Bounds on the Largest of Minimum Degree Eigenvalues of Graphs

Chandrashekar Adiga and C. S. Shivakumar Swamy

Department of Studies in Mathematics
University of Mysore, Manasagangotri
Mysore 570 006, India
c_adiga@hotmail.com
cskswamy@gmail.com

Abstract

In this paper we establish some upper bounds for the largest of minimum degree eigenvalues and a lower bound for the largest of minimum degree eigenvalues of trees.

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1 Introduction

Let G be a simple graph and let its vertex set be \( V(G) = \{v_1, v_2, ..., v_n\} \). The adjacency matrix \( A(G) \) of the graph G is a square matrix of order n whose \((i, j)\)-entry is equal to unity if the vertices \( v_i \) and \( v_j \) are adjacent, and is equal to zero otherwise. The eigenvalues \( \lambda_1, \lambda_2, ..., \lambda_n \) of \( A(G) \), assumed in non increasing order, are the eigenvalues of the graph G.

The energy of G was first defined by I.Gutman[7] in 1978 as the sum of the absolute values of its eigenvalues:

\[
E(G) = \sum_{i=1}^{n} |\lambda_i|.
\]

Ever since the graph energy \( E(G) \) of a simple graph G was introduced by I.Gutman[7], there is a constant stream of papers devoted to this topic. Survey of development before 2001 can be found in [8]. For recent developments one can consult [3]. The energy of a graph has close links to chemistry(see for
instance [9]). Let $G$ be a simple graph with $n$ vertices $v_1, v_2, \ldots, v_n$ and let $d_i$ be the degree of $v_i$, $i = 1, 2, 3, \ldots, n$. Define

$$d_{ij} = \begin{cases} \min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of $G$. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\phi(G; \mu) = \det(\mu I - m(G)) = \mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \ldots + c_{n-1}\mu + c_n,$$

where $I$ is the unit matrix of order $n$. The minimum degree eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph $G$, assumed in the non increasing order, are the eigenvalues of its minimum degree matrix $m(G)$. The minimum degree energy of a graph $G$ is defined as

$$E_m(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since $m(G)$ is real symmetric matrix with zero trace, these minimum degree eigenvalues are all real with sum equal to zero. In literature there are several upper bounds for the spectral radius $\lambda_1$ of a graph $G$. A.Berman and X.D.Zang [1], R.A.Buraldi and A.J.Hoffman [4], R.C.Brigham and R.D.Dutton [2], R.P.Stanley[10], K.C.Das and Pavan Kumar[6], have obtained upper bounds for Spectral radius $\lambda_1$.

This paper is organized as follows. In Section 2, we derive some upper bounds for $\mu_1$, the largest of minimum degree eigenvalues and in Section 3, we provide a lower bound for the largest of minimum degree eigenvalues of a tree.

2 UPPER BOUND FOR SPECTRAL RADIUS

The largest of minimum degree eigenvalues is called the spectral radius of $G$. In this section we give some upper bounds for the spectral radius $\mu_1$.

**Theorem 2.1.** Let $G$ be a connected graph and $\mu_1$ be the spectral radius of $G$. Then

$$\mu_1 \leq \max_i \left\{ \sum_{j=1}^{n} \left\{ \min(d_i, d_j) \frac{d_j m_j}{d_i m_i} : v_i v_j \in E \right\} \right\},$$

where $d_i$ is the degree of the vertex $v_i$ and $m_i$ is the average of the degrees of the vertices adjacent to $v_i$.

**Proof.** Let us consider the matrix $A(G)^{-1}m(G)A(G)$, where $A(G)$ is the
diagonal matrix with diagonal elements \(d_i m_i, \ i = 1, 2, \ldots, n\).

Now \(A(G)^{-1}m(G)A(G) = [c_{ij}]\), where

\[
c_{ij} = \begin{cases} 
\min(d_i, d_j) \frac{d_j}{d_i}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\
0, & \text{otherwise.}
\end{cases}
\]

If \(\mu_1, \mu_2, \ldots, \mu_n\) are the minimum degree eigenvalues of \(m(G)\) then they are also the minimum degree eigenvalues of \(A(G)^{-1}m(G)A(G)\). Then

\[
\mu_1 \leq \|A(G)^{-1}m(G)A(G)\|
\]

where \(\|\|\) is the maximum absolute row sum norm.

Thus

\[
\mu_1 \leq \max_i \sum_{j=1}^{n} |c_{ij}| = \max_i \left\{ \sum_{j=1}^{n} \left\{ \min(d_i, d_j) \frac{d_j}{d_i} : v_i v_j \in E \right\} \right\}.
\]

**Theorem 2.2.** Let \(G\) be a connected graph and \(\mu_1\) be the spectral radius of \(G\). Then

\[
\mu_1 \leq \max_i \left\{ \sqrt{\frac{1}{d_i} \sum_j d_j^3 m_j} : v_i v_j \in E \right\},
\]

where \(d_i\) is the degree of the vertex \(v_i\), \(m_j\) is the average of the degrees of the vertices adjacent to \(v_j\).

**Proof.** Let us consider the matrix \(D(G)^{-1}m(G)D(G)\), where \(D(G)\) is the diagonal matrix of the vertex degrees. Let \(b_{ij}\) denote the \((i, j)^{th}\) element of \(D(G)^{-1}m(G)D(G)\). Then

\[
b_{ij} = \begin{cases} 
\min(d_i, d_j) \frac{d_j}{d_i}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\
0, & \text{otherwise.}
\end{cases}
\]

Let \(X = (x_1, x_2, \ldots, x_n)^T\) be an eigenvector corresponding to the eigenvalue \(\mu_1\) of \(D(G)^{-1}m(G)D(G)\). We can assume that one of the coordinate (say \(x_i\)) of the eigenvector \(X\) be equal to \(1\) and other coordinates are less than or equal to \(1\), that is, \(x_i = 1\) and \(x_k \leq 1\) for all \(k\) with \(k \neq i\).

We have

\[
\{D(G)^{-1}m(G)D(G)\}X = \mu_1 X.
\]  \hfill (2.1)

From (2.1) we get a system of \(n\) equations whose \(i^{th}\) and \(j^{th}\) equations are

\[
\mu_1 = \sum_j \left\{ \min(d_i, d_j) \frac{d_j x_j}{d_i} : v_i v_j \in E \right\}
\]  \hfill (2.2)
and

$$\mu_1 x_j = \sum_k \left\{ \min(d_k, d_j) \frac{d_k x_k}{d_j} : v_j v_k \in E \right\}. \quad (2.3)$$

Multiplying both sides of (2.3) by $\min(d_i, d_j) \frac{d_i}{d_i}$ and taking summation over $j$ on both sides such that $v_i v_j \in E$, we get

$$\mu_1^2 = \sum_j \left\{ \min(d_i, d_j) \frac{d_i}{d_i} \sum_k \left\{ \min(d_k, d_j) \frac{d_k x_k}{d_j} : v_j v_k \in E \right\} : v_i v_j \in E \right\}. \quad \cdot$$

Since $x_k \leq 1$, we obtain

$$\mu_1^2 \leq \sum_j \left\{ \min(d_i, d_j) \frac{d_i}{d_i} \sum_k \left\{ \min(d_k, d_j) \frac{d_k x_k}{d_j} : v_j v_k \in E \right\} : v_i v_j \in E \right\}$$

$$= \frac{1}{d_i} \sum_j \left\{ \min(d_i, d_j) \sum_k \left\{ \min(d_k, d_j) d_k : v_j v_k \in E \right\} : v_i v_j \in E \right\}.$$ 

Since $\min(d_k, d_j) \leq d_j$ and $\min(d_i, d_j) \leq d_j$, we have

$$\mu_1^2 \leq \frac{1}{d_i} \sum_j \left\{ d_j \sum_k \left\{ d_j d_k : v_j v_k \in E \right\} : v_i v_j \in E \right\}$$

$$= \frac{1}{d_i} \sum_j \left\{ d_j^2 \sum_k \left\{ d_k : v_j v_k \in E \right\} : v_i v_j \in E \right\}$$

$$= \frac{1}{d_i} \sum_j \left\{ d_j^3 m_j : v_i v_j \in E \right\}.$$ 

Thus

$$\mu_1 \leq \sqrt{\frac{1}{d_i} \sum_j d_j^3 m_j : v_i v_j \in E}.$$ 

i.e.,

$$\mu_1 \leq \max_i \left\{ \sqrt{\frac{1}{d_i} \sum_j d_j^3 m_j} : v_i v_j \in E \right\}.$$
In this section, we establish a lower bound for the spectral radius $\mu_1$ of $G$. We first prove three lemmas which are necessary to prove our main result.

**Lemma 3.1.** Let a graph $G$ have some pendent vertices. We separate the pendent vertices into groups such that all the pendent vertices in each group have common neighbor. Let \( \{v_1, v_2, \ldots, v_r\} \) be such a group with neighbor $v_i$, then the first $r$ coordinate of the eigenvectors corresponding to the non zero eigenvalues are equal.

**Proof.** Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $m(G)$ corresponding to a nonzero eigenvalue $\mu$ of $m(G)$. Also let \( \{v_1, v_2, \ldots, v_r\} \) be a group of pendent vertices having common neighbor $v_i$. We have

$$m(G)X = \mu X.$$ 

i.e.,

$$\mu x_k = \sum_j \{\min(d_k, d_j)x_j : v_kv_j \in E\}, \ k = 1, 2, \ldots, n.$$ 

Since $\min(d_k, d_j) = 1$ for $k = 1, 2, \ldots, r$, we have

$$\mu x_k = x_i, \text{ for } k = 1, 2, \ldots, r. \quad (3.1)$$

From (3.1) we conclude that $x_k, k = 1, 2, \ldots, r$ are equal, because $\mu$ is non-zero. As the following lemma is well known we omit the proof.

**Lemma 3.2.** Let $d_1, d_2, \ldots, d_n$ be positive integers and let $x_1, x_2, \ldots, x_n$ be positive real numbers. If $d_i$ increases with $x_i$, then

$$\frac{d_1x_1 + d_2x_2 + \ldots + d_nx_n}{x_1 + x_2 + \ldots + x_n} \geq \frac{d_1 + d_2 + \ldots + d_n}{n}. \quad (3.2)$$

For any two vertices $u$ and $v$ connected by a path in a graph $G$, we define the distance between $u$ and $v$, denoted by $d(u, v)$ to be the length of a shortest $u - v$ path. Let $G$ be a connected graph with vertex set $V$. For each $v \in V$, the eccentricity of $v$ denoted by $e(v)$, defined by

$$e(v) = \max \{d(u, v) : u \in V, u \neq v\}.$$
Lemma 3.3. Let $T$ be a tree with $n$ vertices and suppose there exists a vertex $v_y \in V$ such that $e(v_y) \leq 2$, then

$$\mu_1 \geq \sqrt{\frac{s \sum_{i=1}^{s} \min(d_i, d_y)x_i}{\sum_{i=1}^{s} x_i}} + m - 1,$$

where $d_y$ is the degree of $v_y \in V$ and $m$ is the average of the degrees of the adjacent vertices of $v_y \in V$.

Proof. Let $v_1, v_2, \ldots, v_s$ be the vertices adjacent to $v_y$ having degrees $d_1, d_2, \ldots, d_s$ respectively. Since $e(v_y) \leq 2$, $v_1, v_2, \ldots, v_s$ are adjacent to $(d_1 - 1), (d_2 - 1), \ldots, (d_s - 1)$ pendent vertices respectively. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvectors of $m(G)$ corresponding to a non zero eigenvalue $\mu$ of $m(G)$.

Using the above Lemma 3.1 we obtain,

$$\mu_1 x_k = x_i, \quad i = 1, 2, \ldots, s, \quad k = 1, \ldots, (d_i - 1). \quad (3.3)$$

$$\mu_1 x_y = \sum_{i=1}^{s} \min(d_i, d_y)x_i \quad (3.4)$$

and

$$\mu_1 x_i = x_y + \sum_{k=1}^{d_i - 1} x_k, \quad i = 1, 2, \ldots, s. \quad (3.5)$$

Multiplying by $\mu_1$ on both sides of (3.5) we get

$$\mu_1^2 x_i = \mu_1 x_y + \sum_{k=1}^{d_i - 1} \mu_1 x_k. \quad (3.6)$$

Now, using (3.3) in the above equation we obtain

$$\mu_1 x_y = \mu_1^2 x_i - (d_i - 1)x_i. \quad (3.6)$$

Now taking summation over $i = 1, \ldots, s$ on both sides of (3.6) we get

$$\sum_{i=1}^{s} \mu_1 x_y = \sum_{i=1}^{s} \mu_1^2 x_i - \sum_{i=1}^{s} (d_i - 1)x_i.$$
i.e., \[ s\mu_1 x_y = \mu_1^2 \sum_{i=1}^{s} x_i - \sum_{i=1}^{s} (d_i - 1)x_i. \]

(3.7)

Using (3.4) in (3.7) we deduce that

\[ \mu_1^2 \sum_{i=1}^{s} x_i = s \sum_{i=1}^{s} \min(d_y, d_i)x_i + \sum_{i=1}^{s} d_i x_i - \sum_{i=1}^{s} x_i. \]

i.e., \[ \mu_1^2 = \frac{s \sum_{i=1}^{s} \min(d_i, d_y)x_i}{\sum_{i=1}^{s} x_i} + \frac{\sum_{i=1}^{s} d_i x_i}{\sum_{i=1}^{s} x_i} - 1. \]

Now by Lemma 3.2 we have

\[ \sum_{i=1}^{s} \frac{d_i x_i}{\sum_{i=1}^{s} x_i} \geq \frac{\sum_{i=1}^{s} d_i}{s} = m. \]

Hence

\[ \mu_1 \geq \sqrt{\frac{s \sum_{i=1}^{s} \min(d_i, d_y)x_i}{\sum_{i=1}^{s} x_i} + m - 1}. \]

**Theorem 3.4.** (The Interlacing Theorem) Let \( G \) be an arbitrary graph on \( n \) vertices. Suppose \( H \) is the subgraph of \( G \) obtained by deleting a vertex from \( G \). Let \( \mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_n(G) \) and \( \gamma_1(H) \geq \gamma_2(H) \geq \ldots \geq \gamma_n(H) \) be the minimum degree eigenvalues of \( G \) and \( H \) respectively. Then for \( i = 1, 2, \ldots, n-1 \),

\[ \mu_i(G) \geq \gamma_i(H) \geq \mu_{i+1}(G). \]

This is a well known result and a proof can be found in [5].

**Theorem 3.5.** Let \( T \) be a tree of \( n \) vertices and \( \mu_1(T) \) be the spectral radius
of $m(T)$. Then

$$
\mu_1(T) \geq \sqrt{\frac{\sum_{i=1}^{s} \min(d_i, d_y)x_i}{\sum_{i=1}^{s} x_i} + m - 1},
$$

where $s$ is the highest degree of vertex $v_y \in V$ and $m$ is the average of the degrees of the adjacent vertices of $v_y \in V$.

**Proof.** Using Lemma 3.3 and applying Theorem 3.4 repeatedly, we get

$$
\mu_1(T) \geq \sqrt{\frac{\sum_{i=1}^{s} \min(d_i, d_y)x_i}{\sum_{i=1}^{s} x_i} + m - 1}.
$$

**References**


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