Improved Homotopy Perturbation Method

M. A. Jafari

Department of Mathematics, K. N. Toosi University of Technology
P.O. Box: 16315-1618, Tehran, Iran

A. Aminataei

Department of Mathematics, K. N. Toosi University of Technology,
P.O. Box: 16315-1618, Tehran, Iran

Abstract

In this paper a new treatment for homotopy perturbation method (HPM) is introduced. The new treatment is called improved homotopy perturbation method (IHPM) which improves the results obtained from HPM. At first, the principle of the HPM is described. Then IHPM is proposed, which yields the analytic approximate solution. To illustrate the methods some experiments are provided. In some experiments different approximate solutions are obtained by using HPM. By applying IHPM, we obtain exact solutions. The results show the efficiency, accuracy, and superiority of the new method. Furthermore, the convergency of the method is also considered.

Mathematics Subject Classification: 35G10; 35G15; 35G25; 35G30; 74S30

Keywords: Homotopy perturbation method; Improved homotopy perturbation method; Linear and nonlinear partial differential equations; Convergency of the method

1 Introduction

Recently a great deal of interest has been focused on the application of HPM for the solution of many different problems. The HPM has been applied with great success to obtain the approximate solution of large variety of linear and nonlinear problems in ordinary differential equations (ODEs),

1m-jafari@dena.kntu.ac.ir
2Corresponding Author: ataei@kntu.ac.ir
partial differential equations (PDEs), and integral equations [1-15]. In some PDEs different approximate solutions can be obtained by using HPM. In this paper we improve HPM to obtain exact solutions for these PDEs.

The present study consists of the following sections. In section 2, we introduce HPM and IHPM. In section 3, to illustrate the method, some experiments are provided. In section 4, the convergency of the method is considered and finally in section 5 a short conclusion is given.

2 HPM and its improvement (IHPM)

He presented a homotopy perturbation technique based on the introduction of a homotopy and an artificial parameter for the solution of algebraic equations and ODEs [1]. To describe HPM, we consider the following nonlinear differential equation:

\[ A(\upsilon) - f(r) = 0, r \in \Omega, \]  

with the boundary conditions:

\[ B(\upsilon, \frac{\partial \upsilon}{\partial n}) = 0, r \in \Gamma, \]  

where \( A \) is a differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytic function and \( \Gamma \) is the boundary of the domain \( \Omega \). The operator \( A \) can be divided into two parts \( L \) and \( N \), where \( L \) is a linear operator and \( N \) is a nonlinear operator. Therefore, equation (1) can be rewritten as:

\[ L(\upsilon) + N(\upsilon) - f(r) = 0. \]  

Now we construct a homotopy \( \upsilon(r, p) : \Omega \times [0, 1] \rightarrow R \) which satisfies:

\[ H(\upsilon, p) = (1 - p)[L(\upsilon) - L(u_0)] + p[A(\upsilon) - f(r)] = L(\upsilon) - (1 - p)L(u_0) + p[N(\upsilon) - f(r)] = 0, p \in [0, 1], r \in \Omega, \]  

where \( p \in [0, 1] \) is an embedding parameter, \( u_0 \) is an initial approximation which satisfies the boundary conditions. Therefore, obviously we have:

\[ H(\upsilon, 0) = L(\upsilon) - L(u_0) = 0, \]

\[ H(\upsilon, 1) = A(\upsilon) - f(r) = 0. \]

Changing the process of \( p \) from zero to unity is just that of \( \upsilon(r, p) \) from \( u_0(r) \) to \( \upsilon(r) \). We assume that the solution of equation (4) can be written as a
power series in \( p \); i.e. \( v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \). By setting \( p = 1 \), the approximation solution of (1) is obtained.

In some PDEs we can select the operator \( L \) in different ways. According to the definition of \( H(v, p) \), the approximate solution that is obtained from \( H(v, p) \) depends on operator \( L \) and initial conditions. Therefore, in some problems different approximate solutions are obtained. To show the subject consider the following problem

\[
 u_t + [f(u)]_x = [g(u)]_{xx} + h(u),
\]

with the following conditions:

\[
 u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1,
\]

\[
 u(0, t) = p(t), \quad 0 \leq t \leq 1,
\]

\[
 u(1, t) = q(t), \quad 0 \leq t \leq T,
\]

where \( 0 \leq u(x, t) \leq C \); \( C \) and \( T \) are given constant numbers. Besides, \([f(u)]_x\), \([g(u)]_{xx}\), and \( h(u) \) represent nonlinear advection, nonlinear diffusion, and nonlinear reaction terms respectively. This problem is widely used to describe many important phenomena and dynamics, mechanics, chemistry, biology, and etc. By defining the \( L = L_t \), we can seek the solution of problem based on HPM as:

\[
 \hat{u}(x, t) = \sum_{n=0}^{\infty} \hat{u}_n(x, t).
\]

Likewise, by defining the \( L = L_{xx} \), we can seek another approximate solution of the problem based on HPM as:

\[
 \tilde{u}(x, t) = \sum_{n=0}^{\infty} \tilde{u}_n(x, t).
\]

Let us define \( \hat{u}_n(x, t) = \sum_{n=0}^{n-1} u_n(x, t) \) and \( \tilde{u}_n(x, t) = \sum_{n=0}^{n-1} \tilde{u}_n(x, t) \). By combining these two approximate solutions, the better approximation solution can be obtained. For this purpose, let:

\[
 u_n(x, t) = \alpha_n \hat{u}_n + \beta_n \tilde{u}_n.
\]

Some authors used \( \alpha_n = \beta_n = \frac{1}{2} \) in their works to combine two approximate solutions that are obtained by Adomian decomposition method [16]. The optimum values of \( \alpha_n \) and \( \beta_n \) are discussed in [17] for the approximate solutions that are obtained by Adomian decomposition method. The best values for \( \alpha_n \)
and $\beta_n$ in (11) can be obtained for each $n$. For instance, consider problem (5)-(8). The residual function for this problem is defined as:

$$J_n = \|u_n(0, t) - p(t)\|^2 + \|u_n(1, t) - q(t)\|^2 + \|u_n(x, 0) - \varphi(x)\|^2.$$  

(12)

The values of $\alpha_n$ and $\beta_n$ are chosen such that this residual is minimized. The initial approximation for this problem (i.e. (5)-(8)), can be obtained by using $u_0 = (1 - x)p(t) + xq(t)$.

### 3 Numerical experiments

To show the efficiency of the methods that is described in section 2 (i.e. HPM and IHPM), we test with the following experiments. The convergency of the method is also considered.

**Experiment 1.** Consider linear PDE $u_{tt} + u_{xx} + u_{xt} = 2(x + t)$, with the following conditions: $u(x, 0) = ax$, $u_t(x, 0) = x^2$, $u(0, t) = 0$, and $u_t(0, t) = a$. First we set $L = L_{tt} = \frac{\partial^4}{\partial t^4}$. According to the HPM that is defined in section 2, we have:

$$H(v, p) = \frac{\partial^2}{\partial t^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...) - (1 - p)\frac{\partial^2}{\partial t^2}v_0$$

$$+ p\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...) + \frac{\partial^2}{\partial x\partial t}(v_0 + pv_1 + p^2v_2$$

$$+ p^3v_3 + p^4v_4 + p^5v_5 + ...) - 2(x + t) = 0.$$  

The initial approximation (i.e. $u_0$) can be obtained from initial and boundary conditions $(u_0 = tu_t(x, 0) + xu_x(0, t))$. By equating the coefficients of $p$ to zero, we obtain:

- coefficient of $p^0: \frac{\partial^2v_0}{\partial t^2} - \frac{\partial^2v_0}{\partial x^2} = 0, \Rightarrow v_0 = u_0 = ax + x^2t,$
- coefficient of $p^1: \frac{\partial^2v_1}{\partial t^2} + \frac{\partial^2v_0}{\partial x^2} + \frac{\partial^2v_1}{\partial t^2} + \frac{\partial^2v_0}{\partial x\partial t} - 2(x + t) = 0, \Rightarrow v_1 = 0,$
- coefficient of $p^2: \frac{\partial^2v_2}{\partial t^2} + \frac{\partial^2v_1}{\partial x^2} + \frac{\partial^2v_2}{\partial t^2} = 0, \Rightarrow v_2 = 0,$
- \ldots
- coefficient of $p^n: \frac{\partial^2v_n}{\partial t^2} + \frac{\partial^2v_{n-1}}{\partial x^2} + \frac{\partial^2v_{n-1}}{\partial x\partial t} = 0, \Rightarrow v_n = 0.$

Therefore, we obtain $u(x, t) = ax + x^2t$, which is the exact solution of the problem.

Now we set $L = L_{xx} = \frac{\partial^2}{\partial x^2}$. According to the HPM that is defined in section 2, we have:

$$H(v, p) = \frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...) - (1 - p)\frac{\partial^2}{\partial x^2}v_0$$
Improved homotopy perturbation method

By equating the coefficients of \( p \) to zero, we obtain:

- coefficient of \( p^0 \):
  \[
  \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial t^2} = 0, \Rightarrow u_0 = ax + x^2t,
  \]

- coefficient of \( p^1 \):
  \[
  \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial t^2} - 2(x + t) = 0, \Rightarrow v_1 = 0,
  \]

- coefficient of \( p^2 \):
  \[
  \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_1}{\partial t^2} = 0, \Rightarrow v_2 = 0,
  \]

- coefficient of \( p^n \):
  \[
  \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_{n-1}}{\partial t^2} = 0, \Rightarrow v_n = 0.
  \]

Therefore, we obtain \( u(x, t) = ax + x^2t \), which is the exact solution of the problem. Since finite numbers of \( v_i \) are not zero, the convergence of the solution is trivial and HPM is equivalent to IHPM. Each number in \([0, 1]\), can be the optimum value for \( \alpha_n \). In this case we set \( \beta_n = 1 - \alpha_n \).

In this experiment, by using \( u_0 = x^2t \) (initial condition), and \( L = L_{tt} = \frac{\partial^2}{\partial t^2} \), we obtain:

- \( v_0 = x^2t \),
- \( v_1 = 0 \),
- \( v_2 = 0 \),
- \( v_3 = 0 \),
- \[
  \vdots
  \]
- \( v_n = 0 \).

Therefore, we obtain \( \tilde{u}(x, t) = x^2t \). In the same way, by using \( u_0 = ax \) (initial condition), and \( L = L_{xx} = \frac{\partial^2}{\partial x^2} \), we obtain:

- \( v_0 = ax \),
- \( v_1 = \frac{1}{3}x^3 + x^2t \),
- \( v_2 = -\frac{1}{3}x^3 \),
- \( v_3 = 0 \),
- \[
  \vdots
  \]
- \( v_n = 0 \).

Therefore, we obtain \( \tilde{u}(x, t) = ax + x^2t \). The optimum values of \( \alpha_n \) and \( \beta_n \) are \( \alpha_n = 0 \) and \( \beta_n = 1 \). So, the exact solution is obtained. This experiment shows that IHPM with these initial conditions (i.e. \( u_0 = x^2t \) and \( u_0 = ax \)) can obtain exact solution but HPM with \( u_0 = x^2t \) can not obtain the exact
solution.

**Experiment 2.** Consider linear PDE $u_t = u_{xx} - u_x$, with the following conditions: $u(x, 0) = e^x - x$, $u(0, t) = 1 + t$, and $u_x(1, t) = e - 1$. First we set $L = L_t = \frac{\partial}{\partial t}$. According to the HPM, we have:

$$H(v, p) = \frac{\partial}{\partial t}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)-(1-p)\frac{\partial u_0}{\partial t}$$

$$+ p\left[\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)- \frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...]\right] = 0.$$

By equating the coefficients of $p$ to zero, we obtain:

- coefficient of $p^0: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0$, $\Rightarrow v_0 = u_0 = e^x - x$,
- coefficient of $p^1: \frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial x} + \frac{\partial v_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} = 0$, $\Rightarrow \frac{\partial v_1}{\partial t} = 1$, $\Rightarrow v_1 = t$,
- coefficient of $p^2: \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial x} - \frac{\partial^2 v_0}{\partial x^2} = 0$, $\Rightarrow \frac{\partial v_2}{\partial t} = 0$, $\Rightarrow v_2 = 0$,
- coefficient of $p^3: \frac{\partial v_3}{\partial x} + \frac{\partial v_2}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} = 0$, $\Rightarrow \frac{\partial v_3}{\partial t} = 0$, $\Rightarrow v_3 = 0$,

$\vdots$

- coefficient of $p^n: \frac{\partial v_n}{\partial x} + \frac{\partial v_{n-1}}{\partial x} - \frac{\partial^2 v_{n-1}}{\partial x^2} = 0$, $\Rightarrow \frac{\partial v_n}{\partial t} = 0$, $\Rightarrow v_n = 0$.

Therefore, we obtain $u(x, t) = e^x - x + t$, which is the exact solution of the problem.

Now we set $L = L_x = \frac{\partial}{\partial x}$. According to the HPM, we have:

$$H(v, p) = \frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)-(1-p)\frac{\partial u_0}{\partial x}$$

$$+ p\left[\frac{\partial}{\partial t}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)- \frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...]\right] = 0.$$

By equating the coefficients of $p$ to zero, we obtain:

- coefficient of $p^0: \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial x} = 0$, $\Rightarrow v_0 = u_0 = 1 + t$,
- coefficient of $p^1: \frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial x} + \frac{\partial v_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} = 0$, $\Rightarrow \frac{\partial v_1}{\partial t} = -1$, $\Rightarrow v_1 = -x$,
- coefficient of $p^2: \frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial x} - \frac{\partial^2 v_0}{\partial x^2} = 0$, $\Rightarrow v_2 = 0$,
- coefficient of $p^3: \frac{\partial v_3}{\partial x} + \frac{\partial v_2}{\partial x} - \frac{\partial^2 v_1}{\partial x^2} = 0$, $\Rightarrow v_3 = 0$,

$\vdots$

- coefficient of $p^n: \frac{\partial v_n}{\partial x} + \frac{\partial v_{n-1}}{\partial x} - \frac{\partial^2 v_{n-1}}{\partial x^2} = 0$, $\Rightarrow v_n = 0$.

Therefore, we obtain $u(x, t) = 1 + t - x$. Consequently HPM by using $u_0 = 1 + t$ can not obtain the exact solution of the problem. The optimum values of $\alpha_n$ and $\beta_n$ for this problem, according to the (12), are $\alpha_n = 1$ and
\( \beta_n = 0 \). So, the exact solution is obtained.

**Experiment 3.** Consider linear PDE \((\text{Klein-Gordon equation})\) \( u_{tt} - u_{xx} + u = 0 \), with the following conditions: \( u(x,0) = x + e^{-x}, u_t(x,0) = 0 \). First we set \( L = L_{tt} = \frac{\partial^2}{\partial x^2} \). According to the HPM, we have:

\[
H(v,p) = \frac{\partial^2}{\partial t^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots) - (1 - p)\frac{\partial^2 u_0}{\partial t^2} - p\left[\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots) - (v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots)\right] = 0.
\]

By equating the coefficients of \( p \) to zero, we obtain:

- coefficient of \( p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \), \( \Rightarrow v_0 = u_0 = e^{-x} + x \),
- coefficient of \( p^1 : \frac{\partial^2 v_1}{\partial t^2} + \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial^2 v_3}{\partial x^2} + v_0 = 0 \), \( \Rightarrow v_1 = -\frac{t^2}{24}x \),
- coefficient of \( p^2 : \frac{\partial^2 v_2}{\partial t^2} - \frac{\partial^2 v_2}{\partial x^2} + v_1 = 0 \), \( \Rightarrow v_2 = \frac{t^4}{48}x \),

\[ \vdots \]

- coefficient of \( p^n \) : \( \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 v_n}{\partial x^2} + v_{n-1} = 0 \), \( \Rightarrow v_n = \frac{(-1)^nu_n}{(2n)!}x \).

Therefore, by using Maclaurin’s formula, we obtain \( u(x,t) = e^{-x} + x(1 - \frac{t^2}{24} + \frac{t^4}{48} - \ldots + \frac{(-1)^nu_n}{(2n)!} + \ldots) = e^{-x} + x \cos(t) \), which is the exact solution of the problem. In this experiment, there is a unique way to choose \( L \).

**Experiment 4.** Consider nonlinear PDE \( u_{tt} + u_{xx} + u_x^2 = 2x + t^4 \), with the following conditions: \( u(x,0) = 0, u_t(x,0) = a, u(0,t) = at \), and \( u_x(0,t) = t^2 \). First, we set \( L = L_{tt} = \frac{\partial^2}{\partial x^2} \). According to the HPM, we have:

\[
H(v,p) = \frac{\partial^2}{\partial t^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots) - (1 - p)\frac{\partial^2 u_0}{\partial t^2} + p\left[\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots) + \left(\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots)\right)^2 - 2x - t^4\right] = 0.
\]

The initial approximation can be obtained from initial and boundary conditions (i.e. \( u_0 = tu_t(x,0) + xu_x(0,t) = at + xt^2 \)). By equating the coefficients of \( p \) to zero, we obtain:

- coefficient of \( p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \), \( \Rightarrow v_0 = u_0 = at + xt^2 \),
- coefficient of \( p^1 : \frac{\partial^2 v_1}{\partial t^2} + 2x + t^4 - 2x - t^4 = 0 \), \( \Rightarrow v_1 = 0 \),
- coefficient of \( p^2 : \frac{\partial^2 v_2}{\partial t^2} + \frac{\partial^2 v_1}{\partial x^2} + (\frac{\partial}{\partial x}2v_0v_1)^2 = 0 \), \( \Rightarrow v_2 = 0 \),

\[ \vdots \]

Therefore, we obtain \( u(x,t) = at + xt^2 \), which is the exact solution of the problem.
Now we set $L = L_{xx} = \frac{\partial^2}{\partial x^2}$. According to the HPM, we have:

\[
H(v, p) = \frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + \cdots) - (1 - p)\frac{\partial^2u_0}{\partial x^2}
\]

\[
+ p\left[\frac{\partial^2}{\partial t^2}(v_0 + pv_1 + p^2v_2 + \cdots) + \left(\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + \cdots)\right)^2 - 2x - t^4\right] = 0.
\]

By equating the coefficients of $p$ to zero, we obtain:

- coefficient of $p^0$: $\frac{\partial^2v_0}{\partial x^2} = 0, \Rightarrow v_0 = u_0 = at + xt^2$,
- coefficient of $p^1$: $\frac{\partial^2v_1}{\partial x^2} + \frac{\partial^2v_0}{\partial x^2} + \left(\frac{\partial v_0}{\partial x}\right)^2 - 2x - t^4 = 0, \Rightarrow v_1 = 0$,
- coefficient of $p^2$: $\frac{\partial^2v_2}{\partial x^2} + \frac{\partial^2v_1}{\partial x^2} + \left(\frac{\partial^2v_0}{\partial x^2} 2u_0v_1\right)^2 = 0, \Rightarrow v_2 = 0$,

\vdots

Therefore, we obtain $u(x, t) = at + xt^2$, which is the exact solution of the problem. Since finite numbers of $v_i$ are not zero, the convergency of the solution is trivial and HPM is equivalent to IHPM in this case. Each number in $[0, 1]$, can be the optimum value of $\alpha_n$. In this case, we set $\beta_n = 1 - \alpha_n$.

In this experiment, by using $u_0 = at$, $L = L_{tt} = \frac{\partial^2}{\partial t^2}$, and HPM, we obtain:

\[
v_0 = at,
\]

\[
v_1 = xt^2 + \frac{1}{30}t^6,
\]

\[
v_2 = 0,
\]

\[
v_3 = -\frac{1}{30}t^6,
\]

\[
v_4 = 0,
\]

\vdots

\[
v_n = 0.
\]

Therefore, we obtain $u(x, t) = at + xt^2$. In the same way, by using $u_0 = xt^2$, and HPM, we obtain:

\[
v_0 = xt^2,
\]

\[
v_1 = 0,
\]

\[
v_2 = 0,
\]

\vdots

\[
v_n = 0.
\]

Therefore, we obtain $u(x, t) = xt^2$. The optimum values of $\alpha_n$ and $\beta_n$ are $\alpha_n = 1$ and $\beta_n = 0$. So, the exact solution of the problem is obtained. By
using \( L = L_t \), and \( u_0 = xt^2 \) HPM cannot obtain exact solution but IHPM can obtain the exact solution.

**Experiment 5.** Consider nonlinear PDE (Burger equation) \( u_t = u_{xx} + uu_x \), with the following conditions: \( u(x, 0) = 1 - x, u(0, t) = \frac{1}{1 + t}, u(1, t) = 0 \).

First we set \( L = L_t = \frac{\partial}{\partial t} \). According to the HPM, we have:

\[
H(v, p) = \frac{\partial}{\partial t} \left( v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots \right) - (1 - p) \frac{\partial u_0}{\partial t}
\]

\[+p[-\frac{\partial^2}{\partial x^2} \left( v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots \right) - (v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots)] = 0.\]

By equating the coefficients of \( p \) to zero, we obtain:

- coefficient of \( p^0 \): \( \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \), \( \Rightarrow v_0 = u_0 = 1 - x \),
- coefficient of \( p^1 \): \( \frac{\partial v_1}{\partial t} + \frac{\partial u_1}{\partial t} - \frac{\partial^2 v_2}{\partial x^2} - v_0 \frac{\partial v_0}{\partial x} = 0 \), \( \Rightarrow v_1 = -t(1 - x) \),
- coefficient of \( p^2 \): \( \frac{\partial v_2}{\partial t} + \frac{\partial u_2}{\partial t} - \frac{\partial^2 v_3}{\partial x^2} - v_0 \frac{\partial v_1}{\partial x} - v_1 \frac{\partial v_0}{\partial x} = 0 \), \( \Rightarrow v_2 = t^2(1 - x) \),
- coefficient of \( p^3 \): \( \frac{\partial v_3}{\partial t} + \frac{\partial u_3}{\partial t} - \frac{\partial^2 v_4}{\partial x^2} - v_0 \frac{\partial v_2}{\partial x} - v_1 \frac{\partial v_1}{\partial x} - v_2 \frac{\partial v_0}{\partial x} = 0 \), \( \Rightarrow v_3 = t^3(1 - x) \),

Therefore, we obtain \( u(x, t) = (1 - t + t^2 - t^3 + \ldots)(1 - x) = \frac{1 - x}{1 + t} \), which is the exact solution of the problem.

Now, we set \( L = L_{xx} = \frac{\partial^2}{\partial x^2} \). According to the HPM, we have:

\[
H(v, p) = \frac{\partial^2}{\partial x^2} \left( v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots \right) - (1 - p) \frac{\partial^2 u_0}{\partial x^2}
\]

\[+p[-\frac{\partial}{\partial t} \left( v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots \right) + (v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + \ldots)] = 0.\]

By equating the coefficients of \( p \) to zero, we obtain:

- coefficient of \( p^0 \): \( \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \), \( \Rightarrow v_0 = u_0 = \frac{1 - x}{1 + t} \),
- coefficient of \( p^1 \): \( \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 v_2}{\partial x^2} - v_0 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} = 0 \), \( \Rightarrow v_1 = 0 \),
- coefficient of \( p^2 \): \( \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial^2 v_1}{\partial x^2} + v_0 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_2}{\partial x} = 0 \), \( \Rightarrow v_2 = 0 \),

Therefore, we obtain \( u(x, t) = \frac{1 - x}{1 + t} \), which is the exact solution of the problem. Since finite numbers of \( v_i \) are not zero, the convergency of the solution is trivial and HPM is equivalent to IHPM. Each number in \([0, 1]\), can be the optimum value of \( \alpha_n \). In this case, we set \( \beta_n = 1 - \alpha_n \).
Experiment 6. Consider nonlinear PDE \( u_t = uu_{xx} + u^2_x + u \), with the following conditions: \( u(x,0) = \sqrt{x}, u(0,t) = 0, u(1,t) = e^t \). First we set \( L = L_t = \frac{\partial}{\partial t} \). According to the HPM, we have:

\[
H(v,p) = \frac{\partial}{\partial t}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...) - (1 - p)\frac{\partial u_0}{\partial t}
\]

\[-p[(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)(\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)] + (\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)) \cdot (v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)] = 0.
\]

By equating the coefficients of \( p \) to zero, we obtain:

- coefficient of \( p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \Rightarrow v_0 = u_0 = \sqrt{x}, \)
- coefficient of \( p^1 : \frac{\partial v_1}{\partial t} + v_0 \frac{\partial^2 u_0}{\partial x^2} - (\frac{\partial u_0}{\partial x})^2 + v_0 = 0 \Rightarrow v_1 = t\sqrt{x}, \)
- coefficient of \( p^2 : \frac{\partial v_2}{\partial t} - [v_0 \frac{\partial^2 u_0}{\partial x^2} + v_1 \frac{\partial^2 u_1}{\partial x^2} + 2v_0 \frac{\partial u_0}{\partial x} \frac{\partial u_1}{\partial x} + v_2] = 0 \Rightarrow v_2 = \frac{t^2}{2} \sqrt{x}, \)

Therefore, by using Maclaurin’s formula, we obtain \( \hat{u}(x,t) = \sqrt{x}(1 + t + \frac{t^2}{2} + ...) = \sqrt{x}e^t \), which is the exact solution of the problem. In this experiment, there is a unique way to choose \( L \).

Experiment 7. Consider nonlinear PDE \( u_t + u^2u_x = 0 \), with the following condition: \( u(x,0) = 3x \). We set \( L = L_t = \frac{\partial}{\partial t} \). According to the HPM, we have:

\[
H(v,p) = \frac{\partial}{\partial t}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...) - (1 - p)\frac{\partial u_0}{\partial t}
\]

\[+p[(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...)^2(\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 + ...))] = 0.
\]

By equating the coefficients of \( p \) to zero, we obtain:

- coefficient of \( p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \Rightarrow v_0 = u_0 = 3x, \)
- coefficient of \( p^1 : \frac{\partial v_1}{\partial t} + v_0 \frac{\partial^2 u_0}{\partial x^2} + 2v_0 v_1 \frac{\partial u_0}{\partial x} = 0 \Rightarrow v_1 = -27x^2t, \)
- coefficient of \( p^2 : \frac{\partial v_2}{\partial t} + v_0^2 \frac{\partial^2 u_0}{\partial x^2} + 2v_0 v_1 \frac{\partial u_0}{\partial x} + v_1 \frac{\partial^2 u_0}{\partial x^2} + v_2 = 486x^3t^2, \)
- coefficient of \( p^3 : \frac{\partial v_3}{\partial t} + v_0^2 \frac{\partial^2 u_0}{\partial x^2} + 2v_0 v_1 \frac{\partial u_0}{\partial x} + v_1 \frac{\partial u_0}{\partial x} + v_2 \frac{\partial^2 u_0}{\partial x^2} + v_3 = -10935x^4t^3, \)

Therefore, by using Maclaurin’s formula, we obtain:

\[u(\hat{x},t) = (3x - 27x^2t + 486x^3t^2 - 10935x^4t^3 + ...)\]
\[
\begin{cases}
3x, & \text{for } (t = 0), \\
\frac{1}{4t}(\sqrt{1+36xt} - 1), & \text{for } (t > 0),
\end{cases}
\]

which is the exact solution of the problem. In this experiment, there is a unique way to choose \(L\).

4 Convergency of the method

An approximate solution of problem (1) is expressed as:

\[
v = \sum_{i=0}^{\infty} v_i.
\]

(13)

To consider the convergency of this series, we can apply classical test such as the root and the ratio tests [18].

**Theorem 1** (Root Test) Given (13), put \(\alpha = \lim sup_{n \to \infty} \sqrt[n]{|v_n|}\). Then:

(a) if \(\alpha < 1\), then (13) converges;
(b) if \(\alpha > 1\), then (13) diverges;
(a) if \(\alpha = 1\), the test gives no information.

**Proof.** See [18].

**Theorem 2** (Ratio Test) The series (13),

(a) converges if \(\lim sup_{n \to \infty} \left| \frac{v_{n+1}}{v_n} \right| < 1\),

(b) diverges if \(\left| \frac{v_{n+1}}{v_n} \right| \geq 1\) for all \(n \geq n_0\), where \(n_0\) is some fixed integer.

**Proof.** See [18].

For expressed experiments in section 3, with infinite non-zero \(v_i\), the convergency of the approximate solutions can be obtained easily by ratio or root tests.

5 Conclusions

According to the approximate solutions that we have obtained, we infer that **IHPM** is a powerful tool for solving the linear, and nonlinear equations. Expressed experiments show that **IHPM** is superior to **HPM**. This method is easy to implement and the convergency shows that the method can gives good approximate solution for different types of evolution equations.
References


Received: January, 2010