A Subclass of Uniformly Convex Functions and a Corresponding Subclass of Starlike Functions with Fixed Second Coefficient Defined by Carlson and Shaffer Operator

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Abstract

The main objective of this paper is to obtain necessary and sufficient condition for a subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient defined by Carlson and Shaffer operator for the function $f(z)$ in $UCT(\alpha, \beta)$. Furthermore, we obtain extreme points, distortion bounds and closure properties for $f(z)$ in $UCT(\alpha, \beta)$ by fixing second coefficient.

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1. Introduction

Denote by $S$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

that are analytic and univalent in the unit disc $U = \{ z : |z| < 1 \}$ and by $ST$ and $CV$ the subclasses of $S$ that are respectively, starlike and convex. Goodman [4, 5] introduced and defined the following subclasses of $CV$ and $ST$.

A function $f(z)$ is uniformly convex (uniformly starlike) in $U$ if $f(z)$ is in $CV(ST)$ and has the property that for every circular arc $\gamma$ contained in $U$, with center $\xi$ also in $U$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions is denoted by $UCV$ and the class of uniformly starlike functions by $UST$. It is well known from [[3], [8]] that

$$f \in UCV \Leftrightarrow \left| \frac{zf''(z)}{f'(z)} \right| \leq \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\}.$$  

In [10], Rønning introduced a new class of starlike functions related to $UCV$ defined as

$$f \in S_p \Leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left\{ \frac{zf''(z)}{f'(z)} \right\}.$$  

Note that $f(z) \in UCV \Leftrightarrow zf'(z) \in S_p$. Further, Rønning generalized the class $S_p$ by introducing a parameter $\alpha$, $-1 \leq \alpha < 1$,

$$f \in S_p(\alpha) \Leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \text{Re} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\}.$$  

Now we define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} c_{n-1}}{(c)_{n-1}} z^n,$$  

(1.2)

for $c \neq 0, -1, -2, \ldots$, $a \neq -1; z \in U$ where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(n + \lambda)}{\Gamma(\lambda)}$$

$$= \begin{cases} 
1; \quad n = 0 \\
\lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1), \quad n \in \{1, 2, \ldots\} 
\end{cases}$$  

(1.3)
Carlson and Shaffer [3] introduced a linear operator \( L(a, c) \), defined by

\[
L(a, c)f(z) = \phi(a, c; z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n, \quad z \in U,
\]

(1.4)

where \(*\) stands for the Hadamard product or convolution product of two power series

\[
\varphi(z) = \sum_{n=1}^{\infty} \varphi_n z^n \quad \text{and} \quad \psi(z) = \sum_{n=1}^{\infty} \psi_n z^n
\]

defined by

\[
(\varphi * \psi)(z) = \varphi(z) * \psi(z) = \sum_{n=1}^{\infty} \varphi_n \psi_n z^n.
\]

We note that \( L(a, a)f(z) = f(z), L(2, 1)f(z) = zf(z), L(m + 1, 1)f(z) = D^m f(z) \), where \( D^m f(z) \) is the Ruscheweyh derivative of \( f(z) \) defined by Ruscheweyh [11] as

\[
D^m f(z) = \frac{z}{(1 - z)^{m+1}} * f(z), \quad m > -1,
\]

(1.5)

which is equivalently,

\[
D^m f(z) = \frac{z}{m!} \frac{d^m}{dz^m} \{ z^{m-1} f(z) \}.
\]

**Definition 1.1**: For \( \beta \geq 0, -1 \leq \alpha < 1 \), we define a class \( UCV(\alpha, \beta) \) subclass of \( S \) consisting of functions \( f(z) \) of the form (1.1) and satisfying the analytic criterion

\[
\text{Re} \left\{ \frac{z(L(a, c)f(z))^\nu}{(L(a, c)f(z))'} + 1 - \alpha \right\} \geq \beta \left| \frac{z(L(a, c)f(z))''}{(L(a, c)f(z))'} \right|, \quad z \in U.
\]

(1.6)

We also let \( UCT(\alpha, \beta) \), the subclass of \( S \) consisting of functions of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad \forall \ n \geq 2.
\]

(1.7)

The main objective of this paper is to obtain necessary and sufficient condition for a subclass of uniformly convex functions and corresponding subclass of starlike functions with fixed second coefficient defined by Carlson and Shaffer operator for the function \( f(z) \in UCT(\alpha, \beta) \). Furthermore, we obtain extreme points, distortion bounds and closure properties for \( f(z) \in UCT(\alpha, \beta) \) by fixing second coefficient.
2. The Class $UCT(\alpha, \beta)$

Firstly, we obtain necessary and sufficient condition for functions $f(z)$ in the classes $UCV(\alpha, \beta)$.

**Theorem 2.1** : A function $f(z)$ of the form (1.1) is in $UCV(\alpha, \beta)$ if

$$
\sum_{n=2}^{\infty} n|n(1 + \beta) - (\alpha + \beta)| \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq 1 - \alpha, \quad (2.1)
$$

$-1 \leq \alpha < 1, \beta \geq 0.$

**Proof** : If suffices to show that

$$
\beta \left| \frac{z(L(a, c)f(z))^n}{(L(a, c)f(z))'} \right| - \text{Re} \left\{ \frac{z(L(a, c)f(z))''}{(L(a, c)f(z))'} \right\} \leq 1 - \alpha.
$$

We have

$$
\beta \left| \frac{z(L(a, c)f(z))^n}{(L(a, c)f(z))'} \right| - \text{Re} \left\{ \frac{z(L(a, c)f(z))''}{(L(a, c)f(z))'} \right\} \leq 1 - \alpha,
$$

$$
(1 + \beta) \sum_{n=2}^{\infty} n(n - 1) \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq 1 - \alpha,
$$

and hence the proof is complete.

**Theorem 2.2** : a necessary and sufficient for $f(z)$ of the form (1.7) to be in the class $UCT(\alpha, \beta), -1 \leq \alpha < 1, \beta \geq 0$ is that

$$
\sum_{n=2}^{\infty} n|n(1 + \beta) - (\alpha + \beta)| \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| \leq 1 - \alpha, \quad (2.2)
$$

**Proof** : In view of Theorem 2.1, we need only to prove the necessity. If $f(z) \in UCT(\alpha, \beta)$ and $z$ is a real then

$$
\text{Re} \left\{ \frac{z(L(a, c)f(z))''}{(L(a, c)f(z))'} + 1 - \alpha \right\} \geq \beta \left| \frac{z(L(a, c)f(z))''}{(L(a, c)f(z))'} \right|
$$

which gives

$$
- \sum_{n=2}^{\infty} n(n - 1) \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| z^{n-1} + (1 - \alpha) \left[ \sum_{n=2}^{\infty} n \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| z^{n-1} \right]
$$

$$
\Leftrightarrow 1 - \sum_{n=2}^{\infty} n \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| z^{n-1}
$$
\[ \geq \beta \left| \sum_{n=2}^{\infty} \frac{n(n-1)(a)_{n-1} a_n z^{n-1}}{(c)_{n-1} a_n z^{n-1}} \right| \left( 1 - \sum_{n=2}^{\infty} n \frac{(a)_{n-1} a_n z^{n-1}}{(c)_{n-1} a_n z^{n-1}} \right). \]

Letting \( z \to 1 \) along the real axis, we obtain the desired inequality

\[ \sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha + \beta)] \frac{(a)_{n-1}}{(c)_{n-1}} a_n \leq 1 - \alpha, \]

\(-1 \leq \alpha < 1, \beta \geq 0.\)

**Corollary 2.1**: Let the function \( f(z) \) defined by (1.7) be in the class \( UCT(\alpha, \beta) \). Then

\[ a_n \leq \frac{(1 - \alpha)(c)_{n-1}}{n[n(1+\beta) - (\alpha + \beta)](a)_{n-1}}. \]

**Remark 2.1**: In view of Theorem 2.2, we can see that if \( f(z) \) is of the form (1.7) and is in the class \( UCT(\alpha, \beta) \) then

\[ a_2 = \frac{(1 - \alpha)(c)}{2(2 + \beta - \alpha)(a)}. \tag{2.3} \]

By fixing the second coefficient, we introduce a new subclass \( UCT_b(\alpha, \beta) \) of \( UCT(\alpha, \beta) \) and obtain the following theorems.

Let \( UCT_b(\alpha, \beta) \) denote the class of functions \( f(z) \) in \( UCT(\alpha, \beta) \) and be of the form

\[ f(z) = z - \frac{b(1-\alpha)(c)}{2(2 + \beta - \alpha)(a)} z^2 - \sum_{n=3}^{\infty} a_n z^n \quad (a_n \geq 0), \quad 0 \leq b \leq 1. \tag{2.4} \]

**Theorem 2.3** Let the function \( f(z) \) defined by (2.4). Then \( f(z) \in UCT_b(\alpha, \beta) \) if and only if

\[ \sum_{n=3}^{\infty} n[n(1+\beta) - (\alpha + \beta)] \frac{(a)_{n-1}}{(c)_{n-1}} a_n \leq (1-b)(1-\alpha) \tag{2.5} \]

\(-1 \leq \alpha < 1, \beta > 0.\)

**Proof**: Substituting

\[ a_2 = \frac{b(1-\alpha)(c)}{2(2 + \beta - \alpha)(a)}, \quad 0 \leq b \leq 1 \]

in (2.2), we obtain

\[ 2(2 + \beta - \alpha) \frac{(a)}{(c)} a_2 + \sum_{n=3}^{\infty} n[n(1+\beta) - (\alpha + \beta)] \times \frac{(a)_{n-1}}{(c)_{n-1}} a_n \leq 1 - \alpha \]
which gives
\[
\sum_{n=3}^{\infty} n[n(1 + \beta) - (\alpha + \beta)\](a)_{n-1}^{(a)_{n-1}a_n \leq (1 - b)(1 - \alpha)}
\]
which is the desired result.

**Corollary 2.2**: Let the function \( f(z) \) defined by (2.4) be in the class \( UCT_b(\alpha, \beta) \). Then
\[
a_n \leq \frac{(1 - b)(1 - \alpha)(c)_{n-1}}{n[n(1 + \beta) - (\alpha + \beta)](a)_{n-1}}, \quad n \geq 3, -1 \leq \alpha < 1, \beta \geq 0.
\]

**Theorem 2.4**: The class \( UCT_b(\alpha, \beta) \) is closed under convex linear combination.

**Proof**: Let the function \( f(z) \) be defined by (2.4) and \( g(z) \) defined by
\[
g(z) = z - \frac{b(1 - \alpha)(c)}{2(2 + b - \alpha)(a)}z^2 - \sum_{n=3}^{\infty} d_n z^n,
\]
where \( d_n \geq 0 \) and \( 0 \leq b \leq 1 \).

Assuming that \( f(z) \) and \( g(z) \) are in the class \( UCT_b(\alpha, \beta) \), it is sufficient to prove that the function \( H(z) \) defined by
\[
H(z) = \lambda f(z) + (1 - \lambda)g(z), \quad 0 \leq \lambda \leq 1
\]
is also in the class \( UCT_b(\alpha, \beta) \).

Since
\[
H(z) = z - \frac{b(1 - \alpha)(c)}{2(2 + \beta - \alpha)(a)}z^2
\]
\[- \sum_{n=3}^{\infty} \{\lambda_n + (1 - \lambda)d_n\} z^n,
\]
a_n \geq 0, d_n \geq 0, 0 \leq b \leq 1, we observe that
\[
\sum_{n=3}^{\infty} n[n(1 + \beta) - (\alpha + \beta)](a)_{n-1}^{(a)_{n-1}\lambda a_n + (1 - \lambda)d_n} \leq (1 - b)(1 - \alpha)
\]
which is, in view of Theorem 2.3, implies that \( H(z) \in UCT_b(\alpha, \beta) \).

This completes the proof of the theorem.

**Theorem 2.5**: Let the functions
\[
f_{j}(z) = z - \frac{b(1 - \alpha)(c)}{2(2 + \beta - \alpha)(a)}z^2 - \sum_{n=3}^{\infty} a_{n,j} z^n,
\]

The desired result.
\[ a_{n,j} \geq 0 \] be in the class \( UCT_b(\alpha, \beta) \) for every \( j \) \((j = 1, 2, \cdots, m)\). Then the function \( F(z) \) defined by

\[ F(z) = \sum_{j=1}^{m} \mu_j f_j(z), \quad (2.12) \]

is also in the class \( UCT_b(\alpha, \beta) \), where

\[ \sum_{j=1}^{\infty} \mu_j = 1. \quad (2.13) \]

**Proof**: Combining the definitions \((2.11)\) and \((2.12)\) further by \((2.13)\) we have

\[ F(z) = z - \frac{b(1 - \alpha)(c)}{2(2 + \beta - \alpha)(a)} \geq 2 - \sum_{n=3}^{\infty} \left( \sum_{j=1}^{m} \mu_j a_{n,j} \right) z^n. \quad (2.14) \]

Since \( f_j(z) \in UCT_n(\alpha, \beta) \) for every \( j = 1, 2, \cdots, m \), Theorem 2.3 yields

\[ \sum_{n=3}^{\infty} n[n(1 + \beta) - (\alpha + \beta)] \frac{(a)^{n-1}}{(c)^{n-1}} a_{n,j} \leq (1 - b)(1 - \alpha). \quad (2.15) \]

Thus we obtain

\[
\begin{align*}
\sum_{n=3}^{\infty} n[n(1 + \beta) - (\alpha + \beta)] \frac{(a)^{n-1}}{(c)^{n-1}} \left( \sum_{j=1}^{m} \mu_j a_{n,j} \right) \\
= \sum_{j=1}^{m} \left( \sum_{n=3}^{\infty} n[n(1 + \beta) - (\alpha + \beta)] \frac{(a)^{n-1}}{(c)^{n-1}} a_{n,j} \right) \\
\leq (1 - b)(1 - \alpha)
\end{align*}
\]

in view of Theorem 2.3. So, \( F(z) \in UCT_b(\alpha, \beta) \).

**Theorem 2.6**: Let

\[ f_2(z) = z - \frac{b(1 - \alpha)(c)}{2(2 + \beta - \alpha)(a)} z^2 \quad (2.16) \]

and

\[ f_n(z) = z - \frac{b(1 - \alpha)(c)}{2(2 + \beta - \alpha)(a)} z^2 - \frac{(1 - b(1 - \alpha)(c))^{n-1}}{n[n(1 + \beta) - (\alpha + \beta)(a)]_{n-1}} z^n \quad (2.17) \]

for \( n = 3, 4, \cdots \). Then \( f(z) \) is in the class \( UCT_b(\alpha, \beta) \) if and only if it can be expressed in the form

\[ f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \quad (2.18) \]
where \( \lambda_n \geq 0 \) and \( \sum_{n=2}^{\infty} \lambda_n = 1 \).

**Proof**: we suppose that \( f(z) \) can be expressed in the form (2.18). Then we have

\[
f(z) = z - \frac{b(1 - \alpha)(c)}{2(2 + \beta - \alpha)(a)} z^2
- \sum_{n=3}^{\infty} \lambda_n \frac{(1 - b)(1 - \alpha)(c)_{n-1}}{n[n(1 + \beta) - (\alpha + \beta)][a]_{n-1}} z^n
= z - \sum_{n=2}^{\infty} A_n z^n, \tag{2.19}
\]

where

\[
A_2 = \frac{b(1 - \alpha)(c)}{2(2 + \beta - \alpha)} \tag{2.20}
\]

\[
A_n = \frac{\lambda_n(1 - b)(1 - \alpha)(c)_{n-1}}{n[n(1 + \beta) - (\alpha + \beta)][a]_{n-1}}, \quad n = 3, 4, \ldots. \tag{2.21}
\]

Therefore,

\[
\sum_{n=2}^{\infty} n[n(1 + \beta) - (\alpha + \beta)](a)_{n-1}^{-1} A_n
= b(1 - \alpha) \sum_{n=3}^{\infty} \lambda_n(1 - b)(1 - \alpha)
= (1 - \alpha)[b + (1 - \lambda_2)(1 - b)]
\leq (1 - \alpha), \tag{2.22}
\]

It follows from Theorem 2.2 and Theorem 2.3 that \( f(z) \) is in the class \( \text{UCT}_{b}(\alpha, \beta) \).

Conversely, we suppose that \( f(z) \) defined by (2.4) is in the class \( \text{UCT}_{b}(\alpha, \beta) \). Then by using (2.6), we get

\[
a_n \leq \frac{(1 - b)(1 - \alpha)(c)_{n-1}}{n[n(1 + \beta) - (\alpha + \beta)](a)_{n-1}}, \quad (n \geq 3). \tag{2.23}
\]

Setting

\[
\lambda_n = \frac{n[n(1 + \beta) - (\alpha + \beta)](a)_{n-1}}{(1 - b)(1 - \alpha)(c)_{n-1}}, \quad (n \geq 3) \tag{2.24}
\]

and

\[
\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n,
\]

we have (2.18). This completes the proof of Theorem 2.6.
Corollary 2.3: The extreme points of the class $UCT_b(\alpha, \beta)$ are functions $f_n(z), n \geq 2$ given by Theorem 2.6.

3. The Class $UCT_{b_n,k}(\alpha, \beta)$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $UCT_{b_n,k}(\alpha, \beta)$ be the class of functions of the form

$$f(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)(c)n-1}{n[n(1+\beta) - (\alpha + \beta)](a)n-1}z^n - \sum_{n=k+1}^{\infty} a_nz^n; \quad (3.1)$$

where $0 \leq \sum_{n=2}^{k} b_n = b \leq 1$. Note that $UCT_{b_2,2}(\alpha, \beta) = UCT_b(\alpha, \beta)$.

Theorem 3.1: The extreme points of the class $UCT_{b_n,k}(\alpha, \beta)$ are

$$f_k(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)(c)n-1}{n[n(1+\beta) - (\alpha + \beta)](a)n-1}z^n$$

and

$$f(n(z) = z - \sum_{n=2}^{\infty} \frac{b_n(1-\alpha)(c)n-1}{n[n(1+\beta) - (\alpha + \beta)](a)n-1}z^n - \sum_{n=k+1}^{\infty} \frac{(1-b)(1-\alpha)(c)n-1}{n[n(1+\beta) - (\alpha + \beta)](a)n-1}z^n.$$  

The details of the proof are omitted, since the characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done for $UCT_b(\alpha, \beta)$.

References


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