A Representation of a Class of
Heyting Algebras by Fractions

M. Hosseinyazdi
Payame Noor University, Shiraz, Iran
mhyazdi@pmu.ac.ir, mahbobeh_hosseinyazdi@yahoo.com

A. Hasankhani and M. Mashinchi
Faculty of Mathematics and Computer Sciences
Shahid Bahonar University of Kerman, Kerman, Iran
{abhasan, mashinchi}@mail.uk.ac.ir

Abstract

In this paper, we solve an open problem in a special case. The problem is to give a characterization for Heyting algebras by means of fractions. Here, we give a representation for a class of Heyting algebras by means of fractions. Fractions on a bounded distributive lattice is a new algebraic structure, which was recently studied by the authors.

Mathematics Subject Classification: 06Axx, 06Dxx

Keywords: Heyting algebra, pseudo-Boolean lattices, distributive lattices, ring of sets, field of sets, fractions of lattices

1 Introduction and preliminaries

A characterization of distributive lattices as a ring of sets and Boolean lattices as a field of sets are well known [4, 9]. A characterization of pseudo-Boolean lattices (or a Heyting algebra [4] and [3] ) is given in [1] by means of residuated mappings (see [1], Theorem 7.9). In [7] we have constructed a new algebraic structure, as fractions on a lattice, and by means of it we characterized finite Heyting algebras. But giving a characterization for Heyting algebras in
general, was left as an open problem. In this paper we give a representa-
tion of a class of Heyting algebras by means of fractions which was studied in [7]. Moreover, pseudo-Boolean lattices (or Heyting algebras) have main role 
in some optimization problems over lattices. The optimization problem over 
distributive lattices was studied in [5], and in particular case this problem was 
studied over pseudo-Boolean lattices in [6] (also see [8]).

We give some definitions and theorems which we need in the sequel in order 
to give our representation. For more details see the references.

**Definition 1.1.** Let \( H \) be a non-empty set. A subset \( R \) of \( P(H) \) is called 
a ring of sets if the union \( X \cup Y \) and the intersection \( X \cap Y \) belongs to \( R \) 
for all \( X \) and \( Y \) in \( R \). A ring of sets \( R \) is called a field of sets if 
\( X^c \in R \), for all \( X \in R \), where \( X^c = H \setminus X \). Note that \( (R, \subseteq) \) is a lattice, where \( \subseteq \) is the 
inclusion of sets.

**Example 1.2.** Let \( H = \{2, 3, 4, 9\} \). We can consider a ring of sets \( R \) as 
follows:

\[
R = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{2, 4\}, \{3, 9\}, \{2, 3, 4\}, \{2, 3, 9\}, \{2, 3, 4, 9\}\}.
\]

**Definition 1.3.** A bounded lattice \((L, \leq)\) is called a pseudo-Boolean if for 
all \( a, b \in L \), there exists \( c \in L \) such that

\[
a \land x \leq b \iff x \leq c \quad \forall x \in L.
\]

If such element \( c \) exists then, it is unique and will be denoted by \( b : a \).

**Definition 1.4.** ([3]) A Heyting algebra is an algebraic structure \( A = (A; \lor, 0, \land, 1, \to) \) such that \((A; \lor, 0, \land, 1)\) is a bounded lattice, and \( \to \) gives 
the residual of \( \land \):

\[
a \land x \leq b \iff x \leq a \to b.
\]

**Remark 1.5.** We see that a pseudo-Boolean lattice is exactly a Heyting 
algebra where, \( b : a \) is \( a \to b \).

**Remark 1.6.** ([9]) (i) Every finite distributive lattice is pseudo-Boolean.
(ii) Every Boolean lattice is pseudo-Boolean.
(iii) In general, a pseudo-Boolean lattice may not be Boolean. For example, 
consider a bounded linearly ordered set \((B, \leq)\), where \( a \land b = \min(a, b) \) and 
\( a \lor b = \max(a, b) \) and for all \( a, b \in B \),

\[
b : a = \begin{cases} 
1 & \text{if } b \geq a \\
 b & \text{if } b < a 
\end{cases}
\]

\( B \) is not Boolean, since for any \( a, 0 < a < 1 \), we have \( a \lor (0 : a) = a \lor 0 = a < 1 \).

**Theorem 1.7.** ([9], Proposition 1.17.) Let \((L, \leq)\) be a lattice. If \( L \) is a 
pseudo-Boolean lattice, then it is distributive.
Definition 1.8. ([4]) Let \((L, \leq)\) be a lattice.
(i) A sublattice \(I\) of \(L\) is an ideal if and only if \(i \in I\) and \(a \in L\) implies \(a \land i \in I\).
(ii) A proper ideal \(I\) of \(L\) is prime if and only if \(a, b \in L\) and \(a \land b \in I\) imply that \(a \in I\) or \(b \in I\).

Definition 1.9. ([2]) A lattice \((L, \leq)\) is called infinitely distributive if
\[
a \land (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \land b_i),
\]
and
\[
a \lor (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \lor b_i).
\]
Equality (1) is called join infinite distributive identity or JID. In the same way, (2) is called meet infinite distributive identity or MID (see [4]).

Note that, JID and MID may not hold in every complete distributive lattice. Also they may not imply each other.

Example 1.10. ([2]) Let \((L, \subseteq)\) be the complete lattice of all closed subsets of the plane. Let \(c\) denote the circle \(x^2 + y^2 = 1\) and \(d_k\) denote the disc \(x^2 + y^2 \leq 1 - k^{-2}\), then \(c \land (\bigvee_{k=1}^{\infty} d_k) = c\) and \(\bigvee_{k=1}^{\infty} (c \land d_k)\) is the empty set. Therefore, JID does not hold. On the other hand, MID holds, since in this case \(\lor\) and \(\land\) coincide with the set theoretic operations \(\cup\) and \(\cap\), respectively.

Theorem 1.11. ([4], Corollary II.1.17.) Let \(L\) be a distributive lattice, \(a, b \in L\) and \(a \neq b\). Then, there is a prime ideal \(P\) such that \(a \in P\) or \(b \in P\).

Corollary 1.12. ([4], Corollary II.1.18.) Every ideal \(I\) of a distributive lattice is the intersection of all prime ideals containing it.

The following theorem characterize any distributive lattice.

Theorem 1.13. ([4], Theorem II.1.19) A lattice is distributive if and only if it is isomorphic to a ring of sets.

Definition 1.14. ([4]) A lattice \((L, \leq)\) is called join-complete lattice if \(\bigvee S\) exists for all subset \(S\) of \(L\).

2 The Lattice of fractions

In this section, we will review the construction of a new algebraic structure studied by the authors. Let \(R\) be a non-empty distributive lattice and \(S\) be a non-empty subset of \(R\), which is a complete meet-semilattice. We will see that the set of fractions \(S^{-1}R\) is a lattice and it inherits many lattice properties from \(R\). For more details see [7].

Definition 2.1. Define a binary relation \(\sim_S\) on \(R \times S\) by \((a, b) \sim_S (c, d)\) if and only if there exists \(t \in S\) such that \((a \land d) \land t = (b \land c) \land t\).
Theorem 2.2. The relation $\sim_S$ on $R \times S$ is an equivalence relation.

**Notation 2.3.** The set of all equivalence classes of $\sim_S$ is denoted by $R/\sim_S$.

In other words, $R/\sim_S = \{[(a, b)]_{\sim_S} : a \in R, b \in S\}$.

**Lemma 2.4.** (i) If $[(a, b)]_{\sim_S}$ and $[(a, c)]_{\sim_S}$ are two elements of $R/\sim_S$, then $[(a, b)]_{\sim_S} = [(a, c)]_{\sim_S}$.

(ii) Let $m = \bigwedge_{x \in S} x$. Then:

1. $(a, m) \sim_S (b, m) \iff (a, m) \sim_{\{m\}} (b, m)$,
2. $R/\sim_S = R/\sim_{\{m\}}$.

**Theorem 2.5.** Let $R$ be a non-empty distributive lattice, $S_1$ and $S_2$ be non-empty subsets of $R$ which are complete meet-semilattices and $\bigwedge_{x \in S_1} x = \bigwedge_{x \in S_2} x$. Then, $R/\sim_{S_1} = R/\sim_{S_2}$.

**Remark 2.6.** (i) From now on, $R/\sim_S$ will be denoted by $S^{-1}R$ and it is called the fractions of lattice $R$ with respect to $S$. Any element $[(a, b)]_{\sim_S} \in S^{-1}R$ is shown by $a/b$.

(ii) By Lemma 2.4.(ii), we can consider every $S$ as a singleton $\{m\}$, where $m = \bigwedge_{x \in S} x$. Therefore, from now on we assume $S$ to be the singleton $\{m\}$. So, by (i) we can write $a/m$ for $a/b$.

(iii) For $a/m$ and $b/m \in S^{-1}R$ we have $a/m = b/m$ if and only if $a \wedge m = b \wedge m$.

**Lemma 2.7.** $(S^{-1}R, \leq)$ is a partial ordered set, where $\leq$ is defined as follows:

$$a/m \leq b/m \iff a \wedge m \leq b \wedge m.$$  

The well-defined binary operations $\vee, \wedge : S^{-1}R \times S^{-1}R \longrightarrow S^{-1}R$ are given by

$$a/m \vee b/m = (a \vee b)/m,$$

and

$$a/m \wedge b/m = (a \wedge b)/m.$$  

**Theorem 2.8.** Let $R$ be a join-complete lattice which satisfies JID and $S = \{m\}$. Then, $S^{-1}R$ is so.

**Proof:** One can easily verifies that $\bigvee (a_i/m) = (\bigvee a_i)/m$. Hence, $S^{-1}R$ is join-complete. Moreover, $a/m \wedge (\bigvee (b_i/m)) = a/m \wedge (\bigvee (b_i)/m) = (a \wedge \bigvee b_i)/m = (\bigvee (a \wedge b_i))/m = \bigvee ((a \wedge b_i)/m) = \bigvee (a/m \wedge b_i/m)$. Therefore, $S^{-1}R$ satisfies JID.

**Theorem 2.9.** Let $(R, \leq)$ be a distributive lattice, $S = \{m\}$ and $I$ be an ideal of $R$. Then $S^{-1}I$ is an ideal of $S^{-1}R$. Moreover, any ideal of $S^{-1}R$ can be represented as $S^{-1}I$ where $I$ is an ideal of $R$.

**Proof:** Straightforward.
3 A representation of a class of Heyting algebras by fractions

In this section, we will see an application of fractions of a lattice, by giving a representation of a class of pseudo-Boolean lattices (or Heyting algebras) (Theorem 3.8). The following characterization for a finite Heyting algebra is well-known.

**Theorem 3.1.** ([4]) Let \( H \) be a finite lattice. Then, \( H \) is a Heyting algebra if and only if it is distributive.

If \( L \) is an infinite lattice we have Theorem 3.2, where a characterization of a pseudo-Boolean lattices is given.

**Theorem 3.2.** Let \((L, \leq)\) be a join-complete lattice. Then, \( L \) is pseudo-Boolean if and only if it is distributive and satisfies JID.

**Proof:** Let \( L \) be a pseudo-Boolean lattice. Then, it is distributive by Theorem 1.7(i). Clearly, \( \bigvee_{i \in I}(a \land b_i) \leq a \land (\bigvee_{i \in I} b_i) \) for any arbitrary index set \( I \). Now, let \( \bigvee_{i \in I}(a \land b_i) = t \). We have:

\[ \forall i \in I, \ a \land b_i \leq t \implies \bigvee_{i \in I} b_i \leq t : a \implies a \land (\bigvee_{i \in I} b_i) \leq t. \]

Hence, JID holds.

Conversely, let \( L \) be a distributive lattice which satisfies JID. Suppose, \( a, b \in L \) are given. Since \( L \) is join-complete, \( c = \bigvee_{a \land y \leq b} y \) exists and satisfies Definition 1.3. Therefore, \( L \) is pseudo-Boolean.

The following example shows that JID in Theorem 3.2 is crucial.

**Example 3.3.** Let \((L, \subseteq)\) be the complete lattice of all closed subsets of the plane in Example 1.10. It is shown that \( L \) does not satisfy JID. Suppose, \( L \) is a pseudo-Boolean lattice. Let \( a, b \) be circles \( x^2 + y^2 = 2 \) and \( x^2 + y^2 = 1 \), respectively. Since \( L \) is pseudo-Boolean, for \( a, b \) there exists \( c \) such that it satisfies Definition 1.3. Let \( d \) be a closed subset such that \( a \land d = \varnothing \). Then, \( x = b \lor d \) satisfies \( a \land x \leq b \). Hence, \( b \lor d \leq c \) for all \( d \) such that \( a \land d = \varnothing \). Therefore, \( b \lor (\bigvee_{a \land d = \varnothing} d) = \bigvee_{a \land d = \varnothing} (b \lor d) \leq c \). Let \( x = b \lor (\bigvee_{a \land d = \varnothing} d) \). On the other hand, \( \bigvee_{a \land d = \varnothing} d = R^2 \setminus \{(x, y)|x^2 + y^2 < 2\} \) and clearly \( x \) does not satisfy \( a \land x \leq b \).

**Corollary 3.4.** Let \( R \) be a join-complete ring of sets. Then, \( S^{-1}R \) is pseudo-Boolean for all \( S \), where \( S = \{m\} \) and \( m \in R \).

**Proof:** Clearly, any join-complete ring of sets satisfies JID. Hence, \( S^{-1}R \) is pseudo-Boolean by Theorems 2.8 and 3.2.

**Definition 3.5.** Let \((L, \leq)\) be a lattice. Then, \( L \) satisfies the join-completed prime ideals (JCPI) condition if every prime ideal of \( L \) is join-completed.

**Example 3.6.** (i) Every finite lattice satisfies the JCPI condition.
(ii) Let \( N \) be the chain of natural numbers with usual ordering \( \leq \). Then, \( N \) satisfies the JCPI condition.

(iii) Let \( I = [0, 1] \) be the chain of real numbers between 0 and 1. Then, \( I \) does not satisfy the JCPI condition. Since \([0, a)\) is a prime ideal of \( I \) for all \( a \in (0, 1] \), and clearly it is not join-completed.

**Theorem 3.7.** A lattice \( L \) satisfies the JCPI condition if and only if every ideal of \( L \) is join-completed.

**Proof:** Let \( I \) be an ideal of \( L \). Then, \( I = \bigcap_{P \supseteq I} P \), by Corollary 1.12. Now, let \( \{a_j\}_{j \in J} \) be an arbitrary family of elements of \( I \). Then, \( \{a_j\}_{j \in J} \subseteq P \) for all \( P \supseteq I \). Since \( L \) satisfies the JCPI condition, \( \bigvee_{j \in J} a_j \in I = \bigcap_{P \supseteq I} P \) and consequently, \( I \) is join-completed. Clearly, if every ideal of \( L \) is join-completed, then \( L \) satisfies the JCPI condition.

Now, we can give a new representation of a pseudo-Boolean lattice which satisfies the JCPI condition, based on our recent results by lattice of fractions [7].

**Theorem 3.8.** Let \((L, \leq)\) be a pseudo-Boolean lattice which satisfies the JCPI condition. Then, it is isomorphic to a lattice of fractions \( S^{-1}R \), where \( R \) is a join-complete ring of sets.

**Proof:** Let \( L \) be a pseudo-Boolean lattice which also satisfies the JCPI condition. Now, let \( X = \{P : P \) be a prime ideal of \( L\} \). For \( a \in L \), define \( r(a) = \{P \in X : a \notin P\} \) and let \( R = \{r(a) : a \in L\} \). We claim that the ring of sets \((R, \subseteq)\), is join-completed. To do this end we show that \( \bigcup_{i \in I} r(a_i) = r(\bigvee_{i \in I} a_i) \). Clearly, \( \bigcup_{i \in I} r(a_i) \subseteq r(\bigvee_{i \in I} a_i) \). Now, let \( P \in r(\bigvee_{i \in I} a_i) \). Hence, \( \bigvee_{i \in I} a_i \notin P \). Since \( L \) satisfies the JCPI condition, there exists \( i_0 \in I \) such that \( a_{i_0} \notin P \). Hence, \( P \in r(a_{i_0}) \subseteq \bigcup_{i \in I} r(a_i) \). Therefore, \( R \) is join-completed. Now, \( L \) is isomorphic to the ring of sets \( R \) as in the proof of Theorem 1.13 in [4]. As every pseudo-Boolean lattice is bounded with the upper bound \( 1 \in L \) we have \( X = r(1) \in R \). Define \( \phi : L \rightarrow S^{-1}R \) by \( a \mapsto r(a)/X \). It can easily verified that \( \phi \) is a lattice isomorphism.

**Remark 3.9.** Note that if \( R \) is a join-complete ring of sets, then \( S^{-1}R \) is pseudo-Boolean, by Theorem 2.8. On the other hand, \( R \) may not satisfy the JCPI. For example, let \( R \) be the complete lattice (and of course, ring of sets) of all subsets of \([0, 1]\) and let \( P \) be \( \mathcal{P}(\{0, 1/2\}) \backslash \{\{0, 1/2\}\} \). Clearly, \( P \) is a prime ideal of \( R \) which is not join-complete.

**Remark 3.10.** By example given in Remark 3.9, the converse of Theorem 3.8 may not hold, generally. However, in finite case, every join-semilattice satisfies
the JCPI and we have the following theorem, which is extended version of Theorem 5.4 of [7].

**Theorem 3.11.** Let $L$ be a finite lattice. Then, $L$ is pseudo-Boolean lattice if and only if it is isomorphic to a lattice of fractions $S^{-1}R$, where $R$ is a finite ring of sets.

We close this section by the following question.

**Open problem.** It is still interesting to find a characterization for Heyting algebras by means of fractions, in general, without JCIP condition.

**References**


Received: January, 2010