Certain Interesting Results on Meromorphic Functions

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Abstract

In this note two subclasses of meromorphic functions $\Sigma^*_{r}(A, B)$ and $\Sigma^c_{r}(A, B)$ are introduced and some interesting properties of functions belonging to these classes are discussed.

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1 Introduction

Let $\Sigma_r$ denote the class of functions of the form

$$ f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n $$

which are regular in the punctured disc $D_r = \{ z \mid 0 < |z| < r \leq 1 \}$ with a simple pole at the origin and residue 1 there. We introduce two classes of meromorphic functions denoted by $\Sigma^*_{r}(A, B)$ and $\Sigma^c_{r}(A, B)$, $-1 < A < B \leq 1$ consisting of functions $f(z) \in \Sigma_r$ obeying the inequalities

$$ -z \frac{f'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, \quad z \in D_r $$

(2)
and
\[- \left( \frac{f''(z)}{f'(z)} + 1 \right) < \frac{1 + Az}{1 + Bz}, \quad z \in D_r \] respectively. For \( A = 2\alpha - 1, B = 1 \) these classes reduce to \( T_r^*(\alpha) \) and \( C_r(\alpha) \) discussed in [1]. We note that \( f(z) \in \Sigma^*_r(A, B) \) if and only if \( -zf'(z) \in \Sigma^*_r(1 + A, B) \).

Many results discussing various properties of classes consisting of univalent, starlike, convex, multivalent and meromorphic functions are contained in the book by Srivastava and Owa [2].

2 Main Results

In this section, we study some interesting results belonging to these classes. Our first result gives a necessary condition for \( f \in \Sigma^*_r(A, B) \).

Theorem 2.1 If \( f(z) \in \Sigma_r \) satisfies
\[
\sum_{n=0}^{\infty} (n + k + |B + A + n - k|) |a_n| r^{n+1} \leq B - A,
\] for \(-1 < A < B \leq 1, \quad A < 2k - 1 \leq 1\), then \( f(z) \in \Sigma^*_r(A, B) \).

Proof: We know that for \( f(z) \in \Sigma_r \),
\[
|zf'(z) + kf(z)| - |zf'(z) + (B + A - k)f(z)| = \left| (k - 1) \frac{1}{z} + \sum_{n=0}^{\infty} (n + k)a_n z^n \right| - \left| (B + A - k - 1) \frac{1}{z} + \sum_{n=0}^{\infty} (B + A + n - k)a_n z^n \right|.
\]
Using (4) we have,
\[
r|zf'(z) + kf(z)| - r|zf'(z) + (2\alpha - k)f(z)| \leq (k - 1) + \sum_{n=0}^{\infty} (n + k)|a_n| r^{n+1} - (k - 1 + B - A) + \sum_{n=0}^{\infty} (B + A + n - k)|a_n| r^{n+1} \leq 0.
\]
Which shows that
\[
\sum_{n=0}^{\infty} (n + k + |B + A + n - k|) |a_n| r^{n+1} \leq B - A.
\]
It follows from the above that,
\[
\left| \frac{zf'(z) + kf(z)}{zf'(z) + (B + A - k)f(z)} \right| \leq 1,
\]
so that
\[-zf'(z) \in \Sigma^*_r(1 + A, B), \quad (z \in D_r).\]
Corollary 2.2 For $A = 2\alpha - 1, B = 1$ we get Theorem 2.1 in [1] which reads as

If $f(z) \in \Sigma_r$ satisfies

$$
\sum_{n=0}^{\infty} (n + k + |2\alpha + n - k|) |a_n| r^{n+1} \leq 2(1 - \alpha),
$$

for some $\alpha, 0 \leq \alpha < 1$ and $k, (\alpha < k \leq 1)$, then $f(z) \in T_r^*(\alpha)$.

Letting $k = 0$ in Theorem 2.1, we have

Corollary 2.3 If $f(z) \in \Sigma_r$ satisfies

$$
\sum_{n=0}^{\infty} (2n + B + A) |a_n| r^{n+1} \leq B - A,
$$

for some $0 \leq A < B < 1$, then $f(z) \in \Sigma_r^*(A, B)$.

The following necessary and sufficient condition can be derived from Theorem 2.1.

Corollary 2.4 Let the function $f(z) \in \Sigma_r$ be given by (1) with $a_n = |a_n|e^{\frac{n+1}{2}\pi i}$, then $f(z) \in \Sigma_r^*(A, B)$ if and only if

$$
\sum_{n=0}^{\infty} (2n + B + A) |a_n| r^{n+1} \leq B - A
$$

for $0 \leq A < B < 1$.

Proof: In view of Theorem 2.1, we see that if the coefficient inequality (7) holds for $0 \leq A < 1$ then $f(z) \in \Sigma_r^*(A, B)$. Conversely, let $f(z)$ be in the class $\Sigma_r^*(A, B)$ then,

$$
\Re \left\{ -zf'(z) \right\} = \Re \left\{ \frac{1 - \sum_{n=0}^{\infty} na_n z^{n+1}}{1 + \sum_{n=0}^{\infty} a_n z^{n+1}} \right\} > \frac{B + A}{2} \text{ for all } z \in D_r.
$$

Letting

$$
z = r e^{\frac{1}{2\pi} i},
$$

we have that

$$
a_n z^{n+1} = |a_n| r^{n+1}
$$

This implies that

$$
1 - \sum_{n=0}^{\infty} n |a_n| r^{n+1} \geq \frac{B + A}{2} \left( 1 + \sum_{n=0}^{\infty} |a_n| r^{n+1} \right)
$$
which is equivalent to (7).

The function \( f(z) \) given by
\[
f(z) = \frac{1}{z} + a_0 + \left( \frac{B - A - (B + A)|a_0|}{2n + B + A} \right) e^{i\theta} z^n
\]
belongs to the class \( \Sigma_r^* (A, B) \) for some real \( \theta \) with
\[
0 \leq A \leq \frac{1 - |a_0|}{1 + |a_0|} < 1.
\]

If \( f(z) \in \Sigma_r \) with \( a_0 = 0 \); then Corollary 2.4 is true for \(-1 \leq A < B \leq 1\).

**Corollary 2.5** Let the function \( f(z) \in \Sigma_r \) be given by (1) with \( a_n \geq 0 \), then \( f(z) \in \Sigma_r^* (A, B) \) if and only if
\[
\sum_{n=0}^{\infty} (2n + B + A)a_n r^{n+1} \leq B - A, \quad 0 \leq A < B < 1.
\] (8)

**Theorem 2.6** If \( f(z) \in \Sigma_r \) satisfies
\[
\sum_{n=0}^{\infty} n(2n + B + A)|a_n| r^{n+1} \leq B - A
\] (9)
for \(-1 \leq A < B \leq 1\), then \( f(z) \in \Sigma_r^* (A, B) \).

**Proof:** We know that \( f(z) \in \Sigma_r^* (A, B) \) if and only if \(-zf'(z) \in \Sigma_r^* (A, B)\)
\[
i.e., \quad -zf'(z) = \frac{1}{z} - \sum_{n=1}^{\infty} na_n z^n.
\]

Using Theorem 2.1 the result is obtained.

**Corollary 2.7** For \( A = 2\alpha - 1, B = 1 \) we get Theorem 2.5 in [1] which states that if \( f(z) \in \Sigma_r \) satisfies
\[
\sum_{n=0}^{\infty} n(n + \alpha)|a_n| r^{n+1} \leq 1 - \alpha
\]
for some \( \alpha (0 \leq \alpha < 1) \), then \( f(z) \) belongs to the class \( C_r^* (\alpha) \).

**Corollary 2.8** Let the function \( f(z) \in \Sigma_r \) be given by (1) with
\[
a_n = |a_n| e^{-\frac{n+1}{2\pi} i},
\]
then \( f(z) \in \Sigma_r^* (A, B) \) if and only if the inequality (9) holds for \(-1 \leq A < B \leq 1\).
The function \( f(z) \) given by
\[
f(z) = \frac{1}{z} + a_0 + \left( \frac{B - A}{n(2n + B + A)} \right) e^{i\theta}z^n
\]
belongs to the class \( \Sigma^e_r(A, B) \) for some real \( \theta \) with \(-1 < A < B \leq 1\).

**Theorem 2.9** A function \( f(z) \in \Sigma_r \) belongs to the class \( \Sigma^*_r(A, B) \) for \( 0 \leq r < r_0 \), where \( r_0 \) is the smallest positive root of the equation.

\[
(B + A)|a_0|^3 - (2\delta + B - A)r^2 - (B + A)|a_0|r + (B - A) = 0 \quad (10)
\]

and
\[
\delta = \sqrt{\frac{\sum_{n=1}^{\infty} n|a_n|^2 + B + A}{2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n} |a_n|^2}}.
\quad (11)
\]

**Proof:** Using Cauchy’s inequality, we see that
\[
\sum_{n=1}^{\infty} \left( \frac{2n + B + A}{2} \right) |a_n|^r^{n+1}
= \frac{B + A}{2} |a_0|r + \sum_{n=1}^{\infty} |a_n|r^{n+1}
\leq \frac{B + A}{2} |a_0|r + \sqrt{\sum_{n=1}^{\infty} n|a_n|^2} \sqrt{\sum_{n=1}^{\infty} n^n^{2n+2}} + \frac{B + A}{2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n} |a_n|^2} \sqrt{\sum_{n=1}^{\infty} n^n^{2n+2}}
= \frac{B + A}{2} |a_0|r + \sqrt{\frac{r}{1 - r}} \left( \sqrt{\sum_{n=1}^{\infty} n|a_n|^2} + \frac{B + A}{2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n} |a_n|^2} \right)
= \frac{B + A}{2} |a_0|r + \frac{r^2}{1 - r}\delta
< \frac{B^2}{2}
\]
where \( \delta \) is given by (10).

Applying Corollary 2.3 we see that \( f(z) \in \Sigma^*_r(A, B) \) for \( 0 \leq r < r_0 \).

Letting \( a_0 = 0 \) in Theorem 2.9, we have

**Corollary 2.10** A function \( f(z) \in \Sigma_r \) with \( a_0 = 0 \) belongs to the class \( \Sigma^*_r(A, B) \) for \( 0 \leq r < r_0 \), where

\[
r_0 = \sqrt{1 - \frac{2\delta}{2\delta + B - A}}
\]

and \( \delta \) is given by (10).

**Corollary 2.11** For \( A = 2\alpha - 1, B = 1 \) we get Theorem 3.1 in[1].

Which is as follows:
A function \( f(z) \in \Sigma_r \) belongs to the class \( \mathcal{T}^*_r(\alpha) \) for \( 0 \leq r < r_0 \), where \( r_0 \) is the smallest positive root of the equation

\[
\alpha|a_0|^3 - (\delta + 1 - \alpha)r^2 - \alpha|a_0|r + 1 - \alpha = 0
\]
and
\[ \delta = \sqrt{\sum_{n=1}^{\infty} n|a_n|^2 + \alpha \sum_{n=1}^{\infty} \frac{1}{n} |a_n|^2}. \]

If we consider the function \( f(z) \) given by
\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n \sqrt{n}} e^{i\theta_n} z^n, \quad \theta_n \text{ is real,} \]
then \( f(z) \in \Sigma_r^c(A, B) \) for \( 0 \leq r < r_0 \) with
\[ \delta = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{B + A}{2} \sum_{n=1}^{\infty} \frac{1}{n^4} - 2 \frac{B + A}{2} \sqrt{\rho(2)} - \frac{B + A}{2} \sqrt{\rho(4)} = \pi \left( \frac{1}{\sqrt{6}} + \frac{\pi(B + A)}{6\sqrt{10}} \right). \]

Further, letting \( A = -B, \quad \delta = \frac{\pi}{\sqrt{6}} \) and \( r_0 = \sqrt{\frac{\sqrt{6}}{\sqrt{6} + \pi}}. \)

**Theorem 2.12** A function \( f(z) \in \Sigma_r \) belongs to the class \( \Sigma_r^c(A, B) \) for \( 0 \leq r < r_1 \), where
\[ r_1 = \sqrt{1 - \frac{2\sigma}{2\sigma + B - A}} \]
and
\[ \sigma = \sqrt{\sum_{n=1}^{\infty} n^3 |a_n|^2 + \frac{B + A}{2} \sum_{n=1}^{\infty} n|a_n|^2}. \]

Let us consider the function \( f(z) \) given by
\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{n}} e^{i\theta_n} z^n, \quad \theta_n \text{ is real.} \]
We see that \( f(z) \in \Sigma_r^c(A, B) \) for \( 0 \leq r < r_0 \) with
\[ \delta = \pi \left( \frac{1}{\sqrt{6}} + \frac{\pi(B + A)}{6\sqrt{10}} \right) \]
For \( A = -B \), we obtain
\[ \delta = \frac{\pi}{\sqrt{6}} \quad \text{and} \quad r_0 = \sqrt{\frac{\sqrt{6}}{\sqrt{6} + \pi}}. \]

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