Some Remarks on Mapping from Semigroup onto Anti-ordered Group

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Abstract

In this article we investigate the subset $H = \{x \in S : (\varphi(x), 1) \in \beta\}$ of semigroup $S$ with apartness, the pre-image of the negative cone of anti-ordered group $G$ by a homomorphism $\varphi$ from semigroup $S$ onto groups $G$. This subset is a reflexive completely prime order anti-ideal of $S$.

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1 Introduction

Setting of this investigation is Constructive Mathematics in sense of the following books [2], [4] and [5]. The investigation is a continuation of author’s papers [9]-[12]. In article [12] author describe a construction of anti-ordered group by given anti-ordered semigroup and embedding the last into the anti-ordered group: Let $((S, =, \neq), \cdot, \alpha)$ be a commutative anti-ordered semigroup with apartness such that $\alpha$ is closed for the semigroup operation. Then we can construct an anti-ordered group $G$ that there exists a strongly extensional isotone and reverse isotone mapping from $S$ into $G$. Here we have intention to analyze opposite situation: Let there exists a homomorphism from semigroup $((S, =, \neq), \cdot)$ onto an anti-ordered group $G = ((G, =, \neq), \cdot, \beta)$. Then, there exists a quasi-antiorder relation $\varphi^{-1}(\beta)$ on $S$, and an anticongruence

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\[ q = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} \]
on S such that \( S/q \) is ordered by anti-order determined by \( \varphi^{-1}(\beta) \) and \( S/q \) is isomorphic to the group \( G \). In this situation we investigate pre-image of negative cone \( H = \{ x \in S : ((x), 1) \in \beta \} \) of \( G \).

This paper is motivated by a class of Dubreil-Jacotin semigroups, in the classical Semigroup Theory ([6]): \((S, \cdot, \leq)\) is called Dubreil-Jacotin semigroup if there an isotone semigroup-homomorphism of \((S, \cdot, \leq)\) onto a partially ordered group \((G, \cdot, \beta)\) such that the pre-image of the negative cone of \( G \) is a principal order ideal of \((S, \leq)\). This concept was introduced in [8] (see also [1] and [3], Theorem 12.1).

Let \( S = ((S, =, \neq), \cdot) \) be a semigroup, \( G = ((G, =, \neq), \cdot, 1, \beta) \) an ordered group under anti-order \( \beta \) compatible with the group operation and \( \varphi : S \longrightarrow G \) a strongly extensional epimorphism. We give a version of the Bigard theorem ([1], [3]) where in this situation we describe pre-image of negative cone \( H \) of group \( G \). This subset is a reflexive completely prime order anti-ideal of \( S \).

### 2 Preliminaries

In this section, following the standard notions and notations in the Constructive Algebra from our articles [9], [10], [11] and [12], readers let us remember some fundamental notions and facts about (quasi-) anti-ordered sets and semigroups.

Let \( X = (X, =, \neq) \) be a set with apartness. For a subset \( Y \) of \( X \) we say that it is strongly extensional if \( y \in Y \) and \( x \in X \) follows \( y \neq x \vee x \in Y \) for any \( x, y \in X \). For a relation \( \tau \subseteq X \times X \) which is consistent and cotransitive i.e. that satisfies the following conditions:

\[
\tau \subseteq \neq, \quad \tau \subseteq \tau \ast \tau,
\]

where ‘\( \ast \)’ is the fulfillment operation between relations, we called quasi-antiorder relation. For a subset \( Y \) of quasi-antiordered set \( X = ((X, =, \neq), \tau) \) we say that it is an order anti-ideal of \( X \) if the following implication \((x, y) \in \tau \ast \tau \Rightarrow ((x, y) \in \tau \vee y \in Y) \) holds for any \( x, y \in X \). If a quasi-antiorder relation \( \beta (\subseteq X \times X) \) is linear, i.e. if \( \beta \) satisfies the following conditions \( \neq \subseteq \beta^{-1} \cup \beta \), we say that it is an anti-order relation on set \( X \) and for set \( X \) we sat that it is ordered under this anti-order \( \beta \) or that it is anti-ordered under \( \beta \). If \( X = ((S, =, \neq), \cdot) \) is a semigroup, compatibility of the relation \( \beta \) and the semigroup operation ‘\( \cdot \)’ means

\[
(\forall a, b, x \in S)(((ax, bx) \in \beta \vee (xa, xb) \in \beta) \Rightarrow (a, b) \in \beta).
\]

For relation \( q \subseteq X \times X \) we say that it is coequality on \( X \) if it is consistent, symmetric and cotransitive relation on \( X \). In the case, if \( X = ((S, =, \neq), \cdot) \)
is a semigroup, compatibility of the relation \( q \) and the semigroup operation means
\[
(\forall a, b, x \in S)((ax, bx) \in q \lor (xa, xb) \in q) \implies (a, b) \in q).
\]

In that case, the relation \( q \) we called \textit{anti-congruence} on semigroup \( S \). Further on, for a subset \( Y \) of a semigroup \( S \) we say that it is a \textit{completely prime subset} of \( S \) if the following implication \( ab \in Y \implies a \in Y \lor b \in Y \) holds for any \( a, b \in S \). At least, subset \( Y \) of semigroup \( S \) is \textit{reflexive subset} of \( S \) if \( ab \in Y \) implies \( ba \in Y \) \((a, b \in S)\).

3 Remarks

Remark A:
Let \( S = ((S, =, \neq), \cdot) \) be a semigroup, \( G = ((G, =, \neq), \cdot, 1, \beta) \) an anti-ordered group compatible with the group operation and \( \varphi : S \longrightarrow G \) a strongly extensional epimorphism. It is known (see [10] and [11]) that \( \varphi^{-1}(\beta) \) is a quasi-antiorder relation on \( S \) (see: [10], Lemma 2 or [11], Theorem 4, Point 1), \( q = \text{Coker} \varphi = \{(a, b) \in S \times S : \varphi(a) \neq \varphi(b)\} \) is an anticongruence on \( S \), \( S/q \) is ordered by anti-order \( \Theta \), defined by \((aq, bq) \in \Theta \) if and only if \((a, b) \in \varphi^{-1}(\beta)\) ([10], Lemma 1), and there exists the strongly extensional mapping \( \psi : S/q \longrightarrow G \), given by \( \psi(aq) = \varphi(a) \) for any \( a \in S \), such that it is an injective and embedding iso-
tone and reverse isotone homomorphism from \( S/q \) onto \( \text{Im} \varphi (\subseteq G) \).

Remarks B:
If \((G, =, \neq), \cdot, 1\) is a group with compatible anti-order relation \( \beta \) on \( G \) and \( \varphi : S \longrightarrow G \) is a homomorphism from \( S \) into \( G \). Let \( H = \{x \in S : (\varphi(x), 1) \in \beta\} \). Then:
(1) \( H \) is a \textit{strongly extensional subset} of \( S \). Indeed: Suppose that \( a \in H \) and \( b \in S \), i.e. suppose \((\varphi(a), 1) \in \beta \land b \in S \). Then, \((\varphi(a), \varphi(b)) \in \beta \lor (\varphi(b), 1) \in \beta \). Thus, by consistency of \( \beta \) and strongly extensionality of \( \varphi \), we have \( a \neq b \lor b \in H \).

(2) If \( \beta \cap \beta^{-1} = \emptyset \), \( H \) is a \textit{subsemigroup} of \( S \). In fact, for \( a \in H \), we have \((\varphi(a), 1) \in \beta \land (\varphi(b), 1) \in \beta \). Thus, \(((\varphi(a), \varphi(ab)) \in \beta \lor (\varphi(ab), 1) \in \beta \land (\varphi(b), 1) \in \beta \) and \(((1, \varphi(b)) \in \beta \lor (\varphi(b), 1) \in \beta) \lor (\varphi(ab), 1) \in \beta \). Therefore, we have \((\varphi(ab), 1) \in \beta \) i.e. \( ab \in H \).

(3) \( H \) is \textit{order anti-ideal} of \( S \). Indeed: For \( a \in H \) and \( b \in S \), we have \((\varphi(a), 1) \in \beta \lor \beta \in S \). Thus, \((\varphi(a), \varphi(b)) \in \beta \lor (\varphi(b), 1) \in \beta \) and \((a, b) \in \varphi^{-1}(\beta) \lor (\varphi(b), 1) \in \beta \). This means \( (a, b) \in \varphi^{-1}(\beta) \lor b \in H \).

(4) \( H \) is \textit{completely prime subset} of \( S \) because for \( ab \in H \), i.e. for \((\varphi(ab), 1) \in \beta \), we have \((\varphi(a) \varphi(b), \varphi(b)1) \in \beta \lor (\varphi(b), 1) \in \beta \). Thus, \((\varphi(a), 1) \in \beta \lor
(\varphi(b), 1) \in \beta \text{ which means } a \in H \lor b \in H.

(5) \(H\) is reflexive subset of \(S\). Indeed, for \(ab \in H\), we have \((\varphi(ab), 1) \in \beta\). This is equivalent with \((\varphi(a)\varphi(b), \varphi(b)^{-1}\varphi(b)) \in \beta\). Thus, \((\varphi(a), \varphi(b)^{-1}) \in \beta\). Further on, we have \((\varphi(b)^{-1}\varphi(b)\varphi(a), \varphi(b)^{-1}) \in \beta\). The least we conclude \((\varphi(ba), 1) \in \beta\), which means \(ba \in H\).

Finally, according to remarks mentioned above, we can conclude that the subset \(H\) is a strongly extensional, reflexive, completely prime, order antireflective relation of \(S\). Especially, if the anti-order \(\beta\) on \(G\) satisfies the following condition \(\beta \cap \beta^{-1} = \emptyset\), then the set \(H\) is a subsemigroup of \(S\).

\textbf{Remark C:}
Let \(x\) be an arbitrary element of \(S\) such that \(\varphi(x) \neq 1\). Then there exists an element \(y\) of \(S\) such that \(\varphi(xy) \neq 1\) because \(\varphi\) is surjective. Thus, \((\varphi(xy), 1) \in \beta\) or \((1, \varphi(xy)) \in \beta\). From the first case, we have \(y \in \{x \in H : x = \{u \in S : yu \in H\}\}\).

In the second case, from \((1, \varphi(xy)) \in \beta\) we have \((\varphi(x)^{-1}\varphi(y)^{-1}, 1) \in \beta\) and \((\varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)^{-1}\varphi(x), 1) \in \beta\). Finally, again \(\varphi\) is onto, there exists an element \(t\) of \(S\) such that \(\varphi(t) = \varphi(x)^{-1}\varphi(y)^{-1}\varphi(x)^{-1}\varphi(x)\) and we have \(t \in \{H : x\}\).

Therefore, for any \(x \in S\) such that \(\varphi(x) \neq 1\), holds \([H : x] \neq \emptyset\). Since \(\varphi\) is surjective, then there exists an element \(t \in S\) such that \(\varphi(t) = 1\). Thus, we conclude \((\varphi(t), 1) \ni \beta \cup \beta^{-1}\).

\textbf{Example:} Let \(B\) be a band, \((G, \beta)\) an anti-ordered group and let \(S = B \times G\) be their direct product with the internal operation '◦' defined by \((e, a) \circ (f, b) = (ef, ab)\). Then, the set \(((S, =, \neq, \circ)\) is a semigroup. The natural anti-order \(\gamma\) on \(S\) is given by \(((e, a), (f, b)) \in \gamma\) if \((e, f) \in \alpha\) or \((a, b) \in \beta\) where \(\alpha\) is a natural anti-order in band \(([7], \text{Lemma 1})\). Notice that, \(\gamma\) is not compatible with multiplication operation in \(S\), in general. The projection \(\varphi : S \rightarrow G\), defined by \(\varphi((e, a)) = a\), is a reverse isotone epimorphism because the implication \((a, b) \in \beta \implies ((e, a), (f, b)) \in \gamma\) holds for any \(e, f \in B\). Further on, the relation

\[\varphi^{-1}(\beta) = \{(e, a), (f, b) : e \in B \land f \in B \land (a, b) \in \beta\}\]

is a quasi-antiiorder relation on \(S\) compatible with the semigroup operation. So, the relation

\[\Theta = \{(e, a)q, (f, b)q \in S/q \times S/q : e \in B \land f \in B \land (a, b) \in \beta\}\]

is induced anti-order on factor-semigroup \(S/q\), where

\[q = \text{Coker } \{((e, a), (f, b)) : e \in B \land f \in B \land a \neq b\}\].

The mapping \(\psi : S/q \rightarrow G\) is isotone and reverse isotone isomorphism and the set \(H = \{\varphi((e, a)), 1) \in B \times G : e \in B \land (a, 1) \in \beta\} = \{(e, a) : (a, 1) \in \beta\}\) is an negative cone of \(S\).
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References


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