The Category $\text{VRel}(H)$

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Abstract

We introduce the new category $\text{VRel}(H)$ consisting of $H$-fuzzy relation spaces and $H$-fuzzy mappings between them satisfying a certain condition, where the concept of $H$-fuzzy mapping is the modification of one of fuzzy mapping introduced by Demirci[6]. And we investigate $\text{VRel}(H)$ in the sense of a topological universe and show that $\text{VRel}(H)$ is Cartesian closed over $\text{Set}$. Moreover, we construct the category $\text{VFRel}(H)$ consisting of all $H$-fuzzy relational spaces over $H$-fuzzy sets and relation preserving mappings between them, and we find some properties of the category $\text{VFRel}(H)$. And we study the relations between the categories $\text{Rel}(H)$ and $\text{VRel}(H)$.

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1. Introduction

Nel[18] introduced the concept of a topological universe which implies concrete guasitopos[1]. Thus every topological universe satisfies all the conditions

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of a topos except one condition of the subobject classifier. The notion of a topological universe has already been put to effective use in several areas of mathematics\[16,17,19\].

Zadeh\[21\] introduced the notion of a fuzzy relation naturally as a generalization of crisp relations in fuzzy set theory. After that time, Cerruti\[4\] made categories of L-fuzzy relations and studies their some properties. In particular, Hur\[11\] introduced the category \( \text{Rel}(H) \) consisting of \( H \)-fuzzy relations and proved that \( \text{Rel}(H) \) is topological universe and Cartesian closed.

Up to now, almost all the researchers studying the categories of fuzzy relations have used morphisms between fuzzy relations as crisp mappings satisfying a certain condition. Recently, Demirci\[6\] introduced the notion of fuzzy mappings and obtained many results. In particular, Hur et al.\[12\] studied relations between a fuzzy mapping and a fuzzy equivalence relation. Furthermore, also Hur et al.\[13\] investigated the category \( \text{VSet}(H) \) consisting of \( H \)-fuzzy spaces and \( H \)-fuzzy mappings between them satisfying a certain condition. In this paper, we introduce the new category \( \text{VRel}(H) \) consisting of \( H \)-fuzzy spaces and \( H \)-fuzzy mappings between them satisfying a certain condition, where the concept of \( H \)-fuzzy mapping is the modification of one of fuzzy mapping introduced by Demirci\[6\]. And we investigate \( \text{VRel}(H) \) in the sense of a topological universe and show that \( \text{VRel}(H) \) is Cartesian closed over \( \text{Set} \). And we study the relations between the categories \( \text{Rel}(H) \) and \( \text{VRel}(H) \).

2. Preliminaries

In this section, we will introduce some basic definitions and well-known results from\[2,8,15,18,20\] which are needed in the next section.

**Definition 2.1**[15]. Let \( A \) be a concrete category and \( ((Y_\alpha, \xi_\alpha))_\Gamma \) a family of objects in \( A \) indexed by a class \( \Gamma \). For any set \( X \), let \( (f_\alpha : X \to Y_\alpha)_\Gamma \) be a source of maps indexed by \( \Gamma \). An \( A \)-structure \( \xi \) on \( X \) is called *initial with respect to* \( (X, (f_\alpha), ((Y_\alpha, \xi_\alpha))) \) provided that the following conditions hold:

1. For each \( \alpha \in \Gamma \), \( f_\alpha : (X, \xi) \to (Y_\alpha, \xi_\alpha) \) is an \( A \)-morphism.
2. If \( (Z, \rho) \) is an \( A \)-object and \( g : Z \to X \) is a map such that for each \( \alpha \in \Gamma \), the map \( f_\alpha \circ g : (Z, \rho) \to (Y_\alpha, \xi_\alpha) \) is an \( A \)-morphism, then \( g : (Z, \rho) \to (X, \xi) \)
The category \( \text{VRel}(H) \)

is an \( A \)-morphism. In this case, \( (f_{\alpha} : (X, \xi) \rightarrow (Y_{\alpha}, \xi_{\alpha}))_{\Gamma} \) is called an initial source in \( A \).

**Dual notions**: final structure ; final sink.

**Definition 2.2[15]**. A concrete category \( A \) is called topological over \( \text{Set} \) provided that for each set \( X \), for any family \( ((Y_{\alpha}, \xi_{\alpha}))_{\Gamma} \) of \( A \)-objects, and for any source \( (f_{\alpha} : X \rightarrow Y_{\alpha})_{\Gamma} \) of maps, there exists a unique \( A \)-structure \( \xi \) on \( X \) which is initial with respect to \( (X, (f_{\alpha}), ((Y_{\alpha}, \xi_{\alpha}))) \).

**Dual notion**: cotopological category.

**Result 2.A[15, Theorem 1.5]**. A concrete category \( A \) is topological if and only if \( A \) is cotopological.

**Result 2.B[15, Theorem 1.6; 9, Proposition in Section 1]**. Let \( A \) be a topological category over \( \text{Set} \). Then \( A \) is complete and cocomplete.

**Definition 2.3[15]**. Let \( A \) be a concrete category.

1. The \( A \)-fibre of a set \( X \) is the class of all \( A \)-structure on \( X \).
2. \( A \) is called properly fibred over \( \text{Set} \) provided that the following conditions hold:
   - (Fibre-smallness) For each set \( X \), the \( A \)-fibre of \( X \) is a set.
   - (Terminal separator property) For each singleton set \( X \), the \( A \)-fibre of \( X \) has precisely one element.
   - If \( \xi, \eta \) and are \( A \)-structures on a set \( X \) such that \( 1_{X} : (X, \xi) \rightarrow (X, \eta) \) and \( 1_{X} : (X, \eta) \rightarrow (X, \xi) \) are \( A \)-morphisms, then \( \xi = \eta \).

**Definition 2.4[8]**. A category \( A \) is called Cartesian closed provided that the following conditions hold:

1. For any \( A \)-objects \( A \) and \( B \), there exists a product \( A \times B \) in \( A \).
2. Exponential objects exist in \( A \), i.e., for any \( A \)-object \( A \), the functor \( A \times - : A \rightarrow A \) has a right adjoint, i.e., for any \( A \)-object \( B \), there exist an \( A \)-object \( B^{A} \) and a \( A \)-morphism \( e_{A,B} : A \times B^{A} \rightarrow B \) (called the evaluation)
such that for any \( A \)-object \( C \) and any \( A \)-morphism \( f : A \times C \to B \), there exists a unique \( A \)-morphism \( \overline{f} : C \to B^A \) such that the diagram

\[
\begin{array}{ccc}
A \times B^A & \xrightarrow{e_{A,B}} & B \\
\downarrow \exists 1_{A \times f} & & \downarrow f \\
A \times C & \xrightarrow{e} & 
\end{array}
\]

commutes.

**Definition 2.5[18].** A category \( A \) is called a *topological universe over* \( \text{Set} \) provided that the following conditions hold:

1. \( A \) is well-structured over \( \text{Set} \), i.e., (i) \( A \) is a concrete category; (ii) \( A \) has the fibre-smallness condition; (iii) \( A \) has the terminal separator property.
2. \( A \) is cotopological over \( \text{Set} \).
3. Final episinks in \( A \) are preserved by pullbacks, i.e., for any final episink \((g_\lambda : X \to Y)_\Gamma\) and any \( A \)-morphism \( f : W \to Y \), the family \((e_\lambda : U_\lambda \to W)_\Gamma\), obtained by taking the pullback of \( f \) and \( g_\lambda \), for each \( \lambda \), is again a final episink.

**Definition 2.6[20].** A category \( A \) is called a *topos* provided that the following conditions hold:

1. There is a terminal object \( U \) in \( A \).
2. \( A \) has equalizers.
3. \( A \) is Cartesian closed.
4. There is a subobject classifier in \( A \), i.e., there is an object \( \Omega \) and morphism \( v \) from \( U \) to \( \Omega \) such that for each monomorphism \( m \) from \( A' \) to \( A \), there exists a unique morphism \( \emptyset_m \) from \( A \) to \( \Omega \) such that the following diagram is a pullback:

\[
\begin{array}{ccc}
A' & \xrightarrow{m} & U \\
\downarrow \phi_m & & \downarrow v \\
A & \xrightarrow{\phi} & \Omega.
\end{array}
\]
Remark. Let \( A \) be any category with a subobject classifier. If \( f \) is any biomorphism in \( A \), then \( f \) is an isomorphism in \( A \) (cf. [3]).

**Definition 2.7**[2]. A lattice \( H \) is called a complete Heyting algebra, if \( H \) satisfies the following conditions hold:

1. \( H \) is a complete lattice.
2. For any \( a, b \in H \), the set \( \{ x \in H : x \land a \leq b \} \) has a greatest element denoted by \( a \rightarrow b \) (called the relative pseudo-complement of \( a \) in \( b \)), i.e., \( x \land a \leq b \) if and only if \( x \leq (a \rightarrow b) \).

In particular, if \( H \) is a complete Heyting algebra with the least element \( 0 \), then for each \( a \in H \), \( N(a) = a \rightarrow 0 \) is called the negation or the pseudo-complement of \( a \).

Throughout this paper, we will use \( H \) as a complete Heyting algebra with the least element \( 0 \) and the largest element \( 1 \).

3. The category \( \text{VRel}(H) \)

In this section, we introduce the category \( \text{VRel}(H) \) of fuzzy relational spaces and show that it has structures similar to those of \( \text{VSet}(H) \) (see [13]).

**Definition 3.1**[6,13]. A mapping \( E_X : X \times X \rightarrow H \) is called an \( H \)-fuzzy equality on \( X \) if it satisfies the following conditions:

(i) \( E_X(x, y) = 1 \iff x = y \ \forall x, y \in X \),
(ii) \( E_X(x, y) = E_X(y, x) \ \forall x, y \in X \),
(iii) \( E_X(x, y) \land E(y, z) \leq E_X(x, z) \ \forall x, y, z \in X \).

We will denote the set of all \( H \)-fuzzy equalities on \( X \) as \( E_H(X) \).

**Definition 3.2**[6,13]. A \( H \)-fuzzy relation \( f \) on \( X \times Y \) is called an \( H \)-fuzzy mapping w.r.t. \( E_X \in E_H(X) \) and \( E_Y \in E_H(Y) \), denoted by \( f : X \rightarrow Y \), if it satisfies the following conditions:

(i) \( \forall x \in X, \exists y \in Y \) such that \( f(x, y) > 0 \),
(ii) \( \forall x, y \in X, \forall z, w \in y, f(x, z) \land f(y, w) \land E_X(x, y) \leq E_Y(z, w) \).
Definition 3.3[6,13]. The identity $H$-fuzzy mapping $I_X$ on $X$ is a $H$-fuzzy relation on $X \times X$ defined by

$$I_X(x,y) = \begin{cases} 
1, & \text{if } x=y, \\
0, & \text{if } x \neq y, \ \forall x,y \in X.
\end{cases}$$

It is clear that $I_X \in E_H(X)$. Also, if $f : X \rightarrow Y$ is an (ordinary) mapping, then it is an $H$-fuzzy mapping w.r.t. $I_X \in E_H(X)$ and $I_Y \in E_H(Y)$.

Definition 3.4[6,13]. Let $f : X \rightarrow Y$ be an $H$-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$. Then $f$ is said to be:

(i) strong if $\forall x \in X, \exists y \in Y$ such that $f(x,y) = 1$,
(ii) surjective if $\forall y \in Y, \exists x \in X$ such that $f(x,y) > 0$,
(iii) strong surjective if $\forall y \in Y, \exists x \in X$ such that $f(x,y) = 1$,
(iv) injective if $f(x,z) \land f(y,w) \land E_Y(z,w) \leq E_X(x,y)$, $\forall x,y \in X$, $\forall z,w \in Y$,
(v) bijective if it surjective and injective,
(vi) strong bijective if it is strong surjective and injective.

In particular, if $f(x,y) = 1$, then we will write $y = f(x)$. It is clear that $I_X$ is a strong $H$-fuzzy mapping w.r.t. $E_X \in E_H(X)$. Moreover, it is strong bijective w.r.t. $E_X \in E_H(X)$.

Definition 3.5[11]. Let $X$ be a set. $R$ is called an $H$-fuzzy relation (or simply, a fuzzy relation) on $X$ if $\mu_R : X \times X \rightarrow H$ is a mapping. In this case, $(X,R)$ is called an $H$-fuzzy relational space (or, simply, a fuzzy relational space).

Definition 3.6[6,13]. Let $R$ and $S$ be $H$-fuzzy relations on $X \times Y$ and $Y \times Z$, respectively. Then

(i) the sup-min composition of $R$ and $S$, denoted by $S \circ R$, is a $H$-fuzzy relation on $X \times Z$ defined by

$$S \circ R(x,z) = \bigvee_{y \in Y} [R(x,y) \land S(y,z)] \ \forall x \in X, \ \forall z \in Z,$$

(ii) the inverse of $R$, denoted by $R$, is a $H$-fuzzy relation on $Y \times X$ defined by
\[ R^{-1}(y, x) = R(x, y), \forall x \in X, \forall y \in Y. \]

**Result 3.A**[13, Proposition 3.6]. Let \( f : X \to Y \) and \( g : Y \to Z \) be \( H \)-fuzzy mapping w.r.t. \( E_X \in E_H(X) \), \( E_Y \in E_H(Y) \) and \( E_Z \in E_H(Z) \). Then the sup-min composition \( g \circ f \) is an \( H \)-fuzzy mapping \( g \circ f : X \to Z \) w.r.t. \( E_X \in E_H(X) \) and \( E_Z \in E_H(Z) \).

**Result 3.B**[13, Corollary 3.6]. Let \( f : X \to Y \) and \( g : Y \to Z \) be \( H \)-fuzzy mappings w.r.t. \( E_X \in E_H(X) \), \( E_Y \in E_H(Y) \) and \( E_Z \in E_H(Z) \). If \( f \) and \( g \) are strong [resp. surjective, strong surjective, injective, bijective and strong bijective], then \( g \circ f \) is strong [resp. surjective, strong surjective, injective, bijective and strong bijective].

**Definition 3.7**[13]. Let \( f : X \to Y \) be an \( H \)-fuzzy mapping w.r.t. \( E_X \in E_H(X) \) and \( E_Y \in E_H(Y) \), let \( A \in H^X \) and let \( B \in H^X \).

(i) The **image of \( A \) under \( f \)**, denoted by \( f(A) \), is an \( H \)-fuzzy set in \( Y \) defined as follows:

\[
 f(A)(y) = \bigvee_{x \in X} [A(x) \wedge f(x, y)] \quad \forall y \in Y.
\]

(ii) The **preimage of \( B \) under \( f \)**, denoted by \( f^{-1}(B) \), is an \( H \)-fuzzy set in \( X \) defined as follows:

\[
 f^{-1}(B)(x) = \bigvee_{y \in Y} [B(y) \wedge f(x, y)] \quad \forall x \in X.
\]

**Definition 3.8**[13, Proposition 3.10]. Let \( f : X \to Y \) be an \( H \)-fuzzy mapping \( E_X \in E_H(X) \) and \( E_Y \in E_H(Y) \). Then \( f^2 = f \times f : X \times X \to Y \times Y \) is called the **fuzzy product mapping of \( f \) w.r.t.** \( E_{X \times X} = E_X \times E_X \in E_H(X \times X) \) and \( E_{Y \times Y} = E_Y \times E_Y \in E_H(Y \times Y) \) if \( f^2 : (X \times X) \times (Y \times Y) \to H \) is the mapping defined as follows:

\[
 f^2((x, x'), (y, y')) = f(x, y) \wedge f(x', y'), \quad \forall (x, x') \in X \times X, \forall (y, y') \in Y \times Y.
\]

**Definition 3.9.** Let \((X, R_X)\) and \((Y, R_Y)\) be \( H \)-fuzzy relational spaces and let \( f : X \to Y \) be an \( H \)-fuzzy mapping w.r.t. \( E_X \in E_H(X) \) and \( E_Y \in E_H(Y) \).
Then $f : (X, R_X) \to (Y, R_Y)$ is called a *relation preserving mapping* if $R_X \subset f^{-1}(R_Y)$, where $f^{-2} = (f \times f)^{-1}$. In particular, a relation preserving mapping $f : (X, R_X) \to (Y, R_Y)$ is called an *epimorphism* [resp. a *monomorphism*, an *isomorphism*] if it is surjective [resp. injective and bijective].

The following is the immediate result of Result 3.A and Definitions 3.7, 3.8 and 3.9.

**Proposition 3.10.** Let $(X, R_X)$, $(Y, R_Y)$ and $(Z, R_Z)$ be $H$-fuzzy relational spaces, and let $f : X \to Y$ and $g : Y \to Z$ be $H$-fuzzy mappings w.r.t. $E_X \in E_H(X)$, $E_Y \in E_H(Y)$ and $E_Z \in E_H(Z)$.

(a) The identity $H$-fuzzy mapping $I_X : (X, R_X) \to (X, R_X)$ w.r.t. $E_X \in E_H(X)$ is a relation preserving mapping.

(b) If $f : (X, R_X) \to (Y, R_Y)$ and $g : (Y, R_Y) \to (Z, R_Z)$ are relation preserving mappings, then $g \circ f : (X, R_X) \to (Z, R_Z)$ is a relation preserving mapping.

From Result 3.B and Proposition 3.10, we can form the concrete category $\mathbf{VRel}(H)$ consisting of $H$-fuzzy relational spaces and strong relation preserving mappings between them. Every $\mathbf{VRel}(H)$ strongmorphism will be called a $\mathbf{VRel}(H)$-mapping.

**Lemma 3.11.** The category $\mathbf{VRel}(H)$ is topological over $\mathbf{Set}$.

**Proof.** Let $X$ be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of $H$-fuzzy relational spaces indexed by a class $\Gamma$. Suppose $(f_\alpha : X \to X_\alpha)_\Gamma$ is a source of strong $H$-fuzzy mappings w.r.t. $E_X \in E_H$ and $E_{X_\alpha} \in E_H(X_\alpha)$. Define $R_X : X \times X \to H$ by $R_X(R)(x, x') = \bigcap_{\alpha \in \Gamma} f_\alpha^{-2}(R_\alpha)(x, x')$, $\forall (x, x') \in X \times X$.

Then clearly $(X, R_X)$ is a $H$-fuzzy relational space and each $f_\alpha : (X, R_X) \to (X_\alpha, R_\alpha)$ is a $\mathbf{VRel}(H)$-mapping. Let $(Y, R_Y)$ be any $H$-fuzzy relational space and suppose $g : Y \to X$ is any strong $H$-fuzzy mapping w.r.t. $E_Y \in E_H(Y)$ and $E_X$ for which $f_\alpha \circ g : (Y, R_Y) \to (X_\alpha, R_\alpha)$ is a $\mathbf{VRel}(H)$-mapping for each $\alpha \in \Gamma$. Then, for each $\alpha \in \Gamma$

$$R_Y \subset (f_\alpha \circ g)^{-2}(R_\alpha) = g^{-2}(f_\alpha^{-2}(R_\alpha)) = g^{-2}(f_\alpha^{-2}(R_\alpha)).$$
Thus $R_Y \subset g^{-2}\left(\bigcap_{\alpha \in \Gamma} f^{-2}(R_{\alpha})\right) = g^{-2}(R_X)$. So $g : (Y, R_Y) \to (X, R_X)$ is a VRel(H)-mapping. Hence $(f_\alpha : (X, A_X) \to (X, A_{\alpha}))_{\Gamma}$ is an initial source in VRel(H). This completes the proof.

Example 3.11. (1) The inverse image of a H-fuzzy relation structure. Let $X$ be a set, let $(Y, R_Y)$ be a H-fuzzy relational space and let $f : X \to Y$ be a strong H-fuzzy mapping w.r.t. $E_X \in E_H$ and $E_Y \in E_H(Y)$. Then there exists a unique H-fuzzy relation $R_X$ in $X$ for which $f : (X, R_X) \to (Y, R_Y)$ is a VRel(H)-mapping. Hence $(f_\alpha : (X, A_X) \to (X, A_{\alpha}))_{\Gamma}$ is an initial source in VRel(H). This completes the proof.

(2) The H-fuzzy product structure. Let $((X_\alpha, R_{\alpha}))_{\Gamma}$ be any family of H-fuzzy spaces, let $X = \prod_{\alpha \in \Gamma} X_\alpha$ and for each $\alpha \in \Gamma$, let $pr_\alpha : X \to X_\alpha$ be the H-fuzzy projection w.r.t. $E_X \in E_H(X)$ and $E_{X_\alpha} \in E_H(X_\alpha)$. Then there exists a unique H-fuzzy relation $R_X$ in $X$ w.r.t. $(X, (pr_\alpha)_{\alpha \in \Gamma}, ((X_\alpha, R_{\alpha}))_{\alpha \in \Gamma})$. In this case, $R_X$ is called the H-fuzzy product of H-fuzzy relation structures in the $X_\alpha$ and denoted by $R_X = \prod_{\alpha \in \Gamma} R_{\alpha}$, and $(\prod_{\alpha \in \Gamma} X_\alpha, \prod_{\alpha \in \Gamma} R_{\alpha})$ is called the H-fuzzy product relational space of $((X_\alpha, R_{\alpha}))_{\Gamma}$. In fact, $\prod_{\alpha \in \Gamma} R_{\alpha} = \bigcap_{\alpha \in \Gamma} pr^{-2}_\alpha(R_{\alpha})$. In particular, if $\Gamma = \{1, 2\}$, then $(R_1 \times R_2)((x_1, y_1), (x_2, y_2)) = R_1(x_1, x_2) \land R_2(y_1, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

The following is the immediate result of Lemma 3.11 and Result 2.B

Corollary 3.11. The category VRel(H) is complete and cocomplete.

It is well-known[15] that a category is topological if and only if it is cotopological. However, we show directly that VRel(H) is cotopological.

Lemma 3.12. The category VRel(H) is cotopological over Set.

Proof. Let $X$ be any set and let $((X_\alpha, R_{\alpha}))_{\Gamma}$ be any family of H-fuzzy relational spaces indexed by a class $\Gamma$. Suppose $(f_\alpha : X_\alpha \to X)_{\Gamma}$ is a sink of strong H-fuzzy mappings w.r.t. $E_{X_\alpha} \in E_H(X_\alpha)$ and $E_X \in E_H(X)$. We define
\( R_X : X \times X \to H \) by \( R_X = \bigcup_{\alpha \in \Gamma} f_\alpha^2(R_\alpha) \). Then clearly \( R_X \) is well-defined and each \( f_\alpha : (X_\alpha, R_\alpha) \to (X, R_X) \) is a \( \text{VRel}(H) \)-mapping. For each \( \text{H-fuzzy relational space} (Y, R_Y) \), let \( g : X \to Y \) be a strong \( \text{H-fuzzy mapping} \) w.r.t. \( E_X \) and \( E_Y \in E_H(Y) \) such that each \( g \circ f_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y) \) is a \( \text{VRel}(H) \)-mapping. Then \( R_\alpha \subset (g \circ f_\alpha)^{-2}(R_Y), \ \forall \alpha \in \Gamma \). Thus, for each \((x_\alpha, x'_\alpha) \in X_\alpha \times X_\alpha\),

\[
R_\alpha(x_\alpha, x'_\alpha) \leq (g \circ f_\alpha)^{-2}(R_Y)(x_\alpha, x'_\alpha)
= f_\alpha^{-2}(g^{-2}(R_Y))(x_\alpha, x'_\alpha)
= \bigvee_{(x, x') \in X \times X} [g^{-2}(R_Y)(x, x') \land f_\alpha(x_\alpha, x) \land f_\alpha(x'_\alpha, x')]
\leq \bigvee_{(x, x') \in X \times X} g^{-2}(R_Y)(x, x'),
\]
i.e., \( R_\alpha(x_\alpha, x'_\alpha) \leq g^{-2}(R_Y)(x, x'), \ \forall (x, x') \in X \times X \). So

\[
R_X(x, x') = \left( \bigcup_{\alpha \in \Gamma} f_\alpha^2(R_\alpha) \right)(x, x') \leq g^{-2}(R_Y)(x, x') \ \forall (x, x') \in X \times X.
\]
i.e., \( R_X \subset g^{-2}(R_Y) \). Hence \( g : (X, R_X) \to (Y, R_Y) \) is a \( \text{VRel}(H) \)-mapping. Therefore \( \text{VRel}(H) \) is cotopological over \( \text{Set} \).

**Result 3.C[13, Proposition 3.11].** Let \( f : X \to Y \) be a strong \( \text{H-fuzzy mapping} \) w.r.t. \( E_X \in E_H(X) \) and \( E_Y \in E_H(Y) \), and let \( g : Z \to Y \) be a strong \( \text{H-fuzzy mapping} \) w.r.t. \( E_Z \in E_H(Z) \) and \( E_Y \). Let \( U = \{ (x, z) \in X \times Z : \exists y \in Y \text{ such that } f(x, y) = 1 = g(z, y) \} \). Then the restriction \( E_U = (E_X \times E_Z) |_{U \times U} : U \times U \to H \) is a \( \text{H-fuzzy equality} \) on \( U \). Moreover, \( \text{pr}_X : U \to X \) and \( \text{pr}_Z : U \to Z \) are projections w.r.t. \( E_U \) and \( E_X \), and \( E_U \) and \( E_Z \), respectively.

**Lemma 3.13.** Final episinks in \( \text{VRel}(H) \) are preserved by pullbacks.

**Proof.** Let \((g_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y))_{\Gamma}\) be any final episink in \( \text{VRel}(H) \) w.r.t. \( E_\alpha \in E_H(X_\alpha) \) and \( E_Y \in E_H(Y) \), and let \((f, W, R_W) \to (Y, R_Y)\) be any \( \text{VRel}(H) \)-mapping w.r.t. \( E_W \in E_H(W) \) any \( E_Y \). For each \( \alpha \in \Gamma \), let

\[
U_\alpha = \{ (w, x_\alpha) \in W \times X_\alpha : \exists y \in Y \text{ such that } f(w, y) = 1 = g_\alpha(x_\alpha, y) \}
\]
and let \( R_{U_\alpha} = (R_W \times R_\alpha) |_{U_\alpha \times U_\alpha} \). Then clearly \( (U_\alpha, R_{U_\alpha}) \) is a \( \text{H-fuzzy relational space} \). By Result 3.C, for each \( \alpha \in \Gamma \), \( e_\alpha : U_\alpha \to W \) and \( p_\alpha : U_\alpha \to X_\alpha \) are projections of \( U_\alpha \) w.r.t. \( E_{U_\alpha} \) and \( E_W \), and \( E_{U_\alpha} \) and \( E_X \), respectively. Furthermore, for each \( \alpha \in \Gamma \), \( e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_W) \) and \( p_\alpha : (U_\alpha, R_{U_\alpha}) \to (X_\alpha, R_\alpha) \)
are \( \text{VRel}(H) \)-mappings and the following diagram is a pullback square in \( \text{VRel}(H) \):

\[
\begin{array}{ccc}
(U_{\alpha}, R_{U_{\alpha}}) & \xrightarrow{\alpha} & (X_{\alpha}, R_{\alpha}) \\
\downarrow e_{\alpha} & & \downarrow g_{\alpha} \\
(W, R_W) & \xrightarrow{f} & (Y, R_Y).
\end{array}
\]

Let \( w \in W \). Since \( f : W \to X \) is a strong \( H \)-fuzzy mapping, \( \exists y_{o} \in Y \) such that \( f(w, y_{o}) = 1 \). Since \( (g_{\alpha})_{\Gamma} \) is a final episink, for each \( \alpha \in \Gamma \), and for \( y_{o} \in Y \), \( \exists x_{\alpha_{o}} \in X_{\alpha} \) such that \( g_{\alpha}(x_{\alpha_{o}}, y_{o}) = 1 \).

Thus \( (w, x_{\alpha_{o}}) \in U_{\alpha} \) and \( e_{\alpha}((w, x_{\alpha_{o}}), w) = 1 \). So \( (e_{\alpha})_{\Gamma} \) is an episink in \( \text{VRel}(H) \).

Moreover \( (e_{\alpha})_{\Gamma} \) is final. Let \( R_{W}^{*} \) be the final structure on \( W \) w.r.t. \( (e_{\alpha})_{\Gamma} \) and let \( (w, w') \in W \times W \). Then

\[
R_{W}(w, w') = R_{W}(w, w') \land R_{W}(w, w')
\]

\[
\leq R_{W}(w, w') \land f^{-2}(R_{Y})(w, w')
\]

[ Since \( f : (W, R_{W}) \to (Y, R_{Y}) \) is a \( \text{VRel}(H) \)-mapping ]

\[
= R_{W}(w, w') \land \left( \bigvee_{(y, y') \in Y \times Y} [R_{Y}(y, y') \in Y \times Y \land f(w, y) \land f(w', y')] \right)
\]

\[
= R_{W}(w, w') \land \left( \bigvee_{(y, y') \in Y \times Y} \bigvee_{\alpha \in \Gamma} [g_{\alpha}^{2}(R_{\alpha})(y, y') \land f(w, y) \land f(w', y')] \right)
\]

[ Since \( (g_{\alpha})_{\Gamma} \) is final ]

\[
= \bigvee_{(x_{\alpha}, y_{\alpha}) \in X_{\alpha} \times X_{\alpha}} [R_{W}(w, w') \land R_{\alpha}(x_{\alpha}, y_{\alpha}) \land \left( \bigvee_{(y, y') \in Y \times Y} g_{\alpha}(x_{\alpha}, y) \land g_{\alpha}(y_{\alpha}, y') \land f(w, y) \land f(w', y') \right)]
\]

\[
= \bigvee_{(x, y, w') \in U_{\alpha} \times U_{\alpha}} [R_{U_{\alpha}}(w, x_{\alpha}) \land R_{U_{\alpha}}(w', y_{\alpha}) \land e_{\alpha}((w, x_{\alpha}), (w', y_{\alpha}), w')]
\]

[ Since \( R_{U_{\alpha}} = (R_{W} \times R_{\alpha}) |_{U_{\alpha} \times U_{\alpha}} \) ]

\( f \) is strong and \( g \) is strong surjective

\[
= R_{W}^{*}(w).
\]

Thus \( R_{W} \subset R_{W}^{*} \). On the other hand, since \( (e_{\alpha} : (U_{\alpha}, R_{U_{\alpha}}) \to (W, R_{W}^{*}))_{\Gamma} \) is final, \( I_{W} : (W, R_{W}^{*}) \to (W, R_{W}) \) is a \( \text{VRel}(H) \)-mapping. So \( R_{W}^{*} \subset R_{W} \). Hence \( A_{W} = A_{W}^{*} \). This completes the proof. \( \square \)
For any singleton set \{a\}, since the \(H\)-fuzzy relation structure \(R_{\{a\}}\) on \{a\} is not unique, the category \(\text{VRel}(H)\) is not properly fibred over \(\text{Set}\). Hence, by Lemmas 3.11 and 3.13, we obtain the following result.

**Theorem 3.14.** The category \(\text{VRel}(H)\) satisfies all the conditions of a topological universe over \(\text{Set}\) except the terminal separator property.

**Theorem 3.15.** The category \(\text{VRel}(H)\) is Cartesian closed over \(\text{Set}\).

**Proof.** It is obvious that \(\text{VRel}(H)\) has products by Corollary 3.11. Then it is sufficient to show that \(\text{VRel}(H)\) has exponential objects.

For any \(H\)-fuzzy spaces \(X = (X, R_X)\) and \(Y = (Y, R_Y)\), let \(Y^X\) be the set of all strong \(H\)-fuzzy mappings from \(X\) to \(Y\). We define a mapping \(R_{Y^X} : Y^X \times Y^X \rightarrow H\) as follows: for each \((f, g) \in Y^X \times Y^X\),

\[
R_{Y^X}(f, g) = \bigvee \{ h \in H : R_X \cap h \subseteq (f^{-1} \times g^{-1})(R_Y) \} = \bigvee \{ h \in H : R_X(x, x') \wedge h \leq \bigvee_{(y, y') \in Y \times Y} [R_Y(y, y') \wedge f(x, y) \wedge g(x', y')] \}
\]

where \(h(x) = h, \forall x \in X\). Since \(f\) and \(g\) are strong,

\[
R_{Y^X}(f, g) = \bigvee \{ h \in H : R_X(x, x') \wedge h \leq \bigvee_{f(x,y)=1,g(x',y')=1} R_Y(y, y') \}.
\]

Then clearly \((Y^X, R_{Y^X}) \in \text{VRel}(H)\). Let \(Y^X = (Y^X, R_{Y^X})\). Then, by the definition of \(R_{Y^X}\),

\[
R_X(x, x') \wedge R_{Y^X}(f, g) \leq \bigvee_{f(x,y)=1,g(x',y')=1} R_Y(x, y), \forall (f, g) \in Y^X \times Y^X,
\]

\(\forall (x, x') \in X \times X\).

We define a mapping \(e_{X, Y^X} : (X \times Y^X) \times Y \rightarrow H\) by

\[
e_{X, Y^X}((x, f), y) = f(x, y) \quad \forall (x, f) \in X \times Y^X, \forall y \in Y.
\]

Then clearly \(e_{X, Y^X}\) is an \(H\)-fuzzy relation on \((X \times Y^X) \times Y\). Now we define a mapping \(E_{X \times Y^X} : (X \times Y^X) \times (X \times Y^X) \rightarrow H\) as follows: For any \((x, f), (x', g) \in X \times Y^X\),

\[
E_{X \times Y^X}((x, f), (x', g)) = (\bigwedge_{y \in Y} f(x, y) \wedge \bigwedge_{y' \in Y} g(x', y')) \wedge E_X(x, x') \wedge E'_X(x, x'),
\]

where \(f : X \rightarrow Y\) and \(g : X \rightarrow Y\) are strong \(H\)-fuzzy mappings w.r.t. \(E_X \in E_H(X)\) and \(E_Y \in E_H(Y)\), and \(E'_X \in E_H(X)\) and \(E'_Y \in E_H(Y)\), respectively.
Thus, by the process of the proof of Theorem 4.8 in [13], $e_{XY} : X \times Y \to Y$ is a strong $H$-fuzzy mapping w.r.t. $E_{X \times Y}$ and $E \in E_H(Y)$, where $E = E_Y \times E_Y$, is an $H$-fuzzy equality on $Y$. Let $((x, f), (x', g)) \in (X \times Y) \times (X \times Y)$. Then

$$e_{XY}^{-2}(R_Y)((x, f), (x', g)) = \bigvee_{(y, h') \in Y \times Y} [R_Y(y, h') \land e_{XY}((x, f), y) \land e_{XY}((x', g), h')]$$

$$= \bigvee_{(y, h') \in Y \times Y} [R_Y(y) \land f(x, y) \land g(x', y')]$$

$$= \bigvee_{\substack{f(x,y) = 1, g(x',y') = 1 \quad \text{[Since } f \text{ and } g \text{ are strong]}}} R_Y(y, h') \quad \text{[Since } f \text{ and } g \text{ are strong]}$$

$$\geq R_X(x, x') \land R_{Y \times X}(f, g) \quad \text{[Since } f \text{ and } g \text{ are strong]}$$

$$= (R_X \times R_{Y \times X})((x, f), (x', g)).$$

Thus $R_X \times R_{Y \times X} \subset e_{XY}^{-2}(R_Y)$. So $e_{XY} : X \times Y \to Y$ is a $\mathbf{VRel}(H)$-mapping.

For any $Z = (Z, R_Z) \in \mathbf{VRel}(H)$, let $h : X \times Z \to Y$ be a $\mathbf{VRel}(H)$-mapping w.r.t. $E_{X \times Z} = E_X \times E_Z \in E_H(X \times Z)$ and $E_Y \in E_H(Y)$. We define a mapping $\bar{h} : Z \times (X \times Y) \to H$ as follows: $\forall z \in Z, \ \forall x \in X, \ \forall y \in Y$,

$$\bar{h}(z)(x, y) = h((x, z), y),$$

where $\bar{h}(z)(x, y) = \bar{h}(z, (x, y)).$

Since $h$ is strong, it is clear that $\bar{h}(z)$ is strong. Thus $\bar{h}(z) \in \mathcal{Y}^X, \forall z \in Z$. So $\bar{h} : Z \to \mathcal{Y}^X$ is a strong $H$-fuzzy mapping. Let $z, z' \in Z$ and let $x, x' \in X$.

Then

$$R_X(x, x') \land R_Z(z, z') = (R_X \times R_Z)((x, z), (x', z'))$$

$$\leq h^{-2}(R_Y)((x, z), (x', z')) \quad \text{[Since } h : X \times Z \to Y \text{ is a } \mathbf{VRel}(H)-\text{mapping]}$$

$$= \bigvee_{(y, h') \in Y \times Y} [R_Y(y, h') \land h((x, z), y) \land h((x', z'), y')]$$

$$= \bigvee_{\substack{f(x,y) = 1, g(x',y') = 1 \quad \text{[Since } h \text{ is strong]}}} R_Y(y, h') \quad \text{[Since } h \text{ is strong]}$$

Thus, by the definition of $R_{Y \times X}$,

$$R_Z(z) \leq \bar{h}^{-2}(R_{Y \times X})(z).$$

So $R_Z \subset \bar{h}^{-2}(R_{Y \times X})$. Hence $\bar{h} : Z \to \mathcal{Y}^X$ is a $\mathbf{VRel}(H)$-mapping. Moreover, $\bar{h}$ is unique $\mathbf{VRel}(H)$-mapping such that $e_{XY} \circ (I_X \times \bar{h}) = h$. This completes the proof. \qed
4. The category $\mathbf{VFRel}(H)$

**Definition 4.1**[13]. The concrete category $\mathbf{VSet}(H)$ is defined by: Objects $(X, A_X)$, called an $H$-fuzzy space, where $X$ is any set and $A_X \in H^X$. A morphism $f : (X, A_X) \to (Y, A_Y)$ is a strong $H$-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$ satisfying $A_X \leq f^{-1}(A_Y)(x)$, $\forall x \in X$. Every $\mathbf{VSet}(H)$-morphism is called a $\mathbf{VSet}(H)$-mapping.

**Definition 4.2**[10]. An $H$-fuzzy relation $R$ on an $H$-fuzzy space $(X, A_X)$ is an $H$-fuzzy set in $X \times X$ satisfying $R(x, y) \leq A_X(x) \land A_X(y)$ for any $x, y \in X$. In this case, the triple $(X, A_X, R)$ is called an $H$-fuzzy relational space over $(X, A_X)$.

**Definition 4.3.** An $H$-fuzzy mapping $f : (X, A_X, R_X) \to (Y, A_Y, R_Y)$ w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$ is called a relation preserving mapping if it satisfies the following conditions:

(i) $f : (X, A_X) \to (Y, A_Y)$ is a $\mathbf{VSet}(H)$-mapping,

(ii) $f : (X, R_X) \to (Y, R_Y)$ is a $\mathbf{VRel}(H)$-mapping.

We denote the category of all $H$-fuzzy relational spaces over $H$-fuzzy spaces and relation preserving strong $H$-fuzzy mappings between them by $\mathbf{VFRel}(H)$, and the mixture of the categories $\mathbf{VSet}(H)$ and $\mathbf{VRel}(H)$ by $\mathbf{VSet}(H) \land \mathbf{VRel}(H)$(cf [14]). Since $\mathbf{VSet}(H)$ and $\mathbf{VRel}(H)$ are topological over $\mathbf{Set}$ by Lemma 4.4 in [13] and Lemma 3.11, so is the mixture $\mathbf{VSet}(H) \land \mathbf{VRel}(H)$ with natural structures by Proposition 2 in [14].

**Lemma 4.4.** $\mathbf{VFRel}(H)$ is a bi(co)reflective subcategory of $\mathbf{VSet}(H) \land \mathbf{VRel}(H)$.

**Proof.** Let $(X, A, R)$ be an object in $\mathbf{VSet}(H) \land \mathbf{VRel}(H)$. We define a mapping $A_X : X \to H$ as follows: For each $x \in X$, $A_X(x) = A(x) \lor \left[ \bigvee_{y \in X} R(x, y) \right]$. Then it is easily seen that $I_X : (X, A, R) \to (X, A_X, R)$ is a $\mathbf{VFRel}(H)$-reflection of $(X, A, R)$. Now we define a mapping $R_X : X \times X \to H$ as follows: For any $x, y \in X$, $R_X(x, y) = R(x, y) \land A(x) \land A(y)$. 

Then \( I_X : (X, A, R_X) \to (X, A, R) \) is a \( \text{VFRel}(H) \)-coreflection of \((X, A, R)\). This completes the proof. \( \square \)

The following is the immediate result of Lemma 4.4 and Theorems 2.6 and 2.8 in [14].

**Theorem 4.5.** (a) The category \( \text{VFRel}(H) \) is topological over \( \text{Set} \).

(b) Final episinks in \( \text{VFRel}(H) \) are preserved by pullbacks.

**Remark 4.6.** (a) Let \( X \) be a set and let \((f_\alpha : X \to (X_\alpha, A_\alpha, R_\alpha))_{\alpha \in \Gamma} \) be a source, where \((X_\alpha, A_\alpha, R_\alpha) \in \text{VFRel}(H)\) for each \( \alpha \in \Gamma \). We define two mappings \( A_X : X \to H \) and \( R_X : X \times X \to H \) as follows, respectively:

\[
A(x) = \bigcap_{\alpha \in \Gamma} f^{-1}(A_\alpha)(x), \, \forall x \in X
\]

and

\[
R_X(x, y) = \bigcap_{\alpha \in \Gamma} f^{-2}(R_\alpha)(x, y), \, \forall x, y \in X.
\]

Then \((X, A_X, R_X)\) is equipped with the initial structure w.r.t. \((f_\alpha)_{\Gamma} \) in \( \text{VFRel}(H) \).

(b) Let \( X \) be a set and let \((f_\alpha : (X_\alpha, A_\alpha, R_\alpha) \to X)_{\alpha \in \Gamma} \) be a sink, where \((X_\alpha, A_\alpha, R_\alpha) \in \text{VFRel}(H)\), \( \forall \alpha \in \Gamma \). We define two mappings \( A_X : X \to H \) and \( R_X : X \times X \to H \) as follows, respectively:

\[
A_X(x) = \bigvee_{\alpha \in \Gamma} f_\alpha(A_\alpha)(x), \, \forall x \in X
\]

and

\[
R_X(x, y) = \bigvee_{\alpha \in \Gamma} f_\alpha(x, y), \, \forall x, y \in X.
\]

Then \((X, A_X, R_X)\) is equipped with the final structure w.r.t. \((f_\alpha)_{\Gamma} \).

(c) Since both \( H \)-fuzzy set structures and \( H \)-fuzzy relational structures on a singleton set are not unique, \( \text{VFRel}(H) \) is not properly fibred.

(d) Let \((g_\alpha : (X_\alpha, A_\alpha, R_\alpha) \to (Y, A_Y, R_Y))_{\alpha \in \Gamma} \) be any final episink in \( \text{VFRel}(H) \) and \( f : (W, A_W, R_W) \to (Y, A_Y, R_Y) \) be any \( H \)-fuzzy mapping w.r.t. \( E_W \in E_H(W) \) and \( E_Y \in E_H(Y) \) in \( \text{VFRel}(H) \). For each \( \alpha \in \Gamma \), let

\[
U_\alpha = \{(w, x_\alpha) \in W \times X_\alpha : \exists y \in Y \text{ such that } f(w, y) = 1 = g_\alpha(x_\alpha, y)\},
\]

let \( A_{U_\alpha} = (A_W \times A_\alpha) \big|_{U_\alpha \times U_\alpha} \) and let \( R_{U_\alpha} = (R_W \times R_\alpha) \big|_{U_\alpha \times U_\alpha} \). Then, for each \( \alpha \in \Gamma \), \( e_\alpha : (U_\alpha, A_{U_\alpha}, R_{U_\alpha}) \to (W, A_W, R_W) \) is the pullback of \( g_\alpha \) along \( f \) in \( \text{VFRel}(H) \), where \( e_\alpha : U_\alpha \to W \) is the \( H \)-fuzzy projection of \( U_\alpha \) w.r.t. \( E_{U_\alpha} \in E_H(U_\alpha) \) and \( E_W \). Moreover, \((e_\alpha : (U_\alpha, A_{U_\alpha}, R_{U_\alpha}) \to (W, A_W, R_W))_{\alpha \in \Gamma} \)
is a final episink in $\text{VFRel}(H)$.

**Remark 4.7.** (a) The category $\text{VFRel}(H)$ is topological over $\text{VSet}(H)$: Let $(X, A_X)$ be any $H$-fuzzy space and let $((X_a, A_a, R_a))_{a \in \Gamma}$ be any family of $H$-fuzzy relational spaces. Let $(f_a : (X, A_X) \rightarrow (X_a, A_a))_{a \in \Gamma}$ be any mapping in $\text{VSet}(H)$. We define a mapping $R_X : X \times X \rightarrow H$ as follows: For any $x, y \in X$, 
$$R_X(x, y) = (\bigwedge \alpha \in \Gamma f_a^{-2}(R_a)(x, y)) \wedge A_X(x) \wedge A_X(y).$$
Then $R_X$ is the initial structure on $(X, A_X)$ w.r.t. $(f_a)_{\Gamma}$.

(b) The category $\text{VFRel}(H)$ is cotopological over $\text{VSet}(H)$: Let $(X, A_X)$ be any $H$-fuzzy space and let $((X_a, A_a, R_a))_{a \in \Gamma}$ be any family of $H$-fuzzy relational spaces. Let $(f_a : (X_a, A_a) \rightarrow (X, A_X))_{a \in \Gamma}$ be any mapping in $\text{VSet}(H)$. We define a mapping $R_X : X \times X \rightarrow H$ as follows: For any $x, y \in X$, 
$$R_X(x, y) = \bigvee_{\alpha \in \Gamma} f^2(R_a).$$
Then $R_X$ is the final structure on $(X, A_X)$ w.r.t. $(f_a)_{\Gamma}$.

**Theorem 4.8.** The category $\text{VFRel}(H)$ is Cartesian closed.

**Proof.** Since $\text{VFRel}(H)$ has finite products by Theorem 4.5, it is enough to show that $\text{VFRel}(H)$ has exponentials.

For any $H$-fuzzy relational spaces $X = (X, A_X, R_X)$ and $Y = (Y, A_Y, R_Y)$, let $Y_X$ be the set of all morphisms from $X$ to $Y$ in $\text{VFRel}(H)$. We define two mappings $A_{Y} : Y^X \rightarrow H$ and $R_{Y} : Y_X \times Y_X \rightarrow H$ as follows, respectively:

$$A_{Y}(f) = \bigvee \{ h \in H : A_X(x) \wedge h \leq \bigvee_{y \in Y} [A_Y(y) \wedge f(x, y)], \forall x \in X \}, \forall f \in Y^X$$

and

$$R_{Y}(f, g) = \bigvee \{ h \in H : R_X(x, x') \wedge h \leq \bigvee_{(y, y') \in Y \times Y} [R_Y(y, y') \wedge f(x, y) \wedge g(x', y')] \wedge (A_{Y}(f) \wedge A_{Y}(g)) \forall x, y \in X \}. \forall (f, g) \in Y_X \times Y_X.$$

Then clearly $R_{Y}$ is an $H$-fuzzy relation on $(Y^X, A_{Y})$. Let $Y^X = (Y_X, A_{Y}, R_{Y})$. We define a mapping $\epsilon_{X,Y} : (X \times Y^X) \times Y \rightarrow H$ as follows:

$$\epsilon_{X,Y}((x, f), y) = f(x, y) \forall (x, f) \in X \times Y^X, \forall y \in Y.$$ 

Then, by the proofs of Theorem 4.8 in [13] and Theorem 3.15, it can be easily seen that $\epsilon_{X,Y} : X \times Y^X \rightarrow Y$ is a $\text{VFRel}(H)$-mapping.

For any $Z = (Z, A_Z, R_Z) \in \text{VFRel}(H)$, let $h : X \times Z \rightarrow Y$ be a
Proof. It is clear that completes the proof.

\[ \text{Rel}(H) \]
\[ (X \text{ mapping from } F \rightarrow R \text{ \bar{VFRel}(H)} \]
\[ (X, R_X) \]
\[ R \subseteq \text{VFRel}(H) \]
\[ \text{VFRel}(H) \text{-mapping. Thus } (I_X \times \bar{h}) = h. \text{ This completes the proof.} \]

5. The relations between Rel(H) and VRel(H)

Definition 5.1 [11]. The concrete category Rel(H) is defined by: Objects are \((X, R_X)\), called an \(H\text{-fuzzy relational space}\)(or simply, a \(fuzzy relational sapce\), where \(X\) is any set and \(R_X \in H^{X \times X}\). A morphism \(f : (X, R_X) \rightarrow (Y, R_Y)\) is a mapping from \(X\) to \(Y\) satisfying \(R_X(x, y) \leq R_Y(f(x), f(y)), \forall (x, y) \in X \times X\), i.e., \(R_X \subseteq f^{-2}(R_Y)\) where “\(\leq\)” means the order induced by the operation “\(\land\)” or “\(\lor\)” in \(H\). Every Rel(H)-morphism is called a Rel(H)-mapping.

Lemma 5.2. Define \(F : \text{Rel}(H) \rightarrow \text{VRel}(H)\) by \(F(X, R_X) = (X, R_X)\) and \(F(f) = f\). Then \(F\) is a functor.

Proof. It is clear that \(F(X, R_X) = (X, R_X) \in \text{VRel}(H), \forall (X, R_X) \in \text{Rel}(H)\). Let \((X, R_X), (Y, R_Y) \in \text{Rel}(H)\) and let \(f : (X, R_X) \rightarrow (Y, R_Y)\) be a Rel(H)-mapping. Then \(R_X(x, y) \leq R_Y(f(x), f(y)), \forall (x, y) \in X \times X\). Since \(f : X \rightarrow Y\) is a mapping, \(f : X \rightarrow Y\) is a strong \(H\)-fuzzy mapping w.r.t. \(I_X \in E_H(X)\) and \(I_Y \in E_H(Y)\). Moreover, for each \((x, x') \in X \times X\),
\[
\begin{align*}
\text{f}^{-2}(R_Y)(x, x') &= \bigvee_{(y, y') \in Y \times Y} [R_Y(y, y') \land f(x, y) \land f(x', y') ] \\
&\geq R_Y(y_o, y_o') \\
&[\text{Since } f \text{ is strong, } \exists y_o \in Y \text{ and } y_o' \in Y \text{ such that} ] \\
&f(x, y_o) = 1 \text{ and } f(x', y_o') = 1] \\
&= R_Y(f(x), f(x')) \\
&\geq R_X(x, x').
\end{align*}
\]
Thus \(R_X \subseteq f^{-2}(R_Y)\). So \(F(f) = f \in \text{VRel}(H)\). Hence \(F(f) = f : (X, R_X) \rightarrow (Y, R_Y)\) is a VRel(H)-mapping. Therefore \(F\) is a functor. \(\square\)
Lemma 5.3. We define \( G : \text{VRel}(H) \rightarrow \text{Rel}(H) \) by \( G(X, R_X) = (X, R_X) \) and \( G(f) = f_\ast \), where if \( f : X \rightarrow Y \) is an \( H \)-fuzzy mapping w.r.t. \( E_X \in E_H(X) \) and \( E_Y \in E_H(Y) \), then \( f_\ast : X \times Y \rightarrow 2 = \{0,1\} \) is a mapping defined by \( f_\ast(x, y) = f(x, y), \ \forall (x, y) \in X \times Y \), and \( E_X^* \) and \( E_Y^* \) are \( H \)-fuzzy equalities on \( X \) and \( Y \) defined by \( E_X^* = I_X \) and \( E_Y^* = I_Y \), respectively. Then \( G \) is a functor.

Proof. It is clear that \( G(X, R_X) = (X, R_X) \in \text{Rel}(H), \ \forall (X, R_X) \in \text{VRel}(H) \). Let \( (X, R_X), (Y, R_Y) \in \text{VRel}(H) \) and let \( f : (X, R_X) \rightarrow (Y, R_Y) \) be a \( \text{VRel}(H) \)-mapping. Then \( R_X \subset f^{-2}(R_Y) \). By the definition of \( G(f) \), \( G(f) = f_\ast : X \rightarrow Y \) is a mapping. Let \( (x, x') \in X \times X \). Since \( f \) is strong, \( \exists (y_0, y'_0) \in Y \times Y \) such that \( f(x, y_0) = 1 = f(x', y'_0) \). Thus
\[
R_Y(f_\ast(x), f_\ast(x')) = R_Y(f(x), f(y)),
\]
\[
= f^{-2}(R_Y)(x, x')
\]
\[
\text{[ Since } f^{-2}(R_Y)(x, x') = \bigvee_{(y, y') \in Y \times Y} [R_Y(y, y') \land f(x, y) \land f(x', y')] = R_Y(f(x), f(x')) \]\n\[
\geq R_X(x, x').
\]
So \( f_\ast : (X, R_X) \rightarrow (Y, R_Y) \) is a \( \text{Rel}(H) \)-mapping. Hence \( G \) is a functor.

Lemma 5.4. The functor \( F \) is a left adjoint of the functor \( G \).

Proof. For each \( (X, R_X) \in \text{Rel}(H) \), \( I_X : (X, R_X) \rightarrow GF(X, R_X) = (X, R_X) \) is a \( \text{Rel}(H) \)-mapping. Let \( (Y, R_Y) \in \text{VRel}(H) \) and let \( f : (X, R_X) \rightarrow G(Y, R_Y) \) be a \( \text{Rel}(H) \)-mapping. Then \( R_X(x, x') \leq R_Y(f(x), f(x')), \ \forall (x, x') \in X \times X \). Thus \( R_X \subset f^{-2}(R_Y) \). So \( f : F(X, R_X) = (X, R_X) \rightarrow (Y, R_Y) \) is a \( \text{VRel}(H) \)-mapping. Hence \( I_X \) is a \( G \)-universal map for \( (X, R_X) \) in \( \text{Rel}(H) \). This completes the proof.

Let \( \text{VRel}_s(H) \) denote the category with \( \text{Mor}(\text{VRel}_s(H)) = \{ f_\ast : f \in \text{Mor}(\text{VRel}(H)) \} \). Then clearly \( \text{VRel}_s(H) \) is a full subcategory of \( \text{VRel}(H) \).

Theorem 5.5. Two categories \( \text{Rel}(H) \) and \( \text{VRel}_s(H) \) are isomorphic.

Proof. By Lemma 5.2, it is clear that \( F : \text{Rel}(H) \rightarrow \text{VRel}_s(H) \) is a functor. Also, By Lemma 5.3, \( G : \text{VRel}_s(H) \rightarrow \text{Rel}(H) \) is a functor. Let \( (X, R_X) \in \text{Rel}(H) \). Then clearly \( F(X, R_X) = (X, R_X) \). Thus \( GF(X, R_X) = (X, R_X) \).
Thus $G \circ F = 1_{\text{Rel}(H)}$. Similarly, we can easily see that $F \circ G = 1_{\text{VRel}_*(H)}$. So $F : \text{Rel}(H) \to \text{VRel}_*(H)$ is an isomorphism. This completes the proof. □

References


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