Congestion with Fuzzy Data;

An Application of DEA

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Abstract

In most applications of Data Envelopment Analysis (DEA), the models presented are designed to obtain a measure of efficiency. Therefore, DEA is a mathematical programming approach that uses the production frontiers to assess relative efficiency. DEA based congestion in input occurs whenever reducing one or some inputs can increase one or some outputs and increasing one or some inputs decreases some outputs without improving other inputs or outputs.

In this paper we provide an extension to the DEA based congestion concept for applying an empirical application with fuzzy data.

Keywords: Data Envelopment Analysis, Efficiency, Benchmark
1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric method for evaluating the relative efficiency of Decision Making Units (DMUs) of multiple inputs and multiple outputs. The original DEA models (Charnes et al (1978), Banker et al. (1984), Charnes et al. (1985)) assume that inputs and outputs are measured by exact values and the value of efficiency be assessed on base relationship between the evaluated unit and its projection point on efficient frontier.

The original DEA-based congestion assumes that input and outputs are measured by exact values on a ratio scale. However, this assumption may not be true, in the sense that some inputs and outputs may be only known as in forms of bounded or fuzzy data. This paper is organized as follows: in section 2 we give a concept of congestion, section 3 introduces fuzzy data also an application example is given in section 4, and in section 5, conclusion is put forward.

2 Preliminaries

The most frequently used DEA model is the CCR model, name after Charnes, Cooper and Rhodes (1978) and BCC model, name after Banker, Charnes, Cooper (1984). Assume \( n \) decision making units \( DMU_j, j \in \{1, \ldots, n\} \) that each using \( m \) inputs produce \( s \) outputs. Also assume that \( X_j = (x_{1j}, \ldots, x_{mj}) \) and \( Y_j = (y_{1j}, \ldots, y_{sj}) \) be inputs and outputs vectors \( DMU_j \), where, \( X_j \geq 0, X_j \neq 0, Y_j \geq 0, Y_j \neq 0 \). The relative efficiency score of the \( DMU_o, o \in \{1, \ldots, n\} \) is obtained from the following model which is called output-oriented \( \epsilon - BCC \) envelopment model.

\[
\begin{align*}
\max & \quad \phi + \epsilon (\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \ldots, m \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \phi y_{ro}, \quad r = 1, \ldots, s \\
& \quad \sum_{j=1}^{n} \lambda_j = 1, \\
& \quad \lambda_j, s_i^-, s_r^+ \geq 0, \quad j = 1, \ldots, n, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s.
\end{align*}
\]

Where, \( \epsilon > 0 \) is a non-Archimedean element defined to be smaller than any positive real number. This means that \( \epsilon \) is not a real number. If \( \phi^* \) be the optimal value of evaluating \( DMU_o \), then \( DMU_o \) is called (strong) efficient if and only if \( \phi^* = 1 \) and all slack variables be zero in all optimal solutions of model (1).
2.1 Congestion

Congestion accrues whenever reducing some inputs can increase outputs, or increasing some inputs can decrease outputs. The first approach due to Fare et al. (FGL, 1985) and the latter approach, due to Cooper et al. (CTT, 1996) was extended by Brockett et al. (BCSW, 1998), to treat tradeoff possibilities between employment and output in Chinese production when congestion is present.

Cooper et al. (2002) introduced the one-model approach for evaluating congestion as follows:

$$\begin{align*}
\max & \quad z = \phi + \epsilon (\sum_{r=1}^{s} s_{r}^{+} - \epsilon \sum_{i=1}^{m} s_{i}^{-c}) \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_{j} x_{ij} + s_{i}^{-c} = x_{io}, \quad i = 1, \ldots, m \\
& \quad \sum_{j=1}^{n} \lambda_{j} y_{rj} - s_{i}^{+} = \phi y_{ro}, \quad r = 1, \ldots, s \\
& \quad \sum_{j=1}^{n} \lambda_{j} = 1, \\
& \quad \lambda_{j}, s_{i}^{-c}, s_{i}^{+} \geq 0, \quad j = 1, \ldots, n, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s.
\end{align*}$$

Note that, in fact, model (2) is composed of three models. Suppose that in evaluating DMU$_{o}$, ($\phi^{\ast}, \lambda^{\ast}, S^{\ast}, S^{-c\ast}$) is an optimal solution of model (2), then $S^{-c\ast}$ is the congestion amount it (Fig. 1.).

3 Fuzzy Linear Programming
3.1 Preliminaries

3.1.1 Definition
An ordered pair of functions \( \tilde{u} = (u(r), \overline{u}(r)) \), \( 0 \leq r \leq 1 \), is called a fuzzy number if and only if it satisfied in the following requirements.
1. \( u(r) \) is a bounded left continues non-decreasing function over \([0,1]\).
2. \( \overline{u}(r) \) is a bounded left continues non-increasing function over \([0,1]\).
3. \( u(r) \) and \( \overline{u}(r) \) are right continues in 0.
4. \( u(r) \leq \overline{u}(r), 0 \leq r \leq 1. \)
A crisp number \( \alpha \) is simply represented by \( u(r) = \overline{u}(r) = \alpha, 0 \leq r \leq 1. \)

3.1.2 Definition
Let \( \tilde{a} = (a^l, a^u, \alpha, \beta) \) denoted the trapezoidal fuzzy number, where \([a^l - \alpha, a^u + \beta]\) is the support of \( \tilde{a} \) and \([a^l, a^u]\) is modal set.

3.1.1 Remark
We denote the set of trapezoidal fuzzy numbers by \( F(R) \). If \( a = a^l = a^u \) then we obtain a triangular fuzzy number, and we show it as follows:
\[
\tilde{a} = (a(r), \overline{a}(r)) = (a + \alpha(r - 1), a + \beta(1 - r))
\]

3.1.1 Theorem
If \( \tilde{a} = (a(r), \overline{a}(r)), \tilde{b} = (b(r), \overline{b}(r)) \) be two fuzzy numbers, \( k \in R \) then:
1. \( \tilde{a} \preceq \tilde{b} \) if and only if \( \tau(\tilde{a}) \geq \tau(\tilde{b}). \)
2. if \( \tilde{a} \succ \tilde{b} \) and only if \( \tau(\tilde{a}) > \tau(\tilde{b}). \)
3. if \( \tilde{a} \asymp \tilde{b} \) and only if \( \tau(\tilde{a}) = \tau(\tilde{b}). \)

Several methods for solving fuzzy linear programming problems have represented, Fang (1999), Lai and Hwang (1992), Maleki et al. (2000). One of the most convenient of these methods is based on the concept of comparison of fuzzy numbers by use of ranking functions. In fact, an efficient approach for ordering the elements of \( F(R) \) is to define a ranking function \( \tau : F(R) \rightarrow R \) which maps each fuzzy number into the line, where a natural order exist. We define orders on \( F(R) \) by

1. \( \tilde{a} \geq \tilde{b} \) if and only if \( \tau(\tilde{a}) \geq \tau(\tilde{b}) \).
2. if \( \tilde{a} \succ \tilde{b} \) and only if \( \tau(\tilde{a}) > \tau(\tilde{b}) \).
3. if \( \tilde{a} \asymp \tilde{b} \) and only if \( \tau(\tilde{a}) = \tau(\tilde{b}) \).
Congestion with fuzzy data; an application of DEA

Where \( \tilde{a} \) and \( \tilde{b} \) are in \( F(R) \). The following lemma is now immediate.

### 3.1.1. Lemma

Let \( \tau \) be any linear ranking function. Then

1. \( \tilde{a} \succeq \tilde{b} \) iff \( \tilde{a} - \tilde{b} \succeq 0 \) iff \( -\tilde{b} \succeq -\tilde{a} \).
2. \( \tilde{a} \succeq \tilde{b} \) and \( \tilde{c} \succeq \tilde{d} \) iff \( \tilde{a} + \tilde{c} \succeq \tilde{b} + \tilde{d} \).

We restrict our attention to linear ranking function \( \tau : \tau(k\tilde{a} + \tilde{b}) = k\tau(\tilde{a}) + \tau(\tilde{b}) \) for any \( \tilde{a} \) and \( \tilde{b} \) belong to \( F(R) \) and any \( k \in R \).

Here, we introduce a linear ranking function similar to the ranking function adopted by Maleki (FJMS, 2002). For a fuzzy number \( \tilde{a} = (a_l, a_u, \alpha, \beta) \), we use ranking function as follows:

\[
\tau(\tilde{a}) = \frac{1}{2} \int_0^1 (\underline{a}(r) + \overline{a}(r)) dr.
\]

This reduces to \( \tau(\tilde{a}) = \frac{1}{2} (a_l + a_u + \frac{1}{2}(\beta - \alpha)) \).

### 3.1.2 Remark

Suppose that \( \tilde{a} = (a_l, a_u, \alpha, \beta) \), \( \tilde{b} = (b_l, b_u, \gamma, \delta) \) be two fuzzy numbers. Then \( \tilde{a} \succeq \tilde{b} \) \( \Leftrightarrow \) \( (a_l + a_u + \frac{1}{2}(\beta - \alpha)) \geq (b_l + b_u + \frac{1}{2}(\delta - \gamma)) \).

### 3.2 Fuzzy Linear programming Problem

Authors who use ranking function for comparison of fuzzy linear programming problems usually define a crisp model which is equivalent to the fuzzy linear programming and then use optimal solution of this model as the optimal solution of fuzzy linear programming problems. We now define fuzzy linear programming problem and the corresponding crisp model.

#### 3.2.1 Definition

A fuzzy linear programming problem (FLP) is as follows:

\[
\begin{align*}
\text{min} & \quad z \simeq cx \\
\text{S.t.} & \quad A \tilde{x} \simeq \tilde{b} \\
& \quad x \geq 0 \\
\end{align*}
\]

Where “ \( \simeq \) ” and “ \( \geq \) ” mean equality and inequality with respect to the ranking function \( \tau \), and \( \tilde{A} = [\tilde{a}_{ij}]_{mn}, \tilde{c} = (\tilde{c}_1, ..., \tilde{c}_n), \tilde{b} = (\tilde{b}_1, ..., \tilde{b}_m), x = (x_1, ..., x_n), \) and \( \tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j \in F(R), x_j \in R. \) for \( i = 1, ..., m, j = 1, ..., n. \)

#### 3.2.2 Definition

Any \( x \) which satisfies the set of constraints of FLP is called a feasible solution. Let \( \tilde{S} \) be the set of all feasible solution of FLP. We say that \( x^* \in \tilde{S} \) is an optimal feasible solution.
for FLP iff $\tilde{c}x^* \preceq \tilde{c}x$ for all $x \in \tilde{S}$.

3.2.3 Definition
We say that the real number $a$ is corresponds to the fuzzy number $\tilde{a}$, with respect to a given linear ranking function $\tau$, if $a = \tau(\tilde{a})$. However, the following theorem shows that any FLP can be converted to a linear programming problem.

3.2.1 Theorem
The following linear programming problem (LP) and the FLP are equivalent:

$$\min \quad z \simeq cx$$
$$\text{S.t.} \quad \tilde{A}x \simeq \tilde{b}$$
$$x \succeq 0$$

$$\min \quad z = cx$$
$$\text{S.t.} \quad \tilde{A}x = \tilde{b}$$
$$x \geq 0$$

Where $a_{ij}$, $b_i$, $c_j$ are real numbers corresponding to the fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}_j$ with respect to a given linear ranking function $\tau$, respectively.

3.3 Fuzzy DEA (FDEA) and congestion with fuzzy data

Data envelopment analysis is a widely applied approach for measuring the relative efficiencies of a set of decision making units which use multiple inputs to produce multiple outputs. When some observations are fuzzy, the efficiencies become fuzzy as well. Fuzzy DEA (FDEA) models take the form of fuzzy linear programming model. The following formulated model with fuzzy coefficients is in output-oriented envelopment form for evaluating congestion:

$$\max \quad z = \phi + \epsilon(\sum_{r=1}^{s} s_r^+ - \epsilon \sum_{i=1}^{m} s_i^- \zeta)$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- \zeta \simeq x_{io}, \quad i = 1, \ldots, m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ \zeta \simeq \phi y_{ro}, \quad r = 1, \ldots, s$$
$$\sum_{j=1}^{n} \lambda_j \simeq \tilde{1},$$
$$\lambda_j, s_i^-, s_r^+ \geq 0, \quad j = 1, \ldots, n, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s.$$

We know the fuzzy model cannot be solved by a crisp model because coefficients in the fuzzy model are fuzzy sets. By considerate the ranking function it is seen that every optimal feasible solution of FDEA model is an optimal feasible solution of DEA model. Hence we consider below model (4), where $x_{ij}$, $y_{rj}$ are real number corresponding to the
fuzzy number $\tilde{x}_{ij}$, $\tilde{y}_{rj}$ in above model (3) with respect to a given linear ranking function $\tau$, respectively.

$$
\begin{align*}
\max \quad & z = \phi + \epsilon \left( \sum_{r=1}^{s} s_{r}^{+} - \epsilon \sum_{i=1}^{m} s_{i}^{-c} \right) \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_{j} x_{ij} + s_{i}^{-c} = x_{io}, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_{j} y_{rj} - s_{r}^{+} = \phi y_{ro}, \quad r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_{j} = 1, \\
& \lambda_{j}, \ s_{i}^{-c}, \ s_{r}^{+} \geq 0, \quad j = 1, \ldots, n, \ i = 1, \ldots, m, \ r = 1, \ldots, s.
\end{align*}
$$

4 An application example

Consider twenty branches of Tehran Social Security Insurance Organization at this section. Each branch uses four inputs in order to produce four outputs. Tables of inputs and outputs are given in below.

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 The number of personals</td>
<td>The total number of insured persons</td>
</tr>
<tr>
<td>2 The total number of computes</td>
<td>The number of insured persons agreements</td>
</tr>
<tr>
<td>3 The area of the branch</td>
<td>The total number of life-pension receivers</td>
</tr>
<tr>
<td>4 A administrative expenses</td>
<td>The receipt total sum (Income)</td>
</tr>
</tbody>
</table>

The set of data are given as follows. The triangular fuzzy inputs and outputs are given in Tables 2, 3 respectively. It is assumed that “M” as number middle, “U” as number up and “L” as number low:

Table 2. The triangular fuzzy inputs for 20 branches of Tehran Social Security Insurance Organization

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 The number of personals</td>
<td>The total number of insured persons</td>
</tr>
<tr>
<td>2 The total number of computes</td>
<td>The number of insured persons agreements</td>
</tr>
<tr>
<td>3 The area of the branch</td>
<td>The total number of life-pension receivers</td>
</tr>
<tr>
<td>4 A administrative expenses</td>
<td>The receipt total sum (Income)</td>
</tr>
<tr>
<td>$DMU_i$</td>
<td>Im1</td>
</tr>
<tr>
<td>--------</td>
<td>-----</td>
</tr>
<tr>
<td>1</td>
<td>98</td>
</tr>
<tr>
<td>2</td>
<td>78</td>
</tr>
<tr>
<td>3</td>
<td>78</td>
</tr>
<tr>
<td>4</td>
<td>92</td>
</tr>
<tr>
<td>5</td>
<td>91</td>
</tr>
<tr>
<td>6</td>
<td>103</td>
</tr>
<tr>
<td>7</td>
<td>96</td>
</tr>
<tr>
<td>8</td>
<td>86</td>
</tr>
<tr>
<td>9</td>
<td>106</td>
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<tr>
<td>10</td>
<td>107</td>
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<td>11</td>
<td>96</td>
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<tr>
<td>12</td>
<td>78</td>
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<tr>
<td>13</td>
<td>105</td>
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<td>14</td>
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<td>80</td>
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<tr>
<td>16</td>
<td>90</td>
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<td>17</td>
<td>90</td>
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<tr>
<td>18</td>
<td>106</td>
</tr>
<tr>
<td>19</td>
<td>100</td>
</tr>
<tr>
<td>20</td>
<td>85</td>
</tr>
</tbody>
</table>

Table 3. The triangular fuzzy outputs for 20 branches of Tehran Social Security Insurance Organization
After using of ranking function $\tau$, the data is given as crisp in Table 4.

Table 4. The crisp inputs and outputs for 20 branches of Tehran Social Security Insurance Organization
Eventually, with using model (4), we obtain the congestion value for inefficient four units 4, 13, 17 and 19 which is given in Table 5.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>I1</th>
<th>I2</th>
<th>I3</th>
<th>I4</th>
<th>O1</th>
<th>O2</th>
<th>O3</th>
<th>O4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>98.2</td>
<td>84.3</td>
<td>4000</td>
<td>61196521</td>
<td>57.02</td>
<td>40.7</td>
<td>1268</td>
<td>174</td>
</tr>
<tr>
<td>2</td>
<td>78.5</td>
<td>95</td>
<td>2565</td>
<td>66287095</td>
<td>3687.5</td>
<td>18.4</td>
<td>8543</td>
<td>313</td>
</tr>
<tr>
<td>3</td>
<td>78.2</td>
<td>88</td>
<td>1344</td>
<td>47612873</td>
<td>3869</td>
<td>20.5</td>
<td>6594</td>
<td>274.8</td>
</tr>
<tr>
<td>4</td>
<td>91.8</td>
<td>94.2</td>
<td>1502</td>
<td>349278137</td>
<td>3593.5</td>
<td>32.6</td>
<td>10516</td>
<td>248.5</td>
</tr>
<tr>
<td>5</td>
<td>90.5</td>
<td>87.3</td>
<td>1682</td>
<td>68356758</td>
<td>5445.9</td>
<td>30.9</td>
<td>9684</td>
<td>221.6</td>
</tr>
<tr>
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<td>103</td>
<td>97.5</td>
<td>3750</td>
<td>75508662</td>
<td>7226.5</td>
<td>11.6</td>
<td>8025</td>
<td>329.5</td>
</tr>
<tr>
<td>7</td>
<td>96</td>
<td>94.3</td>
<td>3312</td>
<td>114264318</td>
<td>3661</td>
<td>101.2</td>
<td>14514</td>
<td>264.6</td>
</tr>
<tr>
<td>8</td>
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<td>95</td>
<td>1501</td>
<td>7495092</td>
<td>4635</td>
<td>17.5</td>
<td>1623</td>
<td>226.8</td>
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<tr>
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<td>105</td>
<td>95</td>
<td>1600</td>
<td>106720451</td>
<td>8605</td>
<td>71.4</td>
<td>10644</td>
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</tr>
<tr>
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<td>97.5</td>
<td>1726</td>
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<td>4724.4</td>
<td>26.6</td>
<td>6824</td>
<td>215.2</td>
</tr>
<tr>
<td>11</td>
<td>96.5</td>
<td>80.5</td>
<td>1925</td>
<td>86613046</td>
<td>3897.8</td>
<td>184.2</td>
<td>12226</td>
<td>178.5</td>
</tr>
<tr>
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<td>78</td>
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<td>4435</td>
<td>69942335</td>
<td>3822.5</td>
<td>21.7</td>
<td>7564</td>
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</tr>
<tr>
<td>13</td>
<td>103.6</td>
<td>104</td>
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<td>7583</td>
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<td>146206908</td>
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<td>25.5</td>
<td>7492</td>
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<td>4953</td>
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<tr>
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<td>95</td>
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<td>8305</td>
<td>23.8</td>
<td>4917</td>
<td>142.8</td>
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<td>101.2</td>
<td>4205</td>
<td>68355245</td>
<td>5100</td>
<td>17.8</td>
<td>1529</td>
<td>212.5</td>
</tr>
<tr>
<td>20</td>
<td>85.8</td>
<td>100</td>
<td>1341</td>
<td>92342642</td>
<td>2963</td>
<td>77.5</td>
<td>14764</td>
<td>305.4</td>
</tr>
</tbody>
</table>

Eventually, with using model (4), we obtain the congestion value for inefficient four units 4, 13, 17 and 19 which is given in Table 5.

Table 5. The results of assessment congestion in input-oriented for inefficient units

<table>
<thead>
<tr>
<th>Units</th>
<th>Objective value</th>
<th>$s_{1-c}$</th>
<th>$s_{2-c}$</th>
<th>$s_{3-c}$</th>
<th>$s_{4-c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.1360</td>
<td>3.8056</td>
<td>0</td>
<td>0</td>
<td>265800662.3</td>
</tr>
<tr>
<td>13</td>
<td>1.1027</td>
<td>47.168</td>
<td>8.5575</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>1.2570</td>
<td>0</td>
<td>8.1795</td>
<td>12041025</td>
<td>23820768.94</td>
</tr>
<tr>
<td>19</td>
<td>1.3785</td>
<td>0</td>
<td>4.5675</td>
<td>1316308</td>
<td>0</td>
</tr>
</tbody>
</table>

5 Conclusion
In this research twenty branches of Tehran Social Security Insurance Organization of Iran were selected. The results show that some of the branches as 4, 13, 17 and 19 had congestion. For example, as showed in Table 5, the unit 4 has congestion in input indexes,
the number of personals and administrative expenses, but it does not congestion in the other indexes, that is, the number of computes and the area of the branch.

References

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