**G-Frames and Operator-Valued Frames in Hilbert Spaces**

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**Abstract**

G-frame in Hilbert spaces have been defined by Sun in [14] and operator-valued frames in Hilbert spaces have been defined by Kaftal et al in [10]. We show that tensor product of g-frames is a g-frame, we get some relations between their g-frame operators, and we show that if they are tight, Riesz bases or orthonormal, then so is their tensor product. We study tensor product of operator-valued frames in Hilbert spaces and tensor product of frame representations.

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**1 Introduction**

Frames were first introduced in 1946 by Gabor [7], reintroduced in 1986 by Daubechies, Grossman and Meyer [5], and popularized from then on. Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing, image processing, data compression, sampling theory and many other fields.

Many generalization of frames were introduced, e.g., frames of subspaces [1], [2] pseudo frames[12], oblique frames [4], fusion frames [3]and g-frames in [14]. Tensor product is useful in the approximation of multi-variate functions of combination of univariate ones. Khosravi and Asgari in [11] introduced frames in tensor product of Hilbert spaces. g-frame in Hilbert spaces have been defined...
by Sun in [14] and operator-valued frames in Hilbert spaces have been defined by Kaftal et al in [10]. In this article, we generalize g-frame and operator-valued frame to tensor product of Hilbert spaces.

In section 2 we briefly recall the definitions and basic properties of Hilbert spaces.

In section 3 subsection 3.1 we show also that tensor product of complete, Riesz basis and orthonormal g-frames for Hilbert spaces $H$ and $F$, present complete, Riesz basis and orthonormal g-frames for $H \otimes F$. In subsection 3.2, we briefly recall the definitions and some properties of operator valued frame in Hilbert spaces which were introduced in [10], and we get some results. We show that tensor product of operator-valued frames for Hilbert spaces $H$ and $F$ present operator-valued frame for $H \otimes F$ and we show that tensor product of their analysis operators is an analysis operator. In subsection 3.3, we study the frame representation and we show that tensor product of frame generators is a frame generator.

Throughout this paper, $\mathbb{N}$ and $\mathbb{C}$ will denote the set of natural numbers and the set of complex numbers, respectively. $I$ and $J$ and $I_j$'s will be countable index sets.

## 2 Preliminaries

In this section we briefly recall the definition and basic properties of Hilbert spaces. Throughout this paper, $H, F$ are two Hilbert spaces and $\{V_i : V_i \subseteq F, i \in I\}$ is a sequence of Hilbert spaces, $B(H, F)$ is the collection of all bounded linear operators from $H$ into $F$.

**Definition 2.1** A family $\{\Lambda_i \in B(H, V_i), i \in I\}$ is said to be a g-frame for $H$ with respect to $\{V_i, i \in I\}$, if there are real constants $0 < A \leq B < \infty$, such that for all $x \in H$,

$$A \| x \|^2 \leq \sum_{i \in I} \| \Lambda_i x \|^2 \leq B \| x \|^2 \quad (1).$$

The optimal constants (i.e. maximal for $A$ and minimal for $B$) are called g-frame bounds. The g-frame $\{\Lambda_i \in B(H, V_i), i \in I\}$is said to be a tight g-frame if $A = B$, and said to be Parseval g-frame if $A = B = 1$. The family $\{\Lambda_i \in B(H, V_i), i \in I\}$ is said to be a g-Bessel sequence for $H$ with respect to $\{V_i, i \in I\}$, if the right-hand side inequality of (1) holds.

**Definition 2.2** Let $H$ be a Hilbert space and $(V_i)_{i \in I}$ is a family of Hilbert spaces and let $\Lambda_i \in B(H, V_i)$ for each $i \in I$. Then

(i) If $\{x : \Lambda_i x = 0 \ 0 \in I\} = \{0\}$then we say that $\{\Lambda_i\}_{i \in I}$ is g-complete for Hilbert space $H$;

(ii) If $\{\Lambda_i\}_{i \in I}$ is g-complete and there are positive constants $A$ and $B$ such that for any finite subset $I_1 \subseteq I$ and $y_i \in V_i, i \in I_1$,

$$A \sum_{i \in I_1} \| y_i \|^2 \leq \sum_{i \in I_1} \Lambda^*_i y_i \|y_i\|^2 \leq B \sum_{i \in I_1} \| y_i \|^2,$$

then we say that $\{\Lambda_i\}_{i \in I}$ is a g-Riesz basis for $H$ with respect to $\{V_i\}_{i \in I}$.
(iii) We say \( \{ \Lambda_i \}_{i \in I} \) is a \( g \)-orthonormal basis for \( H \) with respect to \( \{ V_i \}_{i \in I} \) if it satisfies the following conditions.
\[
< \Lambda_i^* y_1, \Lambda_j^* y_2 > = \delta_{i,j} a < y_1, y_2 >, \forall i, j \in I, y_1 \in V_i, y_2 \in V_j \quad \text{and} \quad \sum_{i \in I} \| \Lambda_i x \|^2 = \| x \|^2 \quad \forall x \in H.
\]

**Definition 2.3** Let \( H \) and \( H_0 \) be Hilbert spaces. \( B(H, H_0) \) is the collection of bounded operators from \( H \) to \( H_0 \). A collection \( \{ A_j \}_{j \in J} \) of operators \( A_j \in B(H, H_0) \) indexed by a countable set \( J \) is called an operator-valued frame on \( H \) with range in \( H_0 \), if the series \( S_A = \sum_{j \in J} A_j^* A_j \) converges in the strong operator topology to a positive bounded invertible operator \( S_A \). The frame bounds \( a \) and \( b \) are the largest number \( a > 0 \) and the smallest number \( b > 0 \) for which \( aI \leq S_A \leq bI \). If \( a = b \), i.e., \( S_A = aI \), then the frame is called tight; if \( S_A = I \), the frame is called Parseval.

**Definition 2.4** Given a Hilbert space \( H_0 \) and an index set \( J \), define the partial isometries \( L_j : H_0 \rightarrow l^2(J) \otimes H_0 \) by \( L_j(h) = e_j \otimes h \), where \( \{ e_j \} \) is the standard basis of \( l^2(J) \). Then \( L_j^* L_i = I_0 \) if \( i=j \), \( L_j^* L_i = 0 \) if \( i \neq j \) and \( \sum_j L_j^* L_j = I \otimes I_0 \) where \( I \) denotes the identity operator on \( l^2(J) \) and \( I_0 \) denotes the identity operator on \( H_0 \). For operator-valued frame \( \{ A_j \}_{j \in J} \) on Hilbert space \( H \) with range in \( H_0 \), the series \( \sum_j L_j A_j \) converges in the strong operator topology to an operator \( \theta_A \in B(H, l^2(J) \otimes H_0) \) and \( S_A = \theta_A^* \theta_A \). Operator \( \theta_A \in B(H, l^2(J) \otimes H_0) \) is called the analysis operator and the projection \( P_A = \theta_A S_A^{-1} \theta_A^* \in B(l^2(J) \otimes H_0) \) is called the frame projection of \( \{ A_j \}_{j \in J} \).

Now we recall some definitions from [9] Tensor product \( K \) of two Hilbert spaces \( H \) and \( F \) is characterized (Up to isomorphism) by the existence of a bilinear mapping \( p \), from the Cartesian product \( H \times F \) into \( K' \), with following property:each suitable bilinear mapping \( L \) from \( H \times F \) into a Hilbert space \( K' \) has a unique factorization \( L = TP \), with \( T \) a bounded linear operator from \( K \) into \( K' \). Proposition 2.6.6 [9] asserts, in effect, that the only finite families of simple tensors that have sum zero are those that are forced to have zero sum by the bilinearity of the mapping \( p : (x, y) \rightarrow x \otimes y \). From this , \( K_0 \) can be identified with the algebraic tensor product of \( H \) and \( F \), which was defined, traditionally, as the quotient of the linear space of all formal finite sums of simple tensors by the subspace consisting of those finite sums that must vanish if \( p \) is to be bilinear. \( K_0 \) has the universal property that characterizes the algebraic tensor product. We can identify \( K \) with the completion of its everywhere-dense subspace \( K_0 \). Accordingly, the Hilbert tensor product \( H \otimes F \) can be viewed as the completion of the algebraic tensor product \( K_0 \), relative to the unique inner product on \( K_0 \) that satisfies
\[
< x_1 \otimes y_1, x_2 \otimes y_2 > = < x_1, x_2 > < y_1, y_2 > \quad \text{for} \quad (x_1, x_2 \in H, y_1, y_2 \in F).
\]
For more details see [9].
3 Frames in Hilbert spaces

Throughout this section $V, W, H$ and $F$ are Hilbert spaces. $\{V_i : V_i \subseteq V\}_{i \in I}$ and $\{W_j : W_j \subseteq W\}_{j \in J}$ are family of Hilbert spaces. $B(H, F)$ is the collection of all bounded linear operators from $H$ into $F$.

3.1 g-frames in Hilbert spaces

Lemma 3.1 Let $\{\Lambda_j\}_{j \in J}$ be a g-frame (g-complete, g-Riesz basis) in Hilbert space $H$ with respect to $\{V_j\}_{j \in J}$ with g-frame operator $S$, let $F$ be a Hilbert space and $Q \in B(H, F)$ be invertible. Then $\{\Lambda_jQ^*\}_{j \in J}$ is a g-frame(g-complete, g-Riesz basis) for $F$ with respect to $\{V_j\}_{j \in J}$ with g-frame operator $QSQ^*$. Moreover if $Q$ is unitary and $\{\Lambda_j\}_{j \in J}$ is a g-orthonormal basis, then $\{\Lambda_jQ^*\}_{j \in J}$ is a g-orthonormal basis.

proof. Since $Q$ is invertible and $Q \in B(H, F)$, then it has an invertible adjoint $Q^* \in B(F, H)$. Then by Theorem 3.2 of [12], $\{\Lambda_jQ^*\}_{j \in J}$ is a g-frame with g-frame operator $QSQ^*$. If $\{\Lambda_j\}_{j \in J}$ is g-complete, then $\{f : \Lambda_jQ^*f = 0, j \in J\} = \{f : Q^*f = 0\} = \{0\}$, because $Q^*$ is one-to-one. Hence $\{\Lambda_jQ^*\}_{j \in J}$ is g-complete. If $\{\Lambda_j\}_{j \in J}$ is a g-Riesz basis with bounds $A$ and $B$, then for every finite subset $I_1$ of $J$ we have

$$A \sum_{j \in I_1} \| g_j \|^2 \leq \| \sum_{j \in I_1} \Lambda_j^*g_j \|^2 \leq B \sum_{j \in I_1} \| g_j \|^2.$$  So

$$\frac{A}{\|Q\|^2} \sum_{j \in I_1} \| g_j \|^2 \leq \frac{1}{\|Q\|^2} \| \sum_{j \in I_1} \Lambda_j^*g_j \|^2 \leq \| \sum_{j \in I_1} \Lambda_j^*g_j \|^2 \leq B \| Q \|^2 \sum_{j \in I_1} \| g_j \|^2.$$  For the moreover statement, for each $j_1, j_2 \in J$ and $x_{j_1}, x_{j_2} \in V_{j_1}, x_{j_2} \in V_{j_2}$ we have

$$< QA_{j_1}^*x_{j_1}, QA_{j_2}^*x_{j_2} > = < QQ^*A_{j_1}^*x_{j_1}, A_{j_2}^*x_{j_2} > = < A_{j_1}^*x_{j_1}, A_{j_2}^*x_{j_2} > = \delta_{j_1j_2} < x_{j_1}, x_{j_2} >$$  and for each

$$f \in H, \sum_{j \in J} \| \Lambda_jQ^*f \|^2 = \| Q^*f \|^2 = \| f \|^2.$$  Theorem 3.2 Let $H, F, W$ and $V$ be Hilbert spaces and $\{V_j\}_{j \in J} \subseteq V$, $\{W_i\}_{i \in I} \subseteq W$ be sequences of closed Hilbert subspaces of $V$ and $W$, respectively. Let $\{\Lambda_j\}_{j \in J}$ be a g-orthonormal basis (resp. g-complete, g-Riesz basis) for $H$ with respect to $\{V_j\}_{j \in J}$ and $\{\Gamma_i\}_{i \in I}$ be a g-orthonormal basis (resp. g-complete, g-Riesz basis) for $F$ with respect to $\{W_i\}_{i \in I}$. Then $\{\Lambda_j \otimes \Gamma_i\}_{i \in I, j \in J}$ is a g-orthonormal basis (resp. g-complete, g-Riesz basis) for Hilbert space $H \otimes F$ with respect to $\{V_j \otimes W_i\}_{i \in I, j \in J}$.
proof. We know that \((\Lambda_j \otimes \Gamma_i)^* = (\Lambda_j^* \otimes \Gamma_i^*)\). Let \(y_{j1} \in V_{j1}, y_{j2} \in V_{j2}\) and \(x_i \in W_{i1}, x_j \in W_{i2}\). Then we have
\[
< \Lambda_{j1}^* y_{j1}, \Lambda_{j2}^* y_{j2} > = \delta_{j1, j2} < y_{j1}, y_{j2} >
\]
and for each \(y \in H\), \(\sum_j < \Lambda_j y, \Lambda_j y > = < y, y >\). Also
\[
< \Gamma_{i1}^* x_i, \Gamma_{i2}^* x_j > = \delta_{i1, i2} < x_i, x_j >
\]
and for each \(x \in F\), \(\sum_i < \Gamma_i x, \Gamma_i x > = < x, x >\).

Therefore
\[
< \Lambda_{j1}^* \otimes \Gamma_{i1}^* (y_{j1} \otimes x_i), \Lambda_{j2}^* \otimes \Gamma_{i2}^* (y_{j2} \otimes x_j) >
\]
\[
= < \Lambda_{j1}^* y_{j1} \otimes \Gamma_{i1}^* x_i, \Lambda_{j2}^* y_{j2} \otimes \Gamma_{i2}^* x_j >
\]
\[
= < \Lambda_{j1}^* y_{j1}, \Lambda_{j2}^* y_{j2} > \otimes < \Gamma_{i1} x_i, \Gamma_{i2} x_j >
\]
\[
= \delta_{j1, j2} \delta_{i1, i2} < y_{j1}, y_{j2} > \otimes < x_i, x_j >
\]
\[
= \delta_{j1, j2} \delta_{i1, i2} < y_{j1} \otimes x_i, y_{j2} \otimes x_j > .
\]

\[
\sum_{i,j} < \Lambda_j \otimes \Gamma_i (x \otimes y), \Lambda_j \otimes \Gamma_i (x \otimes y) = \sum_{i,j} < \Lambda_j x \otimes \Gamma_i y, \Lambda_j x \otimes \Gamma_i y >
\]
\[
= \sum_{i,j} < \Lambda_j x, \Lambda_j x > \otimes < \Gamma_i y, \Gamma_i y > = \sum_j < \Lambda_j x, \Lambda_j x > \otimes \sum_i < \Gamma_i y, \Gamma_i y >
\]
\[
= < x, x > \otimes < y, y > = < x \otimes y, x \otimes y > .
\]

Hence relations (iii) hold and therefore \(\{\Lambda_j \otimes \Gamma_i\}_{j \in J, i \in I}\) is a g-orthonormal basis for \(H \otimes F\).

If \(\{\Lambda_j\}_{j \in J}\) and \(\{\Gamma_i\}_{i \in I}\) are g-complete, then we have \(\{x : \Lambda_j x = 0\} = \{0\}\) and \(\{y : \Gamma_i y = 0\} = \{0\}\). Since \(\|\Lambda_j x \otimes \Gamma_i y\| = \|\Lambda_j x \otimes \Gamma_i y\| = 0\) then \(\Lambda_j x = 0\) or \(\Gamma_i y = 0\) so \(\{x \otimes y : (\Lambda_j \otimes \Gamma_i)(x \otimes y) = 0\} = \{x \otimes y : \Lambda_j x \otimes \Gamma_i y = 0\} = \{x : \Lambda_j x = 0\} \bigcup \{y : \Gamma_i y = 0\} = \{0\}\). Hence \(\{\Lambda_j \otimes \Gamma_i\}_{i \in I, j \in J}\) is g-complete.

If \(\{\Lambda_j\}_{j \in J}\) and \(\{\Gamma_i\}_{i \in I}\) are g-Riesz bases, then there exist constants \(0 < A \leq B < \infty\) such that for each finite subset \(J_1\) of \(J\) and \(g_j \in V_j\)
\[
A \sum_{j \in J_1} < g_j, g_j > \leq \sum_{j \in J_1} < \Lambda_j^* g_j, \Lambda_j^* g_j > \leq B \sum_{j \in J_1} < g_j, g_j > .
\]

Also there exist constants \(0 < C \leq D < \infty\) such that for each finite subset \(I_1\) of \(I\) and \(f_i \in W_i\) we have
\[
C \sum_{i \in I_1} < f_i, f_i > \leq \sum_{i \in I_1} < \Gamma_i^* f_i, \Gamma_i^* f_i > \leq D \sum_{i \in I_1} < f_i, f_i > .
\]
Therefore for each finite subset $J_1$ of $J$ and $I_1$ of $I$ and every $g_j \in V_j$ and $f_i \in W_i$ we have

\[\sum_{i,j} \langle g_j \otimes f_i, g_j \otimes f_i \rangle = AC < g_j, g_j \rangle \otimes < f_i, f_i \rangle \leq C \sum_j \Lambda_j^* g_j, \sum_j \Lambda_j^* g_j \rangle \otimes \sum_i < f_i, f_i \rangle \leq < \sum_j \Lambda_j^* g_j, \sum_j \Lambda_j^* g_j \rangle \otimes < \sum_i \Gamma_i^* f_i, \sum_i \Gamma_i^* f_i \rangle \leq B \sum_j < g_j, g_j \rangle \otimes \sum_i < f_i, f_i \rangle \leq BD \sum_{i,j} < g_j \otimes f_i, g_j \otimes f_i \rangle .\]

### 3.2 operator-valued frames in Hilbert spaces

We show that tensor product of operator valued frame is an operator valued frame.

**Theorem 3.3** Let $H$, $F$, $H_0$ and $F_0$ be Hilbert spaces. Let $\{A_j\}_{j \in J}$ be an operator-valued frame on $H$ with range in $H_0$ and $\{B_i\}_{i \in I}$ be an operator-valued frame on $F$ with range in $F_0$. Then $\{A_j \otimes B_i\}_{j \in J, i \in I}$ is an operator-valued frame on $H \otimes F$ with range in $H_0 \otimes F_0$. In particular, if $\{A_j\}_{j \in J}$ and $\{B_i\}_{i \in I}$ are tight or Parseval frames, then so is $\{A_j \otimes B_i\}_{j \in J, i \in I}$.

**Proof.** There is a positive bounded invertible operator $S_A \in B(H)$ such that $\sum_j A_j^* A_j$ converges in strong operator topology to $S_A$, and there is a positive bounded invertible operator $S_B \in B(F)$ such that $\sum_i B_i^* B_i$ converges in strong operator topology to $S_B$. Let $x \in H, y \in F$. Then

\[\sum_{j \in J, i \in I} (A_j \otimes B_i)^* (A_j \otimes B_i)(x \otimes y) = \sum_{j \in J, i \in I} (A_j^* A_j \otimes B_i^* B_i)(x \otimes y) = \sum_{j \in J} A_j^* A_j x \otimes \sum_{i \in I} B_i^* B_i y\]

which converges to $S_A(x) \otimes S_B(y)$ and we have the result. From these it follows that for all $z = \sum_{k=1}^n x_k \otimes y_k$ in $H \otimes F$,

\[\| \sum_{j \in J, i \in I} (A_j \otimes B_i)^* (A_j \otimes B_i)(z) - (S_A \otimes S_B)(z) \| \rightarrow 0\]

This completes the proof.

**Theorem 3.4** Let $\{A_j\}_{j \in J}$ be an operator-valued frame in $B(H, H_0)$ with analysis operator $\theta_A$, $\{B_i\}_{i \in I}$ be an operator-valued frame in $B(F, F_0)$ with analysis operator $\theta_B$. Then $\theta_A \otimes \theta_B$ is the analysis operator of the operator-valued frame $\{A_j \otimes B_i\}_{j \in J, i \in I}$ and $P_{A \otimes B} = P_A \otimes P_B$. 

proof. As we have in definition 2.4, there are partial isometries \{L_j\}_{j \in J} from \(H_0\) to \(l^2(J) \otimes H_0\), and \{K_i\}_{i \in I} from \(F_0\) to \(l^2(I) \otimes F_0\) such that

\[
L_j^* L_s = \begin{cases} 
I_0 & (s = j) \\
0 & (s \neq j)
\end{cases}
\]

\[
\sum_j L_j L_j^* = I \otimes I_0 \text{ and } \theta_A = \sum_j L_j A_j. \text{ Similarly}
\]

\[
K_i^* K_s = \begin{cases} 
I_0 & (s = j) \\
0 & (s \neq j)
\end{cases}
\]

\[
\sum_i K_i K_i^* = I \otimes I_0 \text{ and } \theta_B = \sum_i K_i B_i. \text{ We define partial isometries}
\]

\[
L_j \otimes K_i : H_0 \otimes F_0 \to l^2(J \times I) \otimes H_0 \otimes F_0
\]

by \((L_j \otimes K_i)(h \otimes k) = e_{ij} \otimes (h \otimes k)\), Where \{e_{ij}\} is the standard basis of \(l^2(J \times I)\)

\[
\theta_{A \otimes B} = \sum_j \sum_i (L_j \otimes K_i)(A_j \otimes B_i) = \sum_j \sum_i L_j A_j \otimes K_i B_i
\]

\[
= \sum_j L_j A_j \otimes \sum_i K_i B_i = \theta_A \otimes \theta_B.
\]

We have

\[
P_{A \otimes B} = \theta_{A \otimes B} (S_{A \otimes B})^{-1} (\theta_{A \otimes B})^* = (\theta_A \otimes \theta_B)(S_A \otimes S_B)^{-1} (\theta_A \otimes \theta_B)^*
\]

\[
= \theta_A S_A^{-1} \theta_A^* \otimes \theta_B S_B^{-1} \theta_B^* = P_A \otimes P_B.
\]

Definition 3.5 Let \(\{A_j\}\) and \(\{B_j\}\) be a pair of operator-valued frames on \(H\). Then \(\{B_j\}\) is called a dual of \(\{A_j\}\) if \(\theta_B \theta_A = I\), the operator-valued frame \(\{A_j S_A^{-1}\}\) is called canonical dual frame of \(\{A_j\}\).

Lemma 3.6 Let \(\{A_j : j \in J\}\) and \(\{B_j : j \in J\}\) be a pair of dual operator-valued frames in \(B(H, H_0)\) and \(\{a_k\}, \{b_k\}\) be a pair of dual frames for \(H_0\), respectively. Then \(\{A_j^* a_k\}\) and \(\{B_j^* b_k\}\) are pair of dual frames for \(H\). Moreover, suppose that \(\{A_j : j \in J\}\) and \(\{B_j : j \in J\}\) are canonical dual operator-valued frames, \(\{a_k\}\), \(\{b_k\}\) are canonical dual frames, and that \(\{a_k\}\) is a tight frame with frame bound \(A\). Then \(\{A_j^* a_k\}\) and \(\{B_j^* b_k\}\) are canonical dual frames.

proof. \(\{A_j : j \in J\}\) is an operator-valued frame, so \(\sum_{j \in J} A_j^* A_j = S_A\) with the frame bounds \(a\) and \(b\) for which \(a I \leq S_A \leq b I\). For every \(\eta \in H\), we have

\[
C < A_j \eta, A_j \eta > \leq \sum_k |< A_j \eta, a_k |^2 \leq D < A_j \eta, A_j \eta > .
\]

Consequently

\[
C \sum_j < A_j \eta, A_j \eta > \leq \sum_j \sum_k |< A_j \eta, a_k |^2 \leq D \sum_j < A_j \eta, A_j \eta > .
\]
So

\[ C < S_A \eta, \eta > \leq \sum_j \sum_k |< \eta, A_j^* a_k >|^2 \leq D < S_A \eta, \eta >, \text{ therefore} \]
\[ aC < \eta, \eta > \leq C < S_A \eta, \eta > \leq \sum_j \sum_k |< \eta, A_j^* a_k >|^2 \leq bD < \eta, \eta >. \]

Hence \( \{ A_j^* a_k \} \) is a frame for \( H \) with frame bounds \( aC \) and \( bD \). Similarly we can get that \( \{ B_j^* b_k \} \) is a frame for \( H \). On the other hand, for any \( \eta \in H \), we have

\[ \sum_j \sum_k < \eta, A_j^* a_k > B_j^* b_k = \sum_j B_j^* \sum_k < A_j \eta, a_k > b_k = \sum_j B_j^* A_j \eta = \theta^* \eta \]
\[ = I \eta = \eta \]

Similarly we can get that \( \sum_j \sum_k < \eta, B_j^* b_k > A_j^* a_k = \eta \). Hence \( \{ A_j^* a_k \} \) and \( \{ B_j^* b_k \} \) are dual frames for \( H \). Now we assume that \( \{ A_j \} \) and \( \{ B_j \} \) are canonical dual operator-valued frames and \( \{ a_k \} \) is a tight frame with frame bound \( A \). Then \( b_k = A^{-1} a_k \). Let \( S_{A,a} \) be frame operator associated with \( \{ A_j^* a_k \} \). Then we have

\[ S_{A,a} \eta = \sum_j \sum_k < \eta, A_j^* a_k > A_j^* a_k = \sum_j A_j^* \sum_k < A_j \eta, a_k > a_k \]
\[ = A \sum_j A_j^* A_j \eta = AS_A \eta \]

Hence \( S_{A,a}^{-1} A_j^* a_k = \frac{1}{A} S_{A,a}^{-1} A_j^* a_k = B_j^* b_k \). This completes the proof.

**Lemma 3.7** Let \( \{ A_j \} \) and \( \{ B_j \} \) be a pair of dual operator-valued frames on \( H \), and let \( \{ C_i \} \), \( \{ D_i \} \) be a pair of dual operator-valued frames on \( F \). Then \( \{ A_j \otimes C_i \} \) and \( \{ B_j \otimes D_i \} \) are dual operator-valued frames on \( \{ H \otimes F \} \).

**proof.** Since \( \{ A_j \otimes C_i \} \) is an operator-valued frame on \( \{ H \otimes F \} \) with \( \theta_{A \otimes C} = \theta_A \otimes \theta_C \) and \( \{ B_j \otimes D_i \} \) is an operator-valued frame on \( \{ H \otimes F \} \) with \( \theta_{B \otimes D} = \theta_B^* \otimes \theta_D^* \), then we have \( \theta_{B \otimes D}^* \theta_{A \otimes C} = (\theta_B^* \otimes \theta_D^*) (\theta_A \otimes \theta_C) = \theta_B^* \theta_A \otimes \theta_D^* \theta_C = I_H \otimes I_F \). This completes the proof.

**Definition 3.8** An operator-valued frame \( \{ A_j : j \in J \} \) for which \( P_A = I \otimes I_0 \) is called Riesz frame and an operator-valued frame that is both parseval and Riesz, i.e., such that \( \theta_A^* \theta_A = I \) and \( \theta_A \theta_A^* = I \otimes I_0 \), is called an orthonormal frame.

**Lemma 3.9** Let \( \{ A_j \}_{j \in J} \) be a Riesz frame (orthonormal frame) in \( B(H, H_0) \) and \( \{ B_i \}_{i \in I} \) be a Riesz frame (orthonormal frame) in \( B(F, F_0) \). Then \( \{ A_j \otimes B_i \}_{j \in J, i \in I} \) is a Riesz frame (orthonormal frame) on \( H \otimes F \) with range in \( H_0 \otimes F_0 \).

**proof.** Since \( \{ A_j \}_{j \in J} \), \( \{ B_i \}_{i \in I} \) are Riesz frames, we have \( P_A = I_H \otimes I_{H_0} \) and \( P_B = I_F \otimes I_{F_0} \) so \( P_{A \otimes B} = P_A \otimes P_B = I_{H \otimes F} \otimes I_{H_0 \otimes F_0} \)
3.3 frame representation

First we recall some definitions and properties of representations from [10]. Let \( G \) be a discrete group, not necessarily countable and let \( \lambda \) be the left regular representation of \( G \) (resp. \( \rho \) the right regular representation of \( G \)). Denote by \( L(G) \subseteq B(l^2(G)) \), (resp. \( R(G) \)) the Von Neumann algebra generated by the unitaries \( \{\lambda_g\}_{g \in G} \), (resp., \( \{\rho_g\}_{g \in G} \)). It is well known that both \( L(G)' = R(G) \) and \( R(G)' = L(G) \) are finite Von Neumann algebras that share a faithful trace vector \( \chi_e \), where \( \{\chi_g\}_{g \in G} \) is the standard basis of \( l^2(G) \).

Let \( H_0 \) be a Hilbert space and \( I_0 \) be the identity of \( B(H_0) \). We call \( \lambda \otimes id : g \in G \rightarrow \lambda_g \otimes I_0 \) the left regular representation of \( G \) with multiplicity \( H_0 \).

**Definition 3.10** Let \((G, \pi, H)\) be a unitary representation of the discrete group \( G \) on the Hilbert space \( H \). Then an operator \( A \in B(H, H_0) \) is called a frame generator or (resp. a Parseval frame generator) with range in \( H_0 \) for the representation if \( \{A_g := A\pi_g^{-1}\}_{g \in G} \) is an operator-valued frame. (resp. a Parseval operator-valued frame).

We show that tensor product of representations is a representation.

**Theorem 3.11** Let \((G_1, \pi_1, H)\) and \((G_2, \pi_2, F)\) be unitary representations of discrete groups \( G_1, G_2 \) on Hilbert spaces \( H, F \), with frame generators \( A \) and \( B \), respectively. Then \( \pi_1 \otimes \pi_2 : G = G_1 \oplus G_2 \rightarrow B(H \otimes F) \) is a unitary representation with frame generator \( A \otimes B \). If \( \theta_A \) and \( \theta_B \) are the analysis operators for frame generators \( A \) and \( B \), respectively, then \( \theta_A \otimes \theta_B \) is the analysis operator for \( A \otimes B \).

**Proof.** \( G = G_1 \oplus G_2 \), the direct sum of \( G_1 \) and \( G_2 \), is a discrete group.

We consider the representation \( \pi_1 \otimes \pi_2 : G \rightarrow B(H \otimes F) \), defined by
\[
(\pi_1 \otimes \pi_2)(g, h) = \pi_1 g \otimes \pi_2 h \quad (g, h) \in G
\]

Since \( \{A_g = A\pi_{1g^{-1}} : g \in G_1\} \) is an operator-valued frame for \( H \) and \( \{B_h = B\pi_{2h^{-1}} : h \in G_2\} \) is an operator valued frame for \( F \), by Theorem 3.3. and the definition of \( \pi_1 \otimes \pi_2 \), \( \{(A \otimes B)(\pi_{1g^{-1}} \otimes \pi_{2h^{-1}}) : (g, h) \in G\} \) is an operator valued frame for \( H \otimes F \). Moreover, if \( \theta_A \) and \( \theta_B \) are the analysis operators on \( H \) and \( F \) for frame generators \( A \) and \( B \), respectively, then by Theorem 3.4. \( \theta_A \otimes \theta_B \) is the analysis operator on \( H \otimes F \) for frame generator \( A \otimes B \).

References


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