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Simplex Codes of Type γ over $F_3 + vF_3$

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Abstract

In this paper, it is constructed simplex linear codes over the ring $F_3 + vF_3$ of type γ, where $v^2 = 1$ and $F_3 = \{0, 1, 2\}$ and obtained the minimum Hamming, Lee and Bachoc weights of this codes.

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1. Introduction

Respectively, in [1] and [2], simplex codes of types α and β over the ring $F_2 + uF_2$ where $u^2 = 0$ and the ring $\sum_{n=0}^{s} u^n F_2$ were given as generalizations and extensions of simplex codes over Z_4 and Z_{2^s}. In [3], it was constructed simplex linear codes over the ring $F_2 + vF_2$ of type α and β, where $v^2 = v$ and $F_2 = \{0, 1\}$. It was also determinated some of their properties. In [6], it was constructed simplex codes over the ring $F_3 + vF_3$ of type α, where $v^2 = 1$ and $F_3 = \{0, 1, 2\}$ and it was obtained the minimum Hamming, Lee and Bachoc weights of this codes.

In this paper, it is constructed simplex codes over the ring $F_3 + vF_3$ of type γ, where $v^2 = 1$ and $F_3 = \{0, 1, 2\}$ and obtained the the minimum Hamming, Lee and Bachoc weights of this codes. .

2. Preliminaries

The alphabet $R = F_3 + vF_3 = \{0, 1, 2, v, 2v, a = 1+v, b = 2+v, c = 1+2v, d = 2+2v\}$ where $v^2 = 1$ and $F_3 = \{0, 1, 2\}$ is a commutative ring with nine elements. The elements $1, 2, v, 2v$ are units. Addition and multiplication operation over R are given in the following tables,

+	0	1	2	v	2v	a	b	c	d
0	0	1	2	v	2v	a	b	c	d
1	1	2	0	a	c	b	v	d	2v
2	2	0	1	b	d	v	a	2v	c
v	v	a	b	2v	0	c	d	1	2
2v	2v	c	d	0	v	1	2	a	b
a	a	b	v	c	1	d	2v	2	0
b	b	v	a	d	2	2v	c	0	1
c	c	d	2v	1	a	2	0	b	v
d	d	2v	c	2	b	0	1	v	a

·	0	1	2	v	2v	a	b	c	d
0	0	0	0	0	0	0	0	0	0
1	0	1	2	v	2v	a	b	c	d
2	0	2	1	2v	v	d	c	b	a
v	0	v	2v	1	2	a	c	b	d
2v	0	2v	v	2	1	d	b	c	a
a	0	a	d	a	d	d	0	0	a
b	0	b	c	c	b	0	b	c	0
c	0	c	b	b	c	0	c	b	0
d	0	d	a	d	a	a	0	0	d

This ring is semi-local ring, it has two maximal ideals $(v-1)$ and $(1+v)$. It can be shown that $R/(v-1)$ and $R/(v+1)$ are isomorphic to F_3. From Chinese Remainder Theorem,

$$R = (v-1) \oplus (v+1)$$

where $(v-1) = \{0, v+2, 1+2v\}$ and $(1+v) = \{0, 1+v, 2+2v\}$ in [4].

In [4], it was shown that,

$$a + vb = (a-b)(v-1) - (a+b)(v+1)$$

for all $a, b \in F_3^n$.

A linear code C of length n over R is an R-submodule R^n. An element of C is called a codeword of C. There are three different weights for codes over R known. Hamming, Lee and

Bachoc weights.

The Hamming weight $wt_H(x)$ of a codeword $x = (x_1, x_2, \ldots, x_n) \in R^n$ is the number of nonzero components. The minimum weight $wt_H(C)$ of a code C is the smallest weight among all its nonzero codewords.

The Lee weight for the codeword $x = (x_1, x_2, \ldots, x_n) \in R^n$ is defined by, $wt_L(x) = \sum_{i=1}^{n} wt_L(x_i)$ where,

$$wt_L(x_i) = \begin{cases} 0 & \text{if} \quad x_i = 0 \\ 1 & \text{if} \quad x_i = 1, 2, v \quad \text{or} \quad 2v \\ 2 & \text{if} \quad x_i = 1+v, 2+v, 1+2v \quad \text{or} \quad 2+2v \end{cases}$$

The Bachoc weight for the codeword $x = (x_1, x_2, \ldots, x_n) \in R^n$ is defined by, $wt_B(x) = \sum_{i=1}^{n} wt_B(x_i)$ where,

$$wt_B(x_i) = \begin{cases} 0 & \text{if} \quad x_i = 0 \\ 1 & \text{if} \quad x_i = 1+v, 2+v, 1+2v \quad \text{or} \quad 2+2v \\ 3 & \text{if} \quad x_i = 1, 2, v \quad \text{or} \quad 2v \end{cases}$$

The minimum Lee weight $wt_L(C)$ and the minimum Bachoc weight $wt_B(C)$ of code C are defined analogously.

For $x = (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in R^n$, $d_H(x, y) = |\{i | x_i \neq y_i\}|$ is called distance between x and $y \in R^n$ is denoted,

$$d_H(x, y) = wt_H(x - y)$$

The minimum Hamming distance between distinct pairs of codewords of a code C is called the minmum distance of C and denoted by $d_H(C)$ or shortly d_H.

The Lee distance and Bachoc distance between x and $y \in R^n$ is defined by,

$$d_L(x, y) = wt_L(x - y) = \sum_{i=1}^{n} wt_L(x_i - y_i)$$

$$d_B(x, y) = wt_B(x - y) = \sum_{i=1}^{n} wt_B(x_i - y_i)$$

respectively.

The minimum Lee and Bachoc distance between distinct pairs of codewords of a code C are called the minimum distance of C and denoted by $d_L(C)$ and $d_B(C)$ or shortly d_L and d_B, respectively.

If C is a linear code, $d_H(C) = wt_H(C)$,$d_L(C) = wt_L(C)$,$d_B(C) = wt_B(C)$.

A generator matrix of C is a matrix whose rows generate C.

Two codes are equivalent if one can be obtained from the other by permuting the coordinates.

The Gray map ϕ from R^n to F_3^{2n} is defined as

$$\phi : R^n \to F_3^{2n}$$

$$x + vy \mapsto (x, y)$$

where $x, y \in F_3^n$. The Lee weight of $x + vy$ is the Hamming weight of its Gray image. Note that ϕ is linear.

By the properties of Chinese Remainder Theorem, any code over R is permutation equivalent to a code generated by the following matrix

$$\begin{pmatrix} I_{k_1} & (1-v)B_1 & (v+1)A_1 & (1+v)A_2 + (1-v)B_2 & (1+v)A_3 + (1-v)B_3 \\ 0 & (1+v)I_{k_2} & 0 & (1+v)A_4 & 0 \\ 0 & 0 & (1-v)I_{k_3} & 0 & (1-v)B_4 \end{pmatrix}$$

where A_i and B_j are ternary matrices. Such a code is said to have rank $\{9^{k_1}, 3^{k_2}, 3^{k_3}\}$. If H is a code over R, let H^+ (resp. H^-) be the ternary code such that $(1+v)H^+$ (resp. $(1-v)H^-$) is read H mod $(1-v)$ (resp. H mod $(1+v)$).

In [4], it is obtained that,

$$H = (1 + v)H^+ \oplus (1 - v)H^-$$

with

$$H^+ = \{s|\exists t \in F_3^n|(1 + v)s + (1 - v)t \in H\}$$
$$H^- = \{t|\exists s \in F_3^n|(1 + v)s + (1 - v)t \in H\}$$

The code H^+ is permutation equivalent to a code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & 0 & 2A_1 & 2A_2 & 2A_3 \\ 0 & I_{k_2} & 0 & A_4 & 0 \end{pmatrix}$$

where A_i are ternary matrices for $i = 1, 2, 3, 4$ and ternary code H^- is permutation equivalent to a code with generator matrix of the form

$$\begin{pmatrix} I_{k_1} & 2B_1 & 0 & 2B_2 & 2B_3 \\ 0 & 0 & I_{k_3} & 0 & B_4 \end{pmatrix}$$

where B_i are ternary matrices for $i = 1, 2, 3, 4$ in [4].

In [6], it was constructed simplex codes over the ring R of type α as in the following

Let G_k^α be a $k \times 3^{2k}$ matrix over R defined inductively by,

$$G_k^\alpha = \left(\begin{array}{c|c|c|c|c|c|c|c|c} 0\dots0 & 1\dots1 & 2\dots2 & v\dots v & 2v\dots2v & a\dots a & b\dots b & c\dots c & d\dots d \\ G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha \end{array} \right)$$

where

$$\mathrm{G}_1^\alpha = \begin{pmatrix} 0 & 1 & 2 & v & 2v & a & b & c & d \end{pmatrix}$$

The columns of G_k^α consist of all distinct k-tuples over R. The code S_k^α generated by G_k^α has length 3^{2k}.

If A_{k-1} denotes the $(9^{k-1} \times 9^{k-1})$ array consisting of all codewords in S_{k-1}^α and $i = (i, i, \dots, i)$ then the $(9^k \times 9^k)$ array of codewords of S_k^α is given by,

$$\begin{pmatrix}
A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} & A_{k-1} \\
A_{k-1} & 1+A_{k-1} & 2+A_{k-1} & v+A_{k-1} & 2v+A_{k-1} & a+A_{k-1} & b+A_{k-1} & c+A_{k-1} & d+A_{k-1} \\
A_{k-1} & 2+A_{k-1} & 1+A_{k-1} & 2v+A_{k-1} & v+A_{k-1} & d+A_{k-1} & c+A_{k-1} & b+A_{k-1} & a+A_{k-1} \\
A_{k-1} & v+A_{k-1} & 2v+A_{k-1} & 1+A_{k-1} & 2+A_{k-1} & a+A_{k-1} & c+A_{k-1} & b+A_{k-1} & d+A_{k-1} \\
A_{k-1} & 2v+A_{k-1} & v+A_{k-1} & 2+A_{k-1} & 1+A_{k-1} & d+A_{k-1} & b+A_{k-1} & c+A_{k-1} & a+A_{k-1} \\
A_{k-1} & a+A_{k-1} & d+A_{k-1} & a+A_{k-1} & d+A_{k-1} & d+A_{k-1} & A_{k-1} & A_{k-1} & a+A_{k-1} \\
A_{k-1} & b+A_{k-1} & c+A_{k-1} & c+A_{k-1} & b+A_{k-1} & A_{k-1} & b+A_{k-1} & c+A_{k-1} & A_{k-1} \\
A_{k-1} & c+A_{k-1} & b+A_{k-1} & b+A_{k-1} & c+A_{k-1} & A_{k-1} & c+A_{k-1} & b+A_{k-1} & A_{k-1} \\
A_{k-1} & d+A_{k-1} & a+A_{k-1} & d+A_{k-1} & a+A_{k-1} & a+A_{k-1} & A_{k-1} & A_{k-1} & d+A_{k-1}
\end{pmatrix}$$

If $R_1, R_2, \dots, R_k$ denote the rows of the matrix G_k^α, then

$$wt_H(R_i) = wt_H(2R_i) = wt_H(vR_i) = wt_H(2vR_i) = 8.3^{2(k-1)}$$

$$wt_H(aR_i) = wt_H(bR_i) = wt_H(cR_i) = wt_H(dR_i) = 6.3^{2(k-1)}$$

$$wt_L(R_i) = wt_L(2R_i) = wt_L(vR_i) = wt_L(2vR_i) = 4.3^{(2k-1)}$$
$$wt_L(aR_i) = wt_L(bR_i) = wt_L(cR_i) = wt_L(dR_i) = 4.3^{(2k-1)}$$
$$wt_B(R_i) = wt_B(2R_i) = wt_B(vR_i) = wt_B(2vR_i) = 16.3^{2(k-1)}$$
$$wt_B(aR_i) = wt_B(bR_i) = wt_B(cR_i) = wt_B(dR_i) = 2.3^{2k-1}$$

in [6].

Let $c = (c_1, \ldots, c_n) \in C$. For each $j \in R$, let $w_i(c) = |\{i|c_i = j\}|$.

Lemma 2.1 Let $c \in S_k^\alpha, c \neq 0$. If for at least one i,c_i is unit, then $\forall j \in R, w_j = 9^{k-1}$, if $\forall i, c_i \in \{0, a, d\}$, then $\forall j \in \{0, a, d\}, w_j = 3^{2k-1}$ in c, if $\forall i, c_i \in \{0, c, b\}$, then $\forall j \in \{0, c, b\}, w_j = 3^{2k-1}$ in c in [6].

Let $A_H(i), A_L(i), A_B(i)$ be the number of codewords Hamming, Lee, Bachoc weight i in the code C respectively, for $i = 1, 2, \ldots, n$. Then $\{A_H(0), A_H(1), \ldots, A_H(n)\}$, $\{A_L(0), A_L(1), \ldots, A_L(n)\}$, $\{A_B(0), A_B(1), \ldots, A_B(n)\}$ is called the Hamming, Lee or Bachoc weight distribution of C respectively.

Hamming , Lee and Bachoc weight distributions of S_k^α are

$$A_H(0) = 1, A_H(8.3^{2(k-1)}) = (3^k - 1)(3^k - 1), A_H(6.3^{2(k-1)}) = 2.(3^k - 1)$$
$$A_L(0) = 1, A_L(4.3^{2k-1}) = (3^k - 1)(3^k - 1), A_L(4.3^{2k-1}) = 2.(3^k - 1)$$
$$A_B(0) = 1, A_B(16.3^{2(k-1)}) = (3^k - 1)(3^k - 1), A_B(2.3^{2k-1}) = 2.(3^k - 1)$$

in [6].

The minimum weights of S_k^α are $d_H = 6.3^{2(k-1)}, d_L = 4.3^{2k-1}$ and $d_B = 2.3^{2k-1}$.

3.Simplex codes of type γ

The length of S_k^α is large and increase fast, so we can omit some columns from G_k^α to obtain good codes over R of smaller length and we will call the simplex codes of type γ.

Let G_k^γ be the $k \times \sum_{n=0}^{k-1} 5^{k-(n+1)} 3^{2n}$ matrix defined inductively by

$$\mathrm{G}_2^\gamma = \begin{pmatrix} 111111111 & | & 0 & | & a & | & b & | & c & | & d \\ 012v2vabcd & | & 1 & | & 1 & | & 1 & | & 1 & | & 1 \end{pmatrix}$$

and for $k > 2$

$$\mathrm{G}_k^\gamma = \begin{pmatrix} 1\ldots1 & | & 0\ldots0 & | & a\ldots a & | & b\ldots b & | & c\ldots c & | & d\ldots d \\ G_{k-1}^\alpha & | & G_{k-1}^\gamma & | & G_{k-1}^\gamma & | & G_{k-1}^\gamma & | & G_{k-1}^\gamma & | & G_{k-1}^\gamma \end{pmatrix}$$

By induction, it is easy to verify that no two columns of G_k^γ are multiple of each other.

Remark 3.1 Each row of G_k^γ has Hamming weight

$$\sum_{n=0}^{k-1} 5^{k-(n+1)}3^{2n} - \sum_{n=0}^{k-2} 5^{(k-1)-(n+1)}3^{2n}$$

Lee weight

$$\sum_{n=0}^{k-1} 5^{k-(n+1)}3^{2n} + 3\sum_{n=0}^{k-2} 5^{(k-1)-(n+1)}3^{2n}$$

and Bachoc weight

$$3^{2k-1} + 2^2\sum_{n=0}^{k-2} 5^{(k-1)-(n+1)}3^{2n}$$

for $k = 2, 3, \ldots$.

Proposition 3.2 Each rows of G_k^γ contains 3^{2k-2} units and $w_a = w_b = w_c = w_d = \sum_{n=0}^{k-1} 5^{k-(n+1)}3^{2n}$.

Proof By induction, for $k = 2$ the result is true. Assume that the result is true for each row of G_{k-1}^γ. The number of units in each row of $G_{k-1}^\gamma$is equal to 3^{2k-4}. From Lemma 2.1, the number of units in any row of G_{k-1}^α is 4.3^{2k-4}. Because $1, 2, v, 2v$ are units. For the total number of units in any row of G_k^γ, we have $4.3^{2k-4} + 5.3^{2k-4} = 3^{2k-2}$. It is similarly shown that for the number of $a's, b's, c's$ and $d's$.

Remark 3.3 Let S_k^γ be the code generated by G_k^γ. The minimum Hamming weight of S_k^γ is

$$3^{2k-2} + 2^{k-2}\sum_{n=0}^{k-2} 5^{(k-1)-(n+1)}3^{2n}$$

the Lee weight of S_k^γ is

$$3^{2k-2} + 2^3\sum_{n=0}^{k-2} 5^{(k-1)-(n+1)}3^{2n}$$

the minimum Bachoc weight of S_k^γ is

$$3^{2k-2} + 2^{k-2}\sum_{n=0}^{k-2} 5^{(k-1)-(n+1)}3^{2n}$$

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