Common Fixed Point Theorem by Altering Distance

Involving under a Contractive Condition of Integral Type

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Abstract

A generalization is obtained for some of the fixed point theorems of for a self-map on a metric space, which involve the idea of alteration of distances between points of under a contractive condition of integral type.

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1 Introduction

Definition 1.1 Let $\Psi_n$ denote the set of all the variables
(1) $\psi$ is continuous ;
(2) $\psi$ is monotone increasing in all the variable ;
(3) $\psi(t_1, t_2, t_3, t_4, \ldots, t_n) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = \ldots = $
we Definitional \( \phi(x) = \psi(x, x, x, ...) \) for \( x \in [0, \infty) \). Clear, \( \phi(x) = 0 \) if and only if \( x = 0 \). Examples of \( \psi \) are:

\[
\psi(t_1, t_2, t_3, \ldots, t_n) = k \max \{t_1, t_2, t_3, \ldots, t_n\}, \text{ for } k > 0 \quad (1)
\]

\[
\psi(t_1, t_2, t_3, \ldots, t_n) = t_1^{a_1} + t_2^{a_2} + \ldots + t_n^{a_n}, \quad a_1, a_2, a_3, a_4, \ldots \geq 1 \quad (2)
\]

## 2 main result

Theorem 2.1 Let \((X, d)\) be a complete metric space and \(S, T\), \(X :\to X\) such that:

\[
\int_0^{\phi_1(d(Sx,Ty))} \varphi(t)dt \leq \int_0^{\psi_1(d(x,y), d(Sx,x), d(Ty,y))} \varphi(t)dt - \int_0^{\psi_2(d(x,y), d(Sx,x), d(Ty,y))} \varphi(t)dt \quad (3)
\]

for all \(x, y \in X\) where \(\psi_1, \psi_2 \in \Psi_3\) and \(\phi_1 = \psi(x, x, x)\), \(x \in [0, \infty)\)

\((i)\) where \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) is a lebesgue integrable mapping which is sum able, non negative and such that for each \(\varepsilon > 0\), \(\int_0^\varepsilon \varphi(t)dt > 0\) Then \(S, T\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0 \in X\), be an arbitrary point. for \(n = 1, 2, 3, \ldots\)

\[
x_{2n+1} = Sx_{2n}
\]

\[
x_{2n+2} = Tx_{2n+1}
\]

Let

\[
a_n = d(x_n, x_{n+1})
\]

petting \(x = x_{2n}\) and \(y = x_{2n+1}\) in (1); For all \(n = 1, 2, 3, \ldots\)

We get:

\[
\int_0^{\phi_1(d(Sx_{2n},Tx_{2n+1}))} \varphi(t)dt = \int_0^{\phi_1(d(x_{2n+1},x_{2n}))} \varphi(t)dt \leq
\]

\[
\int_0^{\psi_1(d(x_{2n+1},x_{2n}), d(x_{2n},Sx_{2n}), d(x_{2n+1},Tx_{2n+1}))} \varphi(t)dt - \int_0^{\psi_2(d(x_{2n+1},x_{2n}), d(x_{2n},Sx_{2n}), d(x_{2n+1},Tx_{2n+1}))} \varphi(t)dt
\]

\[
= \int_0^{\phi_1(d(x_{2n+1},x_{2n}), d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}))} \varphi(t)dt - \int_0^{\psi_2(d(x_{2n+1},x_{2n}), d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}))} \varphi(t)dt
\]
Or by (2) for all $n = 1, 2, 3, \ldots$

$$\int_0^{\phi_1(a_{2n+1})} \varphi(t)dt \leq \int_0^{\psi_1(a_{2n}, a_{2n}, a_{2n+1})} \varphi(t)dt - \int_0^{\psi_1(a_{2n}, a_{2n}, a_{2n+1})} \varphi(t)dt \quad (5)$$

If $a_{2n+1} > a_{2n}$ Then:

$$\int_0^{\phi_1(a_{2n+1})} \varphi(t)dt \leq \int_0^{\psi_1(a_{2n+1}, a_{2n+1}, a_{2n+1})} \varphi(t)dt - \int_0^{\phi_1(a_{2n+1})} \varphi(t)dt \quad (6)$$

This is due to the fact that $\psi_1$ is monotone increasing in all variables and $\psi_2(a_{2n}, a_{2n}, a_{2n+1}) \neq 0$ whenever $a_{2n+1} \neq 0$. Thus we arrive at a contradiction, so that

$$a_{2n+1} \leq a_{2n} \quad n = 0, 1, 2, \ldots \quad (7)$$

Putting $x = x_{2n}$ and $y = x_{2n-1}$ in (1) we obtain

$$\int_0^{\phi(a_{2n})} \varphi(t)dt \leq \int_0^{\psi_1(a_{2n-1}, a_{2n-1}, a_{2n})} \varphi(t)dt - \int_0^{\psi_2(a_{2n-1}, a_{2n-1}, a_{2n})} \varphi(t)dt \quad (8)$$

By an identical argument we obtain

$$a_{2n+2} \leq a_{2n+1} \quad (9)$$

Then (5) and (7) we obtain for all $n = 1, 2, 3, \ldots$

$$a_{n+1} \leq a_n \quad (10)$$

From (3) and (6) for all $n = 1, 2, 3, \ldots$ obtain

$$\int_0^{\phi_1(a_{n+1})} \varphi(t)dt \leq \int_0^{\psi_1(a_n)} \varphi(t)dt - \int_0^{\psi_2(a_n)} \varphi(t)dt$$

Then

$$\int_0^{\phi_2(a_{n+1})} \varphi(t)dt \leq \int_0^{\phi_1(a_n)} \varphi(t)dt - \int_0^{\phi_1(a_{n+1})} \varphi(t)dt$$

Summing up in (8) we obtain

$$\sum_{n=0}^{\infty} \int_0^{\phi_2(a_{n+1})} \varphi(t)dt \leq \int_0^{\phi_1(a_0)} \varphi(t)dt < \infty$$

which implies

$$\phi_2(a) \to 0 \quad which \ imply \ that \quad n \to \infty \quad (11)$$
Again from (8), \( \{a_n\} \) is convergent and let \( a_n \to a \) as \( n \to \infty \) Since \( \phi \) is continuous, from (9) we obtain \( \phi_2(a) = 0 \) which implies that \( a = 0 \), that is

\[
a = d(x_{n+1}, x_n) \to 0 \quad \text{as} \quad n \to \infty
\]

(12)

we next prove that \( \{x_n\} \) is a cauchy sequence. In view of (10) it is sufficient to prove that \( \{x_{2r}\}_{r=1}^{\infty} \subset \{x_n\} \) is cauchy sequence. If \( \{x_{2r}\}_{r=1}^{\infty} \) is not cauchy sequence of natural number \( \{2m(k)\}, \{2n(k)\} \) such that

\[
n(k) > m(k), d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon \\
d(x_{2m(k)}, x_{2n(k)-1}) < \epsilon
\]

(13)

Then by (11)

\[
\epsilon < d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2m(k)}, x_{2n(k)-1}) < \epsilon + d(x_{2n(k)}, x_{2n(k)-1})
\]

Making \( k \to \infty \) in the above inequality by virtue of (10) we obtain

\[
\lim_{n \to \infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon
\]

(14)

for all \( k = 1, 2, 3, \ldots \)

\[
d(x_{2n(k)+1}, x_{2m(k)}) \leq d(x_{2n(k)+1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)})
\]

(15)

Also for all \( k = 1, 2, 3, \ldots \)

\[
d(x_{2n(k)}, x_{2m(k)}) \leq d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)})
\]

(16)

Making \( k \to \infty \) in (13) and (14) respectively, by using of (10) and (12) we have

\[
\lim_{k \to \infty} d(x_{2n(k)+1}, x_{2m}) \leq \epsilon
\]

and

\[
\epsilon \leq \lim_{k \to \infty} d(x_{2n(k)+1}, x_{2m})
\]

\[
\lim_{k \to \infty} d(x_{2n(k)+1}, x_{2m(k)}) = \epsilon
\]

(17)

For all \( k = 1, 2, 3, \ldots \)

\[
d(x_{2n(k), x_{2m(k)-1}}) \leq d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)-1})
\]

\[
d(x_{2n(k)}, x_{2m(k)}) \leq d(x_{2n(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)})
\]
Making $k \to \infty$ in the above two inequalities and using (10) and (12) we obtain
\[
\lim_{n \to \infty} d(x_{2n(k)}, x_{2m(k)-1}) = \epsilon \tag{18}
\]
Putting $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$ in (1), for all $k = 1, 2, 3, \ldots$ we obtain
\[
\int_0^{\phi_1(d(x_{2n(k)+1},x_{2m(k)}))} \varphi(t)dt \leq 
\int_0^{\psi_1(d(x_{2n(k)},x_{2m(k)-1}),d(x_{2n(k)},x_{2m(k)+1}),d(x_{2m(k)-1},x_{2m(k)}))} \varphi(t)dt
\]
\[
- \int_0^{\psi_2(d(x_{2n(k)},x_{2m(k)-1}),d(x_{2n(k)},x_{2m(k)+1}),d(x_{2m(k)-1},x_{2m(k)}))} \varphi(t)dt
\]
Making $k \to \infty$ in the above inequality and taking into account the continuity of and , by virtue of (10), (15), (16) we have
\[
\int_0^{\phi_1(\epsilon)} \varphi(t)dt \leq \int_0^{\psi_1(\epsilon,0,0)} \varphi(t)dt - \int_0^{\psi_1(\epsilon,0,0)} \varphi(t)dt
\]
then
\[
\phi_1(\epsilon) \leq \psi_1(\epsilon,0,0) - \psi_2(\epsilon,0,0) < \phi_1(\epsilon)
\]
This is due the fact that $\psi_1$ is monotone increasing in its variables and by property of $\psi_2$ that $\psi(x,y,z) = 0$ if only if $x = y = z = 0$
The above inequality give a contradiction so that $\epsilon = 0$ This establishes sequence and hence convergence in $(X,d)$
Let
\[
x_n \to z \text{ as } n \to \infty \tag{19}
\]
Putting $x = x_{2n}$ and $y = z$ in (1) for all $n = 1, 2, 3, \ldots$
\[
\int_0^{\phi_1(d(x_{2n+1},Tz))} \varphi(t)dt \leq \int_0^{\psi_1(d(x_{2n},z),d(x_{2n},x_{2n+1}),d(z,Tz))} \varphi(t)dt - \int_0^{\psi_2(d(x_{2n},z),d(x_{2n},x_{2n+1}),d(z,Tz))} \varphi(t)dt
\]
Making $n \to \infty$ in the above inequality , by using (10) and (17) and continuity of $\psi_1$ and $\psi_2$ we obtain
\[
\int_0^{d(z,Tz)} \leq \int_0^{\psi_1(0,0,d(z,Tz))} - \int_0^{\psi_2(0,0,d(z,Tz))}
\]
If $d(z,Tz) \neq 0$ then using property that $\psi_1$ and $\psi_2$ are monotone increasing and $\psi_2(x,y,z) = 0$ if and only if $x = y = z = 0$ we obtain
\[
\int_0^{\phi_1(d(z,Tz))} \varphi(t)dt < \int_0^{d(z,Tz)} \varphi(t)dt
\]
which is contradictions. Hence, we obtain

\[ d(z, Tz) = 0, \quad \text{or} \quad z = Tz \quad (20) \]

In an exactly similar way we prove

\[ z = Sz \quad (21) \]

Equations (18) and (19) show that \( z \) is a common fixed point of \( S \) and \( T \).

References


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