On Strongly Semiclean Rings

Xiaoqing Sun and Yanfeng Luo

Department of Mathematics, Lanzhou University
Lanzhou, Gansu, 730000, P.R. China
sunxq06@lzu.cn

Abstract

Let $R$ be an associative ring with identity. $R$ is called strongly semiclean if every element $r \in R$ can be written as $r = a + u$, where $a$ is periodic, i.e., $a^k = a^l$, $a \in R$ for some positive integers $k$ and $l$ ($k \neq l$) and $u$ is a unit in $R$ such that $au = ua$. Some general properties of semiclean and strongly semiclean rings are given, and strongly semiclean rings of skew Hurwitz series are discussed.

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1 Introduction

Let $R$ be an associative ring with the group of units $U(R)$. $R$ is called clean if for every $r \in R$, $e = e + u$ with $e^2 = e \in R$ and $u \in U(R)$ [7] and $R$ is called strongly clean if in addition, $eu = ue$ [8].

Let $Pri(R)$ denote the set of all periodic elements of $R$. Ye [10] said $R$ is semiclean if for every element $r \in R$, $r = a + u$, where $a$ is periodic, i.e., $a^k = a^l$, $a \in R$ for some positive integers $k$ and $l$ ($k \neq l$) and $u$ is a unit in $R$. Ye has shown that the group ring $\mathbb{Z}_p G$ with $G$ is a cyclic group of order 3 is semiclean for every prime $p$. Matrix extensions of semiclean rings and the relation between semiclean and clean elements are investigated.

A ring $R$ is called strongly semiclean if in addition, $au = ua$. In section 2 and 3, some general properties of semiclean rings and strongly semiclean rings are given. The last section strongly semiclean rings of skew Hurwitz series are explored.

Throughout the paper, rings are associative with identity. We denote by $E(R)$, $M_n(R)$, $(T_n(R))$ the set of idempotent elements of $R$, the matrix (upper triangular matrix) ring of order $n$ over $R$. The symbols $\mathbb{N}$ stands for the set of all positive integers.
2 Basic properties of semiclean rings

An element $r$ of ring $R$ is called semiclean if $r = a + u$, where $a$ is periodic, i.e., $a^k = a'$, $a \in R$ for some positive integers $k$ and $l$ ($k \neq l$) and $u$ is a unit in $R$. Following Ye (2003) [10] a ring $R$ is called a semiclean ring if every element of $R$ is semiclean.

Firstly, some properties of periodic elements are given.

**Lemma 2.1** Let $R$ be a ring. If $F_1$, $F_2$ are periodic matrices over $R$, then the matrix \( \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \) is also periodic.

**Proof.** Since $F_1$, $F_2$ are periodic matrices, there exist $n_1$, $m_1$, $n_2$, $m_2 \in \mathbb{N}(n_1 < m_1, n_2 < m_2)$ such that $F_{m_1}^n = F_{m_1}^{m_1}$ and $F_{m_2}^n = F_{m_2}^{m_2}$. Hence

\[
F_{m_1}^n = F_{m_1}^{m_1} = F_{m_1}^{m_1-n_1+1} = F_{m_1}^{2(m_1-n_1)+n_1} = \cdots = F_{m_1}^{k(m_1-n_1)+n_1},
\]

\[
F_{m_2}^n = F_{m_2}^{m_2} = F_{m_2}^{m_2-n_2+1} = F_{m_2}^{2(m_2-n_2)+n_2} = \cdots = F_{m_2}^{k(m_2-n_2)+n_2},
\]

where $k \in \mathbb{N}$. Thus $F_{m_1}^n = F_{m_1}^{m_1-n_1+1} = F_{m_2}^{k(m_1-n_1)+n_2}$, $F_{m_2}^n = F_{m_2}^{k(m_2-n_2)+n_2}$.

Let $m = (m_1 - n_1)(m_2 - n_2) + n_2$ and $n = n_2$. Then

\[
\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}^n = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}^n.
\]

**Lemma 2.2** Let $R$ be a ring. If $f_i (1 \leq i \leq n)$ is periodic over $R$ and $f_if_j = 0$ if $i \neq j$, then $f = \sum_i f_i$ is also periodic.

**Proof.** Since each $f_i (1 \leq i \leq n)$ is periodic, there exist $k_i$, $l_i \in \mathbb{N} (k_i > l_i)$ such that $f_i^{k_i} = f_i^{l_i}$. It follows from the proof of Lemma 2.1 that $f_i^n = f_i^{l_i+n-l_i} = f_i^{k_i+n-l_i}$. Let $m = \Pi_{i=1}^n (k_i - l_i) + l_n$, $n = l_n$. Then for any $i$, $f_i^n = f_i^n$. Hence $f^n = (\sum_i f_i)^n = \sum_i f_i^n = \sum_i f_i^n = (\sum_i f_i)^n = f^n$.

**Theorem 2.3** The following hold:

1. Every homomorphic image of a semiclean ring is semiclean;
2. A direct product $R = \Pi_{i=1}^n R_i$ of rings $\{R_i\}$ is semiclean if and only if so is each $R_i$.

**Proof.** (1) It follows from Proposition 2.1 [10].

(2) Suppose that each $R_i$ is a semiclean ring $(1 \leq i \leq n)$. Let $x = (x_i) \in \Pi R_i$. For each $i$, write $x_i = a_i + u_i$ where $a_i \in \operatorname{Pri}(R_i)$ and $u_i \in U(R_i)$. Then $u = (u_i) \in U(\Pi R_i)$. Since $a_i \in \operatorname{Pri}(R_i)$, there exist $k_i$, $l_i (k_i > l_i)$ such that $a_i^{k_i} = a_i^{l_i}$. It follows from the proof of Lemma 2.2 that $(a_i)^m = (a_i)^n$ where
Let $S(R)$ be the nonempty set of all proper ideals of $R$ generated by central idempotents. An ideal $P \in S(R)$ is called a Pierce ideal of $R$ if $P$ is a maximal (with respect to inclusion) element of the set $S(R)$. If $P$ is a Pierce ideal of $R$, then the factor ring $R/P$ is called a Pierce stalk of $R$. The next result shows that the semiclean property needs to be checked only for indecomposable rings or Pierce stalks.

**Theorem 2.4** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is semiclean;
2. Every factor ring of $R$ is semiclean;
3. Every indecomposable factor ring of $R$ is semiclean;
4. Every Pierce stalk of $R$ is semiclean.

**Proof.** (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are directly verified.

(3) $\Rightarrow$ (1) Suppose that (3) holds and $R$ is not semiclean. Then there is an element $a \in R$ which is not semiclean. Now let $M$ be the set of all proper ideals $I$ of $R$ such that $a$ is not semiclean in $R/I$. Clearly, $0 \in M$ and the set $M$ is not empty. Define a partial ordering on $M$ by $\subseteq$. If $\{I_\alpha : \alpha \in \Lambda\}$ is a chain in $M$, let $I = \bigcup_{\alpha \in \Lambda} I_\alpha$. We will show that $\bar{a}$ is not semiclean in $R/I$. Suppose that $\bar{a}$ is semiclean in $R/I$. Then there exist $\bar{f}^k = \bar{f}^l \in R/I$ ($k > l$) and $\bar{u} \in U(R/I)$ (with inverse $\bar{v}$) such that $\bar{a} = \bar{f} + \bar{u}$. Note that $f^k - f^l \in I$ and $uv - 1$, $vu - 1 \in I$, so $f^k - f^l \in I_{a_0}$, $uv - 1 \in I_{a_1}$, and $vu - 1 \in I_{a_2}$ for $a_0$, $a_1$, $a_2 \in \Lambda$. Because $\{I_\alpha : \alpha \in \Lambda\}$ is a chain in $M$, there is a maximal $I_s$ in the set $\{I_{a_0}, I_{a_1}, I_{a_2}\}$ such that $I_{a_0}, I_{a_1}, I_{a_2} \subseteq I_s$. That is, $\bar{a}$ is semiclean in $R/I_s$, a contradiction. This implies that $I \in M$ is an upper bound of the chain. Thus $M$ is an inductive set and, by Zorn’s lemma, $M$ has a maximal element $I_0$. If $R/I_0$ is decomposable as a ring and write $R/I_0 \cong R/I_1 \oplus R/I_2$ where both the ideals $I_1$ and $I_2$ strictly contain $I_0$, and so by the choice of $I_0$, $\bar{a}$ is semiclean in $R/I_1$ and $R/I_2$. Thus $\bar{a}$ is semiclean in $R/I_0$, a contradiction. $R/I_0$ is indecomposable and hence is semiclean. It is contradiction with the definition of $I_0$.

(4) $\Rightarrow$ (1) Let $M$ be the set of all proper ideals $I$ of $R$ such that $I$ is generated by central idempotents and the ring $R/I$ is not semiclean. Suppose that $R$ is not semiclean. Then $0 \in M$ and the set $M$ is not empty. It is directly verified as above that the union of every ascending chain of ideals from $M$ belongs to $M$. By Zorn’s Lemma, the set $M$ contains a maximal
element \( P \). By (4), it is sufficient to prove that \( P \) is a Pierce ideal. Assume the contrary. By the definition of the Pierce ideal, there is a central idempotent \( e \) of \( R \) such that \( P + eR \) and \( P + (1 - e)R \) are proper ideals of \( R \) which properly contain the ideal \( P \). Since ideals \( P + eR \) and \( P + (1 - e)R \) do not belong to \( M \) and are generated by central idempotents, \( R/(P + eR) \) and \( R/(P + (1 - e)R) \) are semiclean. Note that \( R/P \cong (R/(P + eR)) \times (R/(P + (1 - e)R)) \). It follows from Theorem 2.3(2) that \( R/P \) is semiclean, a contradiction. Hence \( P \) is a Pierce ideal of \( R \), a contradiction.

**Theorem 2.5** Let \( R \) be a ring, \( m, n \geq 1 \). If the matrix rings \( M_n(R) \) and \( M_m(R) \) are both semiclean, then so is the triangular matrix ring \( T_{n+m}(R) \).

**Proof.** Let \( A \in T_{n+m}(R) \) be a typical \((n + m) \times (n + m)\) triangular matrix which we will write in the block decomposition form \( A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \), where \( A_{11} \in M_n(R) \), \( A_{22} \in M_m(R) \) and \( A_{12} \) is appropriately sized rectangular matrices. By hypothesis, there exist periodic matrices \( F_1, F_2 \) and invertible matrices \( U, V \) in \( M_n(R) \) and \( M_m(R) \) such that \( A_{11} = F_1 + U \) and \( A_{22} = F_2 + V \). Thus the decomposition

\[
\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \begin{pmatrix} F_1 + U & A_{12} \\ 0 & F_2 + V \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} + \begin{pmatrix} U & A_{12} \\ 0 & V \end{pmatrix}.
\]

It follows from Lemma 2.1 that \( \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \) is periodic. Since \( U, V \) are inverse, then \( \begin{pmatrix} U & A_{12} \\ 0 & V \end{pmatrix} \) is inverse. Thus the above decomposition is semiclean decomposition. 

From Theorem 2.3 and Theorem 2.5, the following corollary is obvious.

**Corollary 2.6** The ring \( R \) is a semiclean ring if and only if the triangular matrix ring \( T_n(R) \) is semiclean for any positive integer \( n \).

**Corollary 2.7** If \( R \) is a semiclean ring, then so is the matrix ring \( M_n(R) \) for any positive integer \( n \).

**Proof.** Let \( A \) be a typical \( n \times n \) matrix of \( R \) which we will write in the block decomposition form \( A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \). From Corollary 2.6,

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} + \begin{pmatrix} U & A_{12} \\ 0 & V \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix},
\]
where \( \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \) is periodic and \( \begin{pmatrix} U & A_{12} \\ 0 & V \end{pmatrix} \) is inverse in \( M_n(R) \). It follows from Lemma 4.1 [6] that \( \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \in J(M_n(R)) \). Since \( J(M_n(R)) = \{ A \in M_n(R) | A + B \) is a unit for every unit \( B \in M_n(R) \} \), we can obtain that
\[
\begin{pmatrix} U & A_{12} \\ 0 & V \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \in U(M_n(R))
\]
Thus \( A \) is semiclean.

\[\square\]

3 On strongly semiclean rings

Definition 3.1 An element \( r \) of a ring \( R \) is called strongly semiclean if \( r = a + u \), where \( a \) is periodic, i.e., \( a^k = a^l \) for some positive integers \( k \) and \( l \) \((k \neq l)\) and \( u \) is a unit in \( R \) such that \( au = ua \). A ring \( R \) is called a strongly semiclean ring if every element of \( R \) is strongly semiclean.

It is obvious that strongly clean rings are strongly semiclean. Obviously the inverse is not true. Therefore, the class of strongly clean rings is a proper subset of the class of strongly semiclean rings.

Let \( C(R) \) denote the center of a ring \( R \) and \( g(x) \) be a polynomial in \( C(R)[x] \). Following Camillo and Simón [1], an element \( r \in R \) is called \( g(x)\)-clean if \( r = s + u \) where \( g(s) = 0 \) and \( u \) is a unit of \( R \). A ring \( R \) is \( g(x)\)-clean if every element of \( R \) is \( g(x)\)-clean. The \((x^2 - x)\)-clean rings are precisely the clean rings. It is easy to see that the \((x^k - x^l)\)-clean rings (for some positive integer \( k, l \) and \( k \neq l \)) are the semiclean rings. The \( g(x)\)-clean rings were researched by Fan and Yang in [2]. If \( r = s + u \) also satisfies \( su = us \), then \( r \in R \) is called strongly \( g(x)\)-clean. The strongly \( g(x)\)-clean rings were explored by Fan and Yang in [3].

Let \( Z_p = \{ \frac{m}{n} \in Q | \text{gcd}(p, n) = 1 \text{ and } p \text{ prime} \} \) be the localization of \( Z \) at the prime ideal \( p\mathbb{Z} \) and \( C_3 \) be the cyclic group of order 3.

Example 3.2 Let \( R \) be a commutative local or commutative semiperfect ring with \( 2 \in U(R) \). By the proof of Theorem 2.5 [9], \( RC_3 \) is strongly semiclean. In particular, the group ring \( \mathbb{Z}_pC_3 \) is strongly semiclean for every prime \( p \). But \( \mathbb{Z}_7C_3 \) is not strongly clean.

Proof. Let \( C_3 = \{ 1, a, a^2 \} \) with \( a^3 = 1 \). Then \( \mathbb{Z}_pC_3 = \{ \frac{m_1}{n_1} + \frac{m_2}{n_2}a + \frac{m_3}{n_3}a^2 \mid m_i, n_i \in \mathbb{Z} \text{ and } p \text{ does not divide each } n_i \} \), and \( \mathbb{Z}_pC_3 \) is commutative. It follows from Theorem 3.1 [10] that \( \mathbb{Z}_pC_3 \) is strongly semiclean. Han and Nicholson (2001) [4] have indicated that \( \mathbb{Z}_7C_3 \) is not strongly clean. \[\square\]
**Lemma 3.3** Every periodic element in a ring $R$ is strongly clean.

**Proof.** Let $a$ be a periodic element in $R$, i.e., $a^k = a^l$, $a \in R$ for some positive integers $k$ and $l$ ($k \neq l$). Thus $a^l = a^l \cdot a^{k-l} = a^l \cdot (a^{k-l})^2 = \ldots = a^l \cdot (a^{k-l})^k = a^{l+1} - a^{k-l} \cdot a^l$. Then $a$ is a strongly $\pi$-regular element of $R$. It follows from Theorem 1 [8] that $a$ is strongly clean. \qed

**Theorem 3.4** The following hold:

1. Every homomorphic image of a strongly semiclean ring is strongly semiclean;
2. A direct product $R = \prod R_\alpha$ of rings $\{R_\alpha\}$ is a strongly semiclean ring if and if each $R_\alpha$ is a strongly semiclean ring;
3. If $2$ is invertible in a strongly semiclean ring $R$, then for any element $a \in R$ there exist $f, u, v \in R$ with $f^2 = 1$ and $u, v \in U(R)$ such that $a = f + u + v$ with $fu = uf$ and $(f + u)v = v(f + u)$.

**Proof.** (1), (2) are obvious from Theorem 2.3.

(3) Let $a \in R$. Since $2$ is invertible in $R$, $\frac{a+1}{2}$ is in $R$. $R$ is a strongly semiclean ring implies that $\frac{a+1}{2} = b + u$ where $b \in Pri(R)$ and $u \in U(R)$ and $bu = ub$. It follows from Lemma 3.3 that $b = e + v$ where $e \in E(R), v \in U(R)$ and $ev = ve$. Then $a = (2e - 1) + 2v + 2u$ where $2v, 2u$ are units in $R$ and $2e - 1$ is a square root of $1$ and $(e + v)u = u(e + v)$. Hence

$$
(2e - 1)2v = 4ev - 2v = 4ve - 2v = 2v(2e - 1)
$$

$$
(2e - 1 + 2v)2u = (2e + 2v - 1)2u = 2(e + v)2u - 2u = 4(e + v)u - 2u = 4u(e + v) - 2u = 2u(2e + 2v - 1).
$$

\qed

**Corollary 3.5** If $R$ is strongly semiclean, then for any ideal $I$ of $R$, $R/I$ is strongly semiclean.

**Corollary 3.6** Let $R$ be a ring and $1 < n \in \mathbb{N}$. If $T_n(R)$ is strongly semiclean, then $R$ is strongly semiclean.

**Proof.** Let $A = (a_{ij}) \in T_n(R)$ with $a_{ij} = 0$ and $1 \leq j \leq i \leq n$. Note that $\theta : T_n(R) \to R$ with $\theta(A) = a_{11}$ is a ring epimorphism. \qed

**Corollary 3.7** Let $R$ be a ring. If the formal power series ring $R[[t]]$ is strongly semiclean, then $R$ is strongly semiclean.
**Proof.** This is because \( \theta : R[[t]] \to R \) with \( \theta(f) = a_0 \) is a ring epimorphism where \( f = \sum_{i \geq 0} a_i t^i \in R[[t]] \). \qed

**Theorem 3.8** \( M = P_1 \oplus \cdots \oplus P_n \) for some \( n \geq 1 \) where \( P_i \) is \( \alpha \)-invariant and \( \alpha_{|P_i} \) is strongly semiclean in \( \text{end}(P_i) \) for each \( i \). Then \( \alpha \) is strongly semiclean in \( E = \text{end}(R M) \).

**Proof.** Extend maps \( \lambda_i \) in \( \text{end}(P_i) \) to \( \hat{\lambda}_i \) in \( \text{end}(M) \) by defining \((p_1 + \cdots + p_n)\hat{\lambda}_i = p_i \lambda_i \). Then \( \hat{\lambda}_i \hat{\lambda}_j = 0 \) if \( i \neq j \) while \( \hat{\lambda}_i \mu_i = \lambda_i \hat{\mu}_i \) and \( \hat{\lambda}_i + \hat{\mu}_i = \lambda_i + \mu_i \) for all \( \mu_i \) in \( \text{end}(P_i) \). Now let \( \alpha_{|P_i} = \beta_i + \sigma_i \) in \( \text{end}(P_i) \) where \( \beta_i \) and \( \sigma_i \) are sums of \( p_i \)-linear combinations of \( \lambda_i \) and \( \mu_i \) respectively. If \( \beta \sigma = 0 \), then \( \beta \sigma = 0 \). Hence \( \alpha = \sum_i \hat{\alpha}_{|P_i} = \sum_i (\hat{\beta}_i + \hat{\sigma}_i) = \beta + \sigma \) is the strongly semiclean expression of \( \alpha \) in \( E = \text{end}(R M) \). \qed

**Lemma 3.9** If \( e \in E(R) \) and \( r \in eRe \) is strongly semiclean in \( eRe \), then \( r \) is strongly semiclean in \( R \).

**Proof.** Since \( r \) is strongly semiclean in \( eRe \), there exist a periodic element \( b \) and a unit \( v \) in \( eRe \) such that \( r = b + v \) and \( bv = vb \). It means that \( b^k = b \) for some positive integers \( k \) and \( l \) \((k \neq l)\) and \( uv = vu = e \) for \( w \in eRe \). Then \( u = v - (1 - e) \) is a unit in \( R \) \((with \ w^{-1} = w - (1 - e))\) and \( r - u = b + (1 - e) \). Since \( (1 - e) \) is periodic by Lemma 2.2, and \( ru = (b + v)u = bu + vu = b[v - (1 - e)] + v(v - (1 - e)) = bv - 0 + v^2 - 0 = vb + v^2 = ur \). Hence \( r \) is strongly semiclean in \( R \). \qed

**Theorem 3.10** Let \( e \) is the central idempotent of \( R \). For an element \( x \in R \), \( x \) is strongly semiclean in \( R \) if and only if \( ex \) is strongly semiclean in \( eRe \) and \( (1 - e)x \) is strongly semiclean in \( (1 - e)R(1 - e) \).

**Proof.** \( \Rightarrow \) Since \( x \) is strongly semiclean in \( R \), then \( x = a + u \) where \( a \in \text{Pri}(R) \), \( u \in U(R) \) and \( au = ua \). It is easy to say that \( ea \in \text{Pri}(eRe) \), \( eu \in U(eRe) \) and \( (ea)(eu) = (eu)(ea) \). Thus \( ex = ea + eu \) is strongly semiclean. Similarly, \( (1 - e)x \) is strongly semiclean in \( (1 - e)R(1 - e) \).

\( \Leftarrow \) Since \( ex \) is strongly semiclean in \( eRe \) and \( (1 - e)x \) is strongly semiclean in \( (1 - e)R(1 - e) \), then \( ex = f_1 + u_1 \), \((1 - e)x = f_2 + u_2 \) where \( f_1 \in \text{Pri}(eRe), f_2 \in \text{Pri}((1 - e)R(1 - e)) \), \( u_1 \in U(eRe) \), \( u_2 \in U((1 - e)R(1 - e)) \) and \( f_1 u_1 = u_1 f_1, f_2 u_2 = u_2 f_2 \). Let \( f = f_1 + f_2 \). It follows from \( f_1 f_2 = f_2 f_1 = 0 \) that \( f \) is periodic by Lemma 2.2. Since \( u_1 v_1 = e = v_1 u_1, u_2 v_2 = 1 - e = v_2 u_2 \) and \( u_1 v_2 = 0 = u_2 v_1 \), then \( (u_1 + u_2)(v_1 + v_2) = 1 \). Similarly \( (v_1 + v_2)(u_1 + u_2) = 1 \). Then \( u = u_1 + u_2 \in U(R) \). Furthermore we have \( fu = (f_1 + f_2)(u_1 + u_2) = f_1 u_1 + f_2 u_2 = u_1 f_1 + u_2 f_2 = (u_1 + u_2)(f_1 + f_2) = uf \) and \( f + u = f_1 + f_2 + u_1 + u_2 = ex + (1 - e)x = x \). \qed
Corollary 3.11 Let $e$ is a central idempotent of $R$. $R$ is strongly semiclean if and only if $eRe$ and $(1 - e)R(1 - e)$ are both strongly semiclean rings.

Corollary 3.12 For an element $x \in R$. Let $1 = e_1 + e_2 + \cdots + e_n$ in $R$, $n \geq 1$, where $e_i$ are orthogonal central idempotents. $x$ is strongly semiclean in $R$ if and only if $e_ix$ is strongly semiclean in $e_iRe_i$ for each $i$.

Corollary 3.13 Let $1 = e_1 + e_2 + \cdots + e_n$ in $R$, $n \geq 1$, where $e_i$ are orthogonal central idempotents. $R$ is strongly semiclean if and only if each $e_iRe_i$ is strongly semiclean.

4 strongly semiclean rings of skew Hurwitz series

The ring $T = (HR, \sigma)$ of skew Hurwitz series over a ring $R$ and $\sigma \in Aut(R)$ is defined as follows: the elements of $T = (HR, \sigma)$ are functions $f : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of all natural numbers, the operation of addition in $T = (HR, \sigma)$ is component wise and the operation of multiplication for each $f, g \in T$ is defined by

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\sigma^k(g(n-k))$$

It can be easily shown that $T$ is a ring with identity $h_1$, defined by $h_1(0) = 1$ and $h_1(n) = 0$ for all $n \geq 1$. For a function $f \in T$, we defined $supp(f) = \{n \in N | f(n) = 0\}$ and $\pi(f)$ denotes the minimal element in $supp(f)$. It is clear that $R$ is canonically embedded as a subring of $T$ via $0 \neq r \in R \mapsto h_r \in T$, where $h_r(0) = r$, $h_r(n) = 0$ for every $n \geq 1$ (then $supp(h_r) = 0$).

Hassanein has proved that $T$ is a clean ring if and only if $R$ is in [5]. For semiclean rings we can also get this result.

Lemma 4.1 (Proposition 2.2 [5]) Let $R$ be a ring. Then $f$ is a unit of $T = (HR, \sigma)$ if and only if $f(0)$ is a unit of $R$.

Theorem 4.2 Let $R$ be a ring and $\sigma$ be an automorphism of $R$. Then $T = (HR, \sigma)$ is a semiclean ring if and only if $R$ is.

Proof. $\Leftarrow$ For any $f \in T$, since $R$ is semiclean, there exist $a, u \in R$ such that $f(0) = a + u$ where $a$ is periodic, i.e., $a^k = a^l$, $a \in R$ for some positive integers $k$ and $l$ ($k \neq l$) and $u$ is a unit in $R$. We define an element $g \in T$:

$$g(n) = \begin{cases} f(n), & n > 0, \\ u, & n = 0. \end{cases}$$
It follows from Lemma 4.1 that $g \in U(T)$. Then $f = g + h_a$ with $h_a^k = h_a^l$. Hence, $T = (HR, \sigma)$ is a semiclean ring.

$\Rightarrow$) Let $W = \{ f \in T \mid f(0) = 0 \}$. For any $f \in W$, $g \in T$, $(gf)(0) = g(0)f(0) = 0 = (fg)(0)$. Thus $W$ is an ideal of $T$. Define $\alpha : R \to T/W$ by $\alpha(r) = h_r + W$. It is easy to proof that $\alpha$ is a ring isomorphism. Then $R \simeq T/W$. Since $T = (HR, \sigma)$ is a semiclean ring, $R$ is semiclean by Theorem 2.3[1].

For strongly semiclean rings we have the following:

**Theorem 4.3** Suppose that $R$ is a ring and $\sigma$ an automorphism of $R$. If $R$ is a semiclean ring such that every periodic element $a$ of $R$ is central and $\sigma(a) = a$, then $T$ is a strongly semiclean ring.

**Proof.** Suppose that $R$ is a semiclean ring. It follows from Theorem 4.2 that $T$ is a semiclean ring and every element $f \in T$ can be represented as $f = (f - h_a) + h_a$ where $a$ is a periodic element of $R$. It is sufficient to show that $fh_a = h_a f$.

Since $a$ is central in $R$, we have $(fh_a)(n) = f(n)\sigma^n(h_a(0)) = f(n)\sigma^n(a) = f(n)a = af(n) = h_a(0)f(n) = (h_a f)(n)$ for each $n \in \text{supp}(fh_a)$. Thus $fh_a = h_a f$. □

The following example shows that Theorem 4.3 is not true without the assumption that $\sigma(a) = a$ for all periodic element $a \in R$.

**Example 4.4** Let $R = \mathbb{Z}_p C_3 \oplus \mathbb{Z}_p C_3$, with the usual operations of component-wise addition and multiplication. Then by Example 3.2 $R$ is strongly semiclean. Now let $\sigma : R \to R$ be defined by $\sigma(a,b) = (b,a)$, then $\sigma$ is an automorphism of $R$ and there exist periodic elements in $R$ with $\sigma(a) \neq a$ (e.g., if $a = (0,1) \in R$). We can easily show that $T = (HR, \sigma)$ is not strongly semiclean by direct computations.

**Corollary 4.5** Let $R$ be a ring and $\sigma$ an automorphism of $R$. If $T = (HR, \sigma)$ is a strongly semiclean ring, then $R$ is strongly semiclean.

**Proof.** It follows from the prove of Theorem 4.2 that $R \simeq T/W$. Then $R$ is strongly semiclean by Theorem 3.4. □

**References**


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