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On Semilinear Equations of Mixed Type in Cone Metric Spaces

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Abstract

In this paper we investigate the existence and uniqueness for Volterra-Fredholm type integral equations and the existence of unique common solution of the Urysohn integral equations in cone metric spaces. The result is obtained by using the some extensions of Banach's contraction principle, common fixed points for two self mappings in complete cone metric space and the theory of cosine family.

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1 Introduction

The purpose of this paper is study the existence and uniqueness of solutions for the Volterra-Fredholm integrodifferential equations of second order and the existence of unique common solution of the Urysohn integral equations.

In Section 3 we consider the consider the following Volterra-Fredholm integrodifferential equation of second order form:

$$x^{''}(t) = Ax(t) + \int_0^t k(t,s,x(s))ds + \int_0^b h(t,s,x(s))ds, \quad t \in J = [0,b], \tag{1}$$

$$x(0) = x_0, \quad x^{'}(0) = y_0, \tag{2}$$

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where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in a Banach space$X$, the functions $k, h : J \times J \times X \to Z$ are continuous and the given x_0, y_0 are elements of X.

Many authors have been studied the problems of existence, uniqueness, continuation and other properties of solutions of these type or special forms of the equations (1)–(2) are studied by different techniques, for example, see [3, 4, 9, 11, 12, 13, 16] and the references given therein.

In Section 4 we study the existence of unique common solution of the Urysohn integral equations of Volterra-Fredholm type:

$$x(t) = \int_a^t k_1(t, s, x(s))ds + \int_a^b h_1(t, s, x(s))ds + g_1(t), \quad t \in [a, b], \tag{3}$$

$$x(t) = \int_a^t k_2(t, s, x(s))ds + \int_a^b h_2(t, s, x(s))ds + g_2(t), \quad t \in [a, b], \tag{4}$$

where $x, g_1, g_2 : [a, b] \to X$; the functions $k_i, h_i : [a, b] \times [a, b] \times X \to X$ $(i = 1, 2)$, are continuous functions.

The objective of the present paper is to study the existence and uniqueness of solution of the system (1)–(2) under the conditions in respect of the cone metric space, fixed point theory and the cosine family. Hence we extend and improve some results reported in [9, 13, 15, 16, 17]. We are motivated by the work of P. Raja and S. M. Vaezpour in [14] and influenced by the work of M. Arshad [2].

The paper is organized as follows: Section 2, we discuss the preliminaries. Section 3, we dealt with study of the mixed Volterra-Fredholm integrodifferential equation and in Section 4, we consider an Urysohn Volterra-Fredholm type equations. Finally in Section 5, we give examples to illustrate the application of our results.

2 Preliminaries

Let us recall the concepts of the cone metric space and we refer the reader to [1, 2, 6, 7] for the more details.

Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1. P is closed, nonempty and $P \neq \{0\}$;
2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
3. $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose E is a real Banach space , P is a cone in E with $intP \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 2.1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of metric space.

The following example is a cone metric space, see [14].

Example 2.2 Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3 Let X be a an ordered space. A function $\Phi : X \to X$ is said to a comparison function if for every $x, y \in X$, $x \leq y$, implies that $\Phi(x) \leq \Phi(y)$, $\Phi(x) \leq x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

Example 2.4 Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$. It is easy to check that $\Phi : E \to E$, with $\Phi(x, y) = (ax, ay)$, for some $a \in (0, 1)$ is a comparison function. Also if Φ_1, Φ_2 are two comparison functions over $\mathbb{R}$, then $\Phi(x, y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

3 Existence and uniqueness of solutions

Let X is a Banach space with norm $\|\cdot\|$. Let $B = C(J, X)$ be the Banach space of all continuous functions from J into X endowed with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

Let $P = \{(x, y) : x, y \geq 0\} \subset E = \mathbb{R}^2$ be a cone and define $d(f, g) = (\|f - g\|_\infty, \alpha\|f - g\|_\infty)$, for every $f, g \in B$. Then it is easily seen that (B, d) is a cone metric space.

In many cases it is advantageous to treat second abstract differential equations directly rather than to convert first order systems. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine family. We refer the reader to [17, 18] for the necessary concepts about cosine functions. If $\{C(t) : t \in \mathbb{R}\}$ is a strongly continuous cosine family in X, then $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family, is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}.$$

Let us assume that $M \geq 1$ and N are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for every $t \in J$.

Definition 3.1 *The function $x \in B$ satisfies the integral equation*

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)[\int_0^s k(s,\tau,x(\tau))d\tau + \int_0^b k(s,\tau,x(\tau))d\tau]ds, \quad t \in J$$

is called the mild solution of the initial value problem (1)–(2).

We need the following lemma for further discussion:

Lemma 3.2 *[14] Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that*

$$d(f(x), f(y)) \leq \Phi(d(x, y)),$$

for every $x, y \in X$. Then f has a unique fixed point.

We list the following hypotheses for our convenience:

(H_1) There exist continuous functions $p_1, p_2 : J \times J \to \mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\|k(t,s,u) - k(t,s,v)\|, \alpha\|k(t,s,u) - k(t,s,v)\|) \leq p_1(t,s)\Phi(d(u,v)),$$

and

$$(\|h(t,s,u) - h(t,s,v)\|, \alpha\|h(t,s,u) - h(t,s,v)\|) \leq p_2(t,s)\Phi(d(u,v)),$$

for every $t, s \in J$ and $u, v \in Z$.

(H_2)

$$N\int_0^b\int_0^b [p_1(t,s)+p_2(t,s)]dsdt \le 1.$$

Theorem 3.3 *Assume that hypotheses* $(H_1)-(H_2)$ *hold. Then the abstract integral equation (1)–(2) has a unique solution* x *on* J.

Proof: The operator $F : B \to B$ is defined by

$$\begin{aligned} Fx(t) &= C(t)x_0 + S(t)y_0 \\ &\quad + \int_0^t S(t-s)[\int_0^s k(s,\tau,x(\tau))d\tau + \int_0^b k(s,\tau,x(\tau))d\tau]ds, \quad t \in J. \end{aligned} \tag{5}$$

By using the hypotheses $(H_1)-(H_2)$, we have

$$\begin{aligned} &(\|Fx(t)-Fy(t)\|, \alpha\|Fx(t)-Fy(t)\|) \\ &\le \int_0^t N(\|\int_0^s k(s,\tau,x(\tau))d\tau + \int_0^b h(s,\tau,x(\tau))d\tau - \int_0^s k(s,\tau,y(\tau))d\tau - \int_0^b h(s,\tau,y(\tau))d\tau\|, \\ &\quad \alpha\|\int_0^s k(s,\tau,x(\tau))d\tau + \int_0^b h(s,\tau,x(\tau))d\tau - \int_0^s k(s,\tau,y(\tau))d\tau - \int_0^b h(s,\tau,y(\tau))d\tau\|)ds \\ &\le \int_0^t N[(\int_0^s \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\|d\tau, \alpha\int_0^s \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\|d\tau) \\ &\quad + (\int_0^b \|h(s,\tau,x(\tau)) - h(s,\tau,y(\tau))\|d\tau, \alpha\int_0^b \|h(s,\tau,x(\tau)) - h(s,\tau,y(\tau))\|d\tau)]ds \\ &\le \int_0^t N[\int_0^b p_1(s,\tau)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)d\tau + \int_0^b p_2(s,\tau)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)d\tau]ds \\ &\le \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)N\int_0^b\int_0^b [p_1(s,\tau)+p_2(s,\tau)]d\tau ds \\ &= \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty), \end{aligned} \tag{6}$$

for every $x, y \in B$. This implies that $d(Fx, Fy) \le \Phi(d(x,y))$, for every $x,\ y \in B$. Now an application of Lemma 3.2, the operator has a unique point in B. This means that the equation (1)–(2) has unique solution. This completes the proof of the Theorem 3.3.

4 Existence of Common solutions

Let X is a Banach space with norm $\|\cdot\|$. Let $Z = C([a,b], X)$ be the Banach space of all continuous functions from J into X endowed with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in [a,b]\}.$$

Let $P = \{(x,y) : x,y \geq 0\} \subset E = \mathbb{R}^2$ be a cone and define $d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$, for every $f,g \in Z$. Then it is easily seen that (Z,d) is a cone metric space.

Definition 4.1 *[8] A pair (S,T) of self-mappings X is said to be weakly compatible if they commute at their coincidence point (i. e. $STx = TSx$ whenever $Sx = Tx$). A point $y \in X$ is called point of coincidence of a family T_j, $j = 1,2,\cdots$, of self-mappings on X if there exists a point $x \in X$ such that $y = T_j x$ for all $j = 1,2,\cdots$.*

We need the following lemma for further discussion:

Lemma 4.2 *[2] Let (X,d) be a complete cone metric space and P be an order cone. Let $S,T,f : X \to X$ be such that $S(X) \cup T(X) \subset f(X)$. Assume that the following conditions hold:*

(i) $d(Sx,Ty) \leq \alpha d(fx,Sx) + \beta d(fy,Ty) + \gamma d(fx,fy)$, for all $x,y \in X$, with $x \neq y$, where α,β,γ are non-negative real numbers with $\alpha+\beta+\gamma < 1$.

(ii) $d(Sx,Tx) < d(fx,Sx) + d(fx,Tx)$, for all $x \in X$, whenever $Sx \neq Tx$.

If $f(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S,T and f have a unique point of coincidence. Moreover, if (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point.

We list the following hypotheses for our convenience:

(H_3) Assume that

$$(F)x(t) = \int_a^t k_1(t,s,x(s))ds + \int_a^b h_1(t,s,x(s))ds,$$

and

$$(G)x(t) = \int_a^t k_2(t,s,x(s))ds + \int_a^b h_2(t,s,x(s))ds,$$

for all $t,s \in [a,b]$.

(H_4) There exist $\alpha,\beta,\gamma,p \geq 0$ such that

$$\begin{aligned}
&(|Fx(t) - Gy(t) + g_1(t) - g_2(t)|, \alpha|Fx(t) - Gy(t) + g_1(t) - g_2(t)|)\\
&\leq \alpha(|Fx(t) + g_1(t) - x(t)|, p|Fx(t) + g_1(t) - x(t)|)\\
&\quad + \beta(|Gy(t) + g_2(t) - y(t)|, p|Gy(t) + g_2(t) - y(t)|)\\
&\quad + \gamma(|x(t) - y(t)|, p|x(t) - y(t)|),
\end{aligned}$$

where $\alpha+\beta+\gamma < 1$, for every $x,y \in Z$ with $x \neq y$ and $t \in [a,b]$.

(H_5) Whenever $Fx + g_1 \neq Gx + g_2$

$$\sup_{t\in[a,b]} (|Fx(t) - Gx(t) + g_1(t) - g_2(t)|, \alpha|Fx(t) - Gx(t) + g_1(t) - g_2(t)|)$$
$$< \sup_{t\in[a,b]} \alpha(|Fx(t) + g_1(t) - x(t)|, p|Fx(t) + g_1(t) - x(t)|)$$
$$+ \beta(|Gx(t) + g_2(t) - x(t)|, p|Gx(t) + g_2(t) - x(t)|),$$

for every $x \in Z$.

Theorem 4.3 *Assume that hypotheses $(H_3) - (H_5)$ hold. Then the integral equations (3)–(4) have a unique common solution x on $[a, b]$.*

Proof: Define $S, T : Z \to Z$ by $S(x) = Fx + g_1$ and $T(x) = Gx + g_2$. Using hypotheses, we have

$$\begin{aligned}(|Sx(t) - Ty(t)|, \alpha|Sx(t) - Ty(t)|) \leq\ & \alpha(|Sx(t) - x(t)|, p|Sx(t) - x(t)|)\\ &+ \beta(|Ty(t) - y(t)|, p|Ty(t) - y(t)|)\\ &+ \gamma(|x(t) - y(t)|, p|x(t) - y(t)|),\end{aligned}$$

for every $x, y \in Z$ and $x \neq y$. Hence

$$\begin{aligned}(\|S - T\|_\infty, \alpha\|S - T\|_\infty) \leq\ & \alpha(\|Sx - x)\|_\infty, p\|Sx - x\|_\infty)\\ &+ \beta(\|Ty - y\|_\infty, p\|Ty - y\|_\infty)\\ &+ \gamma(\|x - y\|_\infty, p\|x - y\|_\infty).\end{aligned}$$

Next, if $S(x) \neq T(x)$, we have

$$\begin{aligned}(\|S - T\|_\infty, \alpha\|S - T\|_\infty) \leq\ & \alpha(\|Sx - x)\|_\infty, p\|Sx - x\|_\infty)\\ &+ \beta(\|Tx - x\|_\infty, p\|Tx - x\|_\infty),\end{aligned}$$

for every $x \in Z$. By Lemma 4.2 , if f is the identity map on Z, the Urysohn integral equations (3)–(4) have a unique common solution. This completes the proof of the Theorem 4.3.

5 Application

In order to illustrate the applications of some of our result established in previous section, we consider the following partial nonlinear differential equation of the form:

$$\frac{\partial^2 w(t,u)}{\partial t^2} = \frac{\partial^2 w(t,u)}{\partial u^2} + \int_0^t [ts + \frac{w(s,u)s}{2}]ds + \int_0^1 [(ts)^2 + \frac{tsw^2(s,u)}{2}]ds,$$

$$t \in [0,1], \quad u \in I = [0,\pi], \tag{7}$$

$$w(t,0) = w(t,\pi) = 0, \quad t \in [0,1], \tag{8}$$

$$w(0,u) = x_0(u), \quad u \in I, \tag{9}$$

$$\frac{\partial w(t,u)}{\partial t}|_{t=0} = y_0(u), \quad u \in I, \tag{10}$$

Let us take $X = L^2([0,\pi])$ and w(t,u)=x(t)(u). Setting

$$k(t,s,x(s)) = ts + \frac{x}{2} \quad \text{and} \quad h(t,s,x(s)) = (ts)^2 + \frac{tsx^2}{2}.$$

Now we have

$$\begin{aligned}
&(|k(t,s,x(s)) - k(t,s,y(s))|, \alpha|k(t,s,x(s)) - k(t,s,y(s))|)\\
&= (|ts + \frac{xs}{2} - ts - \frac{ys}{2}|, \alpha|ts + \frac{xs}{2} - ts - \frac{ys}{2}|)\\
&= (|\frac{xs}{2} - \frac{ys}{2}|, \alpha|\frac{xs}{2} - \frac{ys}{2}|)\\
&= (\frac{s}{2}|x-y|, \alpha\frac{s}{2}|x-y|)\\
&= \frac{s}{2}(|x-y|, \alpha|x-y|)\\
&\le \frac{s}{2}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)\\
&= p_1^*\Phi^*(\|x-y\|_\infty, \alpha\|x-y\|_\infty),
\end{aligned} \tag{11}$$

where $p_1^*(t,s) = s$, which is continuous function of $[0,1]\times[0,1]$ into $\mathbb{R}^+$ and a comparison function $\Phi^* : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi^*(x,y) = \frac{1}{2}(x,y)$. Similarly, we can show that

$$\begin{aligned}
&(|h(t,s,x(s)) - h(t,s,y(s))|, \alpha|h(t,s,x(s)) - h(t,s,y(s))|)\\
&\le p_2^*\Phi^*(\|x-y\|_\infty, \alpha\|x-y\|_\infty),
\end{aligned} \tag{12}$$

where $p_2^*(t,s) = st$, which is continuous function of $[0,1]\times[0,1]$ into $\mathbb{R}^+$.

Moreover,

$$\int_0^1 [p_1^*(t,s) + p_2^*(t,s)]ds = \int_0^1 [s+st]ds = \frac{1}{2}(1+t)$$

and

$$\sup_{t\in[0,1]} \{1/2(1+t)\} = 1.$$

Also

$$\int_0^1\int_0^1 [p_1^*(t,s) + p_2^*(t,s)]dsdt = \int_0^1\int_0^1 [s+st]dsdt = \int_0^1 \frac{1}{2}(1+t)dt \le \frac{3}{4} < 1.$$

We define the operator $A : D(A) \subset X \to X$ by $Aw = w_{uu}$, where $D(A) = \{w(\cdot) \in X : w,\ w^{'}\ \text{ are absolutely contnuous, }\ w(0) = w(\pi) = 0\}$. It is well known that A is the generator of strongly continuous cosine function $\{C(t) : t \in \mathbb{R}\}$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with corresponding normalized characteristics vectors $w_n(u) := \sqrt{\frac{2}{\pi}} \sin(nu),\ n = 1, 2, 3, \cdots$, and the following conditions hold:

(1) $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(2) If $w \in D(A)$, then $Aw = -\sum_{n=1}^{\infty} n^2 < w, w_n > w_n$.

(3) For $w \in X$, $C(t)w = \sum_{n=1}^{\infty} \cos(nt) < w, w_n > w_n$. Moreover, from these expression, it follows that $S(t)w = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < w, w_n > w_n$, that $S(t)$ is compact for every $t > 0$ and that $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in [0, T]$.

(4) If H denotes the group of translations on X defined by $H(t)x(u) = \tilde{x}(u + t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}\Big(H(t) + H(-t)\Big)$. If $G : X \to X$ is defined by $Gx = x^{'}$, $D(G) = \{x \in X : x^{'} \in X\}$, then it follows that $A = G^2$ (see [5, 10]), where G is the infinitesimal generator of the group H.

With these choices of functions, the equations (7)–(10) can be formulated as an abstract semilinear differential equations (1)–(2). Since all the conditions of Theorem 3.3 are satisfied, the problem (7)–(10) has solution w on $[0, 1] \times [0, \pi]$. On same line one can verify the result of Theorem 4.3.

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