof of his theorem is the statements that in the eighth line of the proof: for any \( p(x) \in D(R^n) \), define \( f(x) = p * h(x) \), which follows \( \int_{R^n} g^\alpha_h(x - t)f(x)dx = 0 \). and,

\[
g^\alpha_h * f = \int_{R^n} g^\alpha_h(x - t)f(x)dx = 0.
\]

But

\[
g^\alpha_h * f \neq \int_{R^n} g^\alpha_h(x - t)f(x)dx
\]
for
\[ g_\lambda \ast f = \int_{\mathbb{R}^n} g_\lambda(t-x)f(x)dx. \]

# 2 New Proof of Theorems

## 2.1 A New Proof of Theorem 1.7.

Assuming that \( \sum_{i=0}^{N} c_i g(\lambda_i x + \theta_i) \) is not dense in \( L^p[a, b] \). By Hahn-Banach Theorem, there exists a function \( h \in L^q[a, b] \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), for any \( \lambda, \theta \in \mathbb{R} \), such that
\[
\int_{a}^{b} g(\lambda x + \theta)h(x) = 0
\]

Then for any \( w \in S(R^1) \) we have
\[
\int_{\mathbb{R}^1} w(\theta)d\theta \int_{a}^{b} g(\lambda x + \theta)h(x)dx = 0. \tag{1}
\]

Let that \( h(x) = 0 \) if \( x \in R^1 \setminus [a, b] \) and that \( u = \lambda x + \theta \). The (1) implies that
\[
\int_{\mathbb{R}^1} g(u)du \int_{\mathbb{R}^1} w(u - \lambda x)h(x)dx = 0. \tag{2}
\]

Assuming that \( w_\lambda(t) = w(\lambda t) \), then (2) is as following
\[
< g(u), (w_\lambda \ast h)(\frac{u}{\lambda}) > = 0 \tag{3}
\]

denote that \( \lambda = -\xi \), then
\[
< g(u), (w_{-\xi} \ast h)(-\frac{u}{\xi}) > = 0
\]

According to Lemma 1.4, i.e. \( \hat{\varphi}(u) = \varphi(-u) \) for any \( \varphi(u) \in S'(R^n) \), we have (3) that
\[
< g(u), (w_{-\xi} \ast h)(\frac{\hat{\varphi}(u)}{\xi}) > = 0
\]

and let \( g(u) = g_\xi(\frac{u}{\xi}) \), this implies that
\[
< g_\xi(\frac{u}{\xi}), (w_{-\xi} \ast h)(\frac{\hat{\varphi}(u)}{\xi}) > = 0. \tag{4}
\]

By the definition of Fourier, (4) is that
\[
< \hat{g}_\xi(t), (w_{-\xi} \ast \hat{h})(t) > = 0 \tag{5}
\]

Thus by Lemma 1.2, (5) is really to be
\[
< \hat{g}_\xi(t), \hat{w}_{-\xi}(t) \cdot \hat{h}(t) > = 0 \tag{6}
\]
Since $w(t)$ and $\lambda$ are both arbitrary, $w_\xi(t)$ is also arbitrary. (6) hence that $\text{supp}\{\hat{g}\} \subset \{0\}$. Then $g$ is a polynomial by Lemma 1.1.

2.2 A New Proof of Theorem 1.10.

Assume that $\{\sum_{i=1}^{N} c_i g(\lambda_i \rho_i x + b_i)\}$ is not dense in $L^{p}(K)$. According to H-B Theorem, there exists a nonzero $h(x) \in L^{q}(K)(\frac{1}{p} + \frac{1}{q} = 1)$, for any $\lambda \in R^1$, any $t \in R^n$ and any rotation $\rho$ of $R^n$, such that
\[
\int_{K} g(\lambda \rho x - t)h(x)dx = 0
\]

Suppose that $\lambda > 0$ and that $g^\lambda(x) = g(\lambda \rho x)$. For $h(x) \neq 0$, $\text{supp}\{h\} \subset K$ and $h(x) \in L^q(K)$, then
\[
\int_{K} g(\lambda \rho x - t)h(x) = 0
\]

Let $h(x) = 0$ if $x \in R^n \setminus K$. For any $w \in S(R^n)$, we have that
\[
\int_{R^n} w(t)dt \int_{R^n} g(\lambda \rho x - t)h(x)dx = 0 \tag{7}
\]

Assuming that $u = t - \lambda \rho x$, (7) implies that
\[
\int_{R^n} \int_{R^n} g(-u)w(u + \lambda \rho x)h(x)dxdu = 0 \tag{8}
\]

Denote that $\tilde{w}(\cdot) = w(-\cdot)$, then $w(u + \lambda \rho x) = \tilde{w}(-u - \lambda \rho x)$. Thus
\[
\int_{R^n} \int_{R^n} g(-u)\tilde{w}(-u - \lambda \rho x)h(x)dxdu = 0
\]

and
\[
\int_{R^n} g(-u)(\tilde{w} \ast h)\left(-\rho \frac{u}{\lambda}\right)du = 0
\]

Replacing $u$ with $-v$ and $\lambda$ with $-\xi$, we have that
\[
< g(v), (\tilde{w} \ast h)\left(-\rho \frac{v}{\xi}\right) >= 0. \tag{9}
\]

Let $g^\xi(v) = g(\xi \rho v)$, (9) is as following form:
\[
< g^\xi\left(\rho^{-1}v\right), (\tilde{w} \ast h)\left(-\rho^{-1}v\right) >= 0
\]

i.e.
\[
< g^\xi\left(\rho^{-1}v\right), (\tilde{w} \ast h)\left(-\rho^{-1}\xi\right) >= 0
\]

Thus by Lemma 1.6,
\[
< \hat{g}^\xi, (\tilde{w} \ast h) >= 0
\]
This also is that
\[ < \hat{g}_\xi, \hat{w} \cdot \hat{h} >= 0 \]

Next, we use Chen’s [1–3] way to show that \( g_\xi^\rho(u) \) is a polynomial. In fact, since \( \hat{h}(t) \in C_0(R^n) \), there exists \( t_0 \in R^n\setminus\{0\} \) and \( O(t_0, \delta) = \{ t : |t - t_0| < \delta \} \) such that \( \forall t \in O(t_0, \delta) \) and \( |\hat{h}(t)| > c > 0 \)

For \( t_1 \in R^n\setminus\{0\} \), let \( t_0 = \lambda \rho(t_1) \) i.e. \( x \cdot \rho^{-1}t = t \cdot \rho x \), thus \( |\hat{h}(\lambda \rho^{-1}t)| > c \) for any \( t \in O(t_1, \frac{\delta}{\lambda}) \), which implies that

\[ < \hat{g}_\xi^\rho(u), \hat{w}(u) >= < \hat{g}_\xi^\rho(u), \frac{\hat{w} \cdot \hat{h}(\lambda \rho^{-1}u)}{\hat{h}(\lambda \rho^{-1}u)} >= 0 \]

Since that \( w(t) \in D(R^n) \) is arbitrary and that \( t \neq 0, \hat{w} \) is arbitrary function in \( S(R^n) \). Then \( supp\{\hat{g}_\xi^\rho(u)\} \subset \{0\} \) or \( \emptyset \). By Rudin’s theorem, we have that \( g_\xi^\rho(u) \) is a polynomial. Further more \( g(u) \) is a polynomial, which completes the proof.

References


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