Modified Noor Iterative Scheme in Banach Spaces

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Abstract

In this paper, we establish the weak convergence of the sequence of modified three steps iterates of quasi-nonexpansive maps in the framework of uniformly convex Banach spaces. The results obtained generalize some well known existing results.

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1. INTRODUCTION

Let $K$ be a closed convex bounded subset of a uniformly convex Banach space $(X, \| \cdot \|)$ and $T$ be a self-mapping of $X$. Recall that $T$ is nonexpansive on $K$ if for all $x, y \in K$, we have

$$\|Tx - Ty\| \leq \|x - y\|. \hspace{1cm} (1.1)$$

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A point $f \in K$ is a fixed point of $T$ if $Tf = f$. We denote the set of the fixed points of $T$ by $F(T)$, where $F(T) = \{ f \in K : Tf = f \}$.

A map $T$ satisfying
\[ \|Tx - f\| \leq \|x - f\| \]  
for all $x \in K$ and $f \in F(T)$ is called a quasi-nonexpansive map.

We remark that the class of quasi-nonexpansive maps properly includes the class of nonexpansive maps with $F(T) \neq \emptyset$ (see also, [9]). The condition (1.2) was introduced by Tricomi [11] for the real functions and thereafter it has been extended and studied independently by a number of authors in different settings. Diaz and Metcalf [2] and Dotson [3] independently studied this for mappings in Banach spaces. The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert spaces. Petryshyn and Williamson [9] have presented the results regarding the necessary and sufficient condition for the convergence of iterates of quasi-nonexpansive maps in the Banach spaces. Ghosh and Debnath [4] studied the convergence of iterates of the family of nonexpansive mappings in a uniformly convex Banach spaces. Kirk [5] has extended this work to metric spaces which is adapted to normed spaces by a number of authors. Xu and Noor [12] have suggested and analyzed a three step iterative scheme for asymptotically nonexpansive mappings in Banach spaces. Further, Rhoades and Temir [10] established the weak convergence of the sequence of the Mann iterates to a common fixed point of $T$ and $I$ by considering the map $T$ to be $I$-nonexpansive.

Recently Kiziltunc and Ozdemir [6] considered $T$ and $I$ nonself mappings of $K$, where $T$ is an $I$-nonexpansive and $I$ a nonexpansive mappings. They established the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of $T$ and $I$. Kumam, Kumethong and Jewwaiworn [7] have established the weak convergence of three-step Noor iterative scheme for an $I$-nonexpansive mapping in a Banach space satisfying Opial’s condition. Our aim is to establish the weak convergence of the sequence of modified three step Noor iterates to a common fixed point of two maps $T$ and $I$.

First we recall some relevant concepts and results used in the sequel.

**Definition 1.1** [6]. Let $X$ be a real Banach space. A subset $K$ of $X$ is said to be a retract of $X$ if there exists a continuous map $P : X \to K$ such that $Px = x$ for all $x \in K$. A map $P : X \to X$ is said to be a retraction if $P^2 = P$. It follows that if $P$ is a retraction map, then $Py = y$ for all $y$ in the range of $P$.

Let $K$ be a nonempty convex subset of $X$ and $T : K \to K$. For $x_i \in K$ and some $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\} \subset [0, 1]$, we recapitulate the iterative procedures used in the literature.
The Mann iteration formula is given as
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 1 \] (1.3)

The Ishikawa iterative scheme is defined by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n \\
\[ y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 1. \] (1.4)

The three step Noor iteration scheme is defined by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n \\
\[ y_n = (1 - \beta_n)x_n + \beta_nTx_n \\
\[ z_n = (1 - \gamma_n)x_n + \gamma_nTx_n, \quad n \geq 1. \] (1.5)

If \( \gamma_n = 0 \) in (1.5) it reduces to (1.4) and (1.4) becomes (1.3) when \( \beta_n = 0 \).

Let \( T, I \) be self maps on \( K \). Motivated by Ghosh and Debnath [4], we extend the scheme (1.5) for two mappings as follows:
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n \\
\[ y_n = (1 - \beta_n)x_n + \beta_nTx_n \\
\[ z_n = (1 - \gamma_n)x_n + \gamma_nTx_n. \] (1.6)

The above iterative procedures may fail to be well defined if the domain \( K \) of \( T \) and \( I \) is a proper subset of \( X \) and \( T, I \) maps \( K \) into \( X \). One method that has been used to overcome this (see [6]) is to introduce a retraction \( P : X \rightarrow K \) in the above recursion formulae (1.3) to (1.6), as follows:
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nPTx_n, \quad n \geq 1 \] (1.7)

The modified Ishikawa iterative scheme \( \{x_n\} \) is defined (cf. [6]) in the following manner
\[ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_nTy_n) \\
\[ y_n = P((1 - \beta_n)x_n + \beta_nTx_n), \quad n \geq 1. \] (1.8)

Let \( X \) be a uniformly convex Banach space and \( K \) be a nonempty convex subset of \( X \) with \( P \) as nonexpansive retraction. Let \( P : X \rightarrow K \) be a given nonself mapping. The modified Noor iterative scheme \( \{x_n\} \) for one and two mappings may be defined, respectively, as
\[ x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_nTy_n) \\
\[ y_n = P((1 - \beta_n)x_n + \beta_nTx_n) \] (1.9)
and
\[ z_n = P((1 - \gamma_n)x_n + \gamma_nTx_n) \]
for certain choices of \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0, 1] \).

Obviously, if \( T \) and \( I \) are self maps, then the iteration schemes (1.7), (1.8), (1.9) and (1.10) respectively reduce to the schemes (1.3), (1.4), (1.5) and (1.6).

**Definition 1.2** [8]. A Banach space \( X \) satisfies Opial’s condition, if for every sequence \( \{x_n\} \) in \( X \) converging to \( x \in X \), the inequality

\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|
\]

holds for all \( y \in X \) with \( x \neq y \).

We recall that all spaces \( l_p (1 \leq p < \infty) \) satisfy Opial’s condition however, \( L_p \) spaces do not unless \( p = 2 \) (see also [7] & [10]).

**Definition 1.3** [6]. \( T \) is called \( I \)-nonexpansive map on \( K \) if

\[
\|Tx - Ty\| \leq \|Ix - Iy\|
\]

for all \( x, y \in K \). \( T \) is called \( I \)-quasi-nonexpansive map on \( K \) if

\[
\|Tx - f\| \leq \|Ix - f\|
\]

for all \( x, y \in K \) and Recall that a point \( x \in K \) is a common fixed point of \( I \) and \( T \) if \( x = Tx = Tx \).

Now we present our main results.

**2. MAIN RESULTS**

**Theorem 2.1.** Let \( K \) be a closed, convex and bounded subset of a uniformly convex Banach space \( X \), which satisfies Opial’s condition, and let \( T \) and \( I \) non-self mapping of \( K \) with \( T \) be an \( I \) quasi- nonexpansive and \( I \) a nonexpansive on \( K \). Then, for \( x_0 \in K \) the sequence \( \{x_n\} \) of iterates defined by (1.10) converges weakly to common fixed point of \( F(T) \cap F(I) \).

**Proof:** If \( F(T) \cap F(I) \) is nonempty and singleton, then the proof is complete.
We will assume that $F(T) \cap F(I) \neq \emptyset$ and that $F(T) \cap F(I)$ is not a singleton.

Since

$$
\left\| x_{n+1} - f \right\| = \left\| P((1 - \alpha_n)x_n + \alpha_nTy_n) - f \right\|
\leq \left\| (1 - \alpha_n)x_n + \alpha_nTy_n - f \right\|
\leq \left\| (1 - \alpha_n)(x_n - f) + \alpha_n(Ty_n - f) \right\|
\leq \left\| (1 - \alpha_n)(x_n - f) + \alpha_n(Iy_n - f) \right\|
\leq \left\| (1 - \alpha_n)(x_n - f) + \alpha_n(y_n - f) \right\|
\leq \left\| (1 - \alpha_n)(x_n - f) + \alpha_n[P((1 - \beta_n)x_n + \beta_nTz_n) - f] \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n\left\| (1 - \beta_n)x_n + \beta_nTz_n - f \right\|

\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n\left\| Tz_n - f \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n\left\| x_n - f \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n\left\| P((1 - \gamma_n)x_n + \gamma_nTx_n) - f \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n\left\| (1 - \gamma_n)(x_n - f) + \gamma_n(Tx_n - f) \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n(1 - \gamma_n)\left\| x_n - f \right\| + \alpha_n\beta_n\gamma_n\left\| Tx_n - f \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n(1 - \gamma_n)\left\| x_n - f \right\| + \alpha_n\beta_n\gamma_n\left\| x_n - f \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n(1 - \gamma_n)\left\| x_n - f \right\| + \alpha_n\beta_n\gamma_n\left\| x_n - f \right\|
\leq (1 - \alpha_n)\left\| x_n - f \right\| + \alpha_n(1 - \beta_n)\left\| x_n - f \right\| + \alpha_n\beta_n(1 - \gamma_n)\left\| x_n - f \right\| + \alpha_n\beta_n\gamma_n\left\| x_n - f \right\|
= \left\| x_n - f \right\|

Thus for $\alpha_n \neq 0$, $\beta_n \neq 0$ and $\gamma_n \neq 0$, $\left\| x_n - f \right\|$ is a nonincreasing sequence.

Then, $\lim_{n \to \infty} \left\| x_n - f \right\|$ exists.

Now we show that the sequence $\{x_n\}$ converges weakly to a common fixed point of $T$ and $I$. The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in $K$. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converge weakly to $f$ and $q$ respectively. We will show that $f = q$. Suppose that $X$ satisfies Opial’s condition and that $f \neq q$ is in weak limit set of the sequence $\{x_n\}$.

Then $\{x_{n_k}\} \to f$ and $\{x_{m_k}\} \to q$ respectively. Since $\lim_{n \to \infty} \left\| x_n - f \right\|$ exists for any $f \in F(T) \cap F(I)$, by Opial’s condition, we conclude that
\[
\lim_{n \to \infty} \|x_n - f\| = \lim_{k \to n} \|x_k - f\| \\
< \lim_{k \to n} \|x_k - q\| = \lim_{j \to n} \|x_j - q\| \\
< \lim_{j \to n} \|x_j - f\| \\
= \lim_{n \to \infty} \|x_n - f\|
\]

This is a contradiction. Thus \(\{x_n\}\) converges weakly to common fixed point of \(F(T) \cap F(I)\).

Let \(T\) and \(I\) be self maps on \(K\) with \(I = I_d\), the identity map. Then we get the following result from above theorem:

**Corollary 2.1** (cf. Theorem 2.1, [7]). Let \(K\) be a closed convex and bounded subset of a uniformly convex Banach space \(X\), which satisfies Opial’s condition, and let \(T, I\) self-mappings of \(K\) with \(T\) be an \(I\)-quasi-nonexpansive mapping, \(I\) a nonexpansive on \(K\). Then, for \(x_0 \in K\), the sequence \(\{x_n\}\) of three-step Noor iterative scheme converges weakly to common fixed point of \(F(T) \cap F(I)\).

In Theorem 2.1, if \(T\) and \(I\) remain self mapping of \(K\) with \(I = I_d\), the identity map, and \(\gamma_n = 0\), we obtain the following theorem of Kumam et al. [7].

**Corollary 2.2** (cf. Theorem 2.2 [7]). Let \(K\) be a closed convex bounded subset of a uniformly convex Banach space \(X\), which satisfies Opial’s condition, and let \(T, I\) self-mappings of \(K\) with \(T\) be an \(I\)-quasi-nonexpansive mapping, \(I\) a nonexpansive on \(K\). Then, for \(x_0 \in K\), the sequence \(\{x_n\}\) of Ishikawa iterative scheme converges weakly to common fixed point of \(F(T) \cap F(I)\).

Further, if \(T\) and \(I\) remain self mapping of \(K\) with \(I = I_d\), the identity map, and \(\beta_n = 0\) and \(\gamma_n = 0\) in Theorem 2.1 we obtain the result of Rhoades and Temir [10].

**Corollary 2.3** (cf. Theorem 2.1 [10]). Let \(K\) be a closed, convex and bounded subset of a uniformly convex Banach space \(X\), which satisfies Opial’s condition, and let \(T, I\) self mapping of \(K\) with \(T\) be an \(I\)-nonexpansive mapping, \(I\) a nonexpansive on \(K\). Then, for \(x_0 \in K\), the sequence \(\{x_n\}\) of Mann iterates converges weakly to common fixed point of \(F(T) \cap F(I)\).
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