A Mixed Lyapunov-Max-Plus Algebra Approach to the Stability Problem for a Two Species Ecosystem Modeled with Timed Petri Nets

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Abstract

Consider the interaction of populations, in which there are exactly two species, one of which the *predators* eats the other the *prey* thereby affecting each other’s growth rates. In the study of this growth rate dynamics Lotka-Volterra models have been used. This paper proposes a new modeling and analysis methodology which consists in considering the growth rate dynamics as a discrete event system. A discrete event system, is a dynamical system whose state evolves in time by the occurrence of events at possibly irregular time intervals. Place-transitions Petri nets (commonly called Petri nets) are a graphical and mathematical modeling tool applicable to discrete event systems in order to represent its states evolution. Timed Petri nets are an extension of Petri nets that model discrete event systems where now the timing at which the state changes is taken into consideration. One of the most important performance issues to be considered in a predator-prey system is its stability. Lyapunov stability theory provides the required tools needed to aboard the stability problem for the predator-prey system treated as a discrete event system modeled with timed petri nets. Employing Lyapunov methods, a sufficient condition for the stabilization problem is obtained. It is shown that it is possible to restrict the discrete event systems’ state space in such a way that boundedness is guaranteed. By proving boundedness one confirms a dominant oscillating behavior of both populations dynamics performance. However, the oscillating frequency results to be unknown. This inconvenience is overcome by considering a specific recurrence equation, in the max-plus algebra, which is assigned to the timed Petri net graph.
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1 Introduction

Consider the interaction of populations, in which there are exactly two species, one of which the predators eats the other the prey thereby affecting each other’s growth rates. Such pairs exist throughout nature: fish and sharks, lions and gazelles, birds and insects, to mention some. In the study of this growth rate dynamics Lotka-Volterra models have been used [5]. This paper proposes a new modeling and analysis methodology which consists in considering the predator-prey system as a discrete event system. A discrete event system, is a dynamical system whose state evolves in time by the occurrence of events at possibly irregular time intervals. Place-transitions Petri nets (commonly called Petri nets) are a graphical and mathematical modeling tool applicable to discrete event systems in order to represent its states evolution. Petri nets are known to be useful for analyzing the systems properties. Timed Petri nets are an extension of Petri nets that model discrete event systems where now the timing at which the state changes is taken into consideration. This is of critical importance since it allows to consider useful measures of performance as for example: how long does the discrete event system spends at a given state etc. One of the most important performance issues to be considered in a predator-prey system is its stability. Lyapunov stability theory provides the required tools needed to aboard the stability problem for the predator-pray system treated as a discrete event system modeled with timed petri nets whose mathematical model is given in terms of difference equations [3]. Employing Lyapunov methods, a sufficient condition for the stabilization problem is also obtained. It is shown that it is possible to restrict the discrete event systems’ state space in such a way that boundedness is guaranteed. By proving boundedness one confirms a dominant oscillating behavior of both populations dynamics performance. However, the oscillating frequency results to be unknown. This inconvenience is overcome by considering a specific recurrence equation, in the max-plus algebra, which is assigned to the the timed Petri net graphical model. The main contribution of the paper consists in combining Lyapunov theory with max-plus algebra to study the stability problem for predator-pray systems treated as discrete event dynamical systems modeled with timed Petri nets. This results in a qualitative approach vs the quanti-
tative approach obtained by solving the Lotka-Volterra differential equations system that models it. The presented methodology is new and results to be innovative. The paper is organized as follows. In section 2, Lyapunov theory for discrete event modeled with Petri nets is addressed. Section 3, presents Max-Plus algebra. In section 4, the stability for discrete event dynamical systems modeled with timed Petri nets is addressed. Section 5, discusses the predator-prey dynamical system's stability using the information and tools previously presented in the past sections. Finally, the paper ends with some conclusions.


The solution to the stability problem for Discrete event systems, whose model is obtained employing timed Petri nets, is achieved thanks to the theory of vector Lyapunov functions and comparison principles. The methodology shows that it is possible to restrict the systems state space in such a way that boundedness is guaranteed.

**NOTATION:** $N = \{0, 1, 2, \ldots \}, R_+ = [0, \infty), N^+_{n_0} = \{n_0, n_0 + 1, \ldots, n_0 + k, \ldots \}, n_0 \geq 0$. Given $x, y \in R^n$, we usually denote the relation “$\leq$” to mean component-wise inequalities with the same relation, i.e., $x \leq y$ is equivalent to $x_i \leq y_i, \forall i$. A function $f(n, x), f : N^+_{n_0} \times R^n \to R^n$ is called nondecreasing in $x$ if given $x, y \in R^n$ such that $x \geq y$ and $n \in N^+_{n_0}$ then, $f(n, x) \geq f(n, y)$.

Consider systems of first ordinary difference equations given by

$$x(n + 1) = f[n, x(n)], x(n_0) = x_0, n \in N^+_{n_0}$$

where $n \in N^+_{n_0}, x(n) \in R^n$ and $f : N^+_{n_0} \times R^n \to R^n$ is continuous in $x(n)$.

**Definition 1** The $n$ vector valued function $\Phi(n, n_0, x_0)$ is said to be a solution of (1) if $\Phi(n_0, n_0, x_0) = x_0$ and $\Phi(n + 1, n_0, x_0) = f(n, \Phi(n, n_0, x_0))$ for all $n \in N^+_{n_0}$.

**Definition 2** The system (1) is said to be practically stable, if given $(\lambda, A)$ with $0 < \lambda < A$, then

$$|x_0| < \lambda \Rightarrow |x(n, n_0, x_0)| < A, \forall n \in N^+_{n_0}, n_0 \geq 0.$$ 

If it is practically stable for every $n_0 \geq 0$ then, is said to be uniformly practically stable.
The following class of function is defined.

**Definition 3** A continuous function \( \alpha : [0, \infty) \to [0, \infty) \) is said to belong to class \( \mathcal{K} \) if \( \alpha(0) = 0 \) and it is strictly increasing.

Consider a vector Lyapunov function \( v(n, x(n)) \), \( v : N_{n_0}^+ \times R^n \to R^p \) and define the variation of \( v \) relative to (1) by

\[
\Delta v = v(n+1, x(n+1)) - v(n, x(n))
\]

Then, the following result concerns the practical stability of (1).

**Theorem 4** Let \( v : N_{n_0}^+ \times R^n \to R^p \) be a continuous function in \( x \), define the function \( v_0(n, x(n)) = \sum_{i=1}^{p} v_i(n, x(n)) \) such that satisfies the estimates

\[
b(|x|) \leq v_0(n, x(n)) \leq a(|x|) \quad \text{for} \quad a, b \in \mathcal{K}
\]

\[
\Delta v(n, x(n)) \leq w(n, v(n, x(n)))
\]

for \( n \in N_{n_0}^+, \ x(n) \in R^n \), where \( w : N_{n_0}^+ \times R^p_+ \to R^p \) is a continuous function in the second argument.

Assume that : \( g(n, e) \triangleq e + w(n, e) \) is nondecreasing in \( e \), \( 0 < \lambda < A \) are given and finally that \( a(\lambda) < b(A) \) is satisfied. Then, the practical stability properties of

\[
e(n+1) = g(n, e(n)), \ e(n_0) = e_0 \geq 0.
\]

imply the practical stability properties of system (1).

**Corollary 5** In Theorem (4) if \( w(n, e) \equiv 0 \) we get uniform practical stability of (1) which implies structural stability.

**Definition 6** A Petri net is a 5-tuple, \( PN = \{ P, T, F, W, M_0 \} \) where:

- \( P = \{ p_1, p_2, ..., p_m \} \) is a finite set of places,
- \( T = \{ t_1, t_2, ..., t_n \} \) is a finite set of transitions,
- \( F \subset (P \times T) \cup (T \times P) \) is a set of arcs,
- \( W : F \to N_1^+ \) is a weight function,
- \( M_0 : P \to N \) is the initial marking,
- \( P \cap T = \emptyset \) and \( P \cup T \neq \emptyset \).
Definition 7 The clock structure associated with a place \( p_i \in P \) is a set \( V = \{ V_i : p_i \in P \} \) of clock sequences \( V_i = \{ v_{i,1}, v_{i,2}, \ldots \} \), \( v_{i,k} \in R^+, k = 1, 2, \ldots \). The positive number \( v_{i,k} \), associated to \( p_i \in P \), called holding time, represents the time that a token must spend in this place until its outputs enabled transitions \( t_{i,1}, t_{i,2}, \ldots \), fire. Some places may have a zero holding time while others not. Thus, we partition \( P \) into subsets \( P_0 \) and \( P_h \), where \( P_0 \) is the set of places with zero holding time, and \( P_h \) is the set of places that have some holding time.

Definition 8 A timed Petri net is a 6-tuple \( TPN = \{ P, T, F, W, M_0, V \} \) where \( \{ P, T, F, W, M_0 \} \) are as before, and \( V = \{ V_i : p_i \in P \} \) is a clock structure. A timed Petri net is a timed event petri net when every \( p_i \in P \) has one input and one output transition, in which case the associated clock structure set of a place \( p_i \in P \) reduces to one element \( V_i = \{ v_i \} \).

A PN structure without any specific initial marking is denoted by \( N \). A Petri net with the given initial marking is denoted by \((N, M_0)\). Notice that if \( W(p, t) = \alpha \) (or \( W(t, p) = \beta \)) then, this is often represented graphically by \( \alpha \), (\( \beta \)) arcs from \( p \) to \( t \) (t to \( p \)) each with no numeric label. Let \( M_k(p_i) \) denote the marking (i.e., the number of tokens) at place \( p_i \in P \) at time \( k \) and let \( M_k = [M_k(p_1), \ldots, M_k(p_m)]^T \) denote the marking (state) of \( PN \) at time \( k \). A transition \( t_j \in T \) is said to be enabled at time \( k \) if \( M_k(p_i) \geq W(p_i, t_j) \) for all \( p_i \in P \) such that \( (p_i, t_j) \in F \). It is assumed that at each time \( k \) there exists at least one transition to fire. If a transition is enabled then, it can fire. If an enabled transition \( t_j \in T \) fires at time \( k \) then, the next marking for \( p_i \in P \) is given by

\[
M_{k+1}(p_i) = M_k(p_i) + W(t_j, p_i) - W(p_i, t_j).
\] (4)

Let \( A = [a_{ij}] \) denote an \( n \times m \) matrix of integers (the incidence matrix) where \( a_{ij} = a_{ij}^+ - a_{ij}^- \) with \( a_{ij}^+ = W(t_i, p_j) \) and \( a_{ij}^- = W(p_j, t_i) \). Let \( u_k \in \{0, 1\}^n \) denote a firing vector where if \( t_j \in T \) is fired then, its corresponding firing vector is \( u_k = [0, \ldots, 0, 1, 0, \ldots, 0]^T \) with the one in the \( j \)th position in the vector and zeros everywhere else. The matrix equation (nonlinear difference equation) describing the dynamical behavior represented by a \( PN \) is:

\[
M_{k+1} = M_k + A^T u_k
\] (5)

where if at step \( k \), \( a_{ij}^- < M_k(p_j) \) for all \( p_i \in P \) then, \( t_i \in T \) is enabled and if this \( t_i \in T \) fires then, its corresponding firing vector \( u_k \) is utilized in the
difference equation to generate the next step. Notice that if $M'$ can be reached from some other marking $M$ and, if we fire some sequence of $d$ transitions with corresponding firing vectors $u_0, u_1, ..., u_{d-1}$ we obtain that

$$M' = M + A^T u, \quad u = \sum_{k=0}^{d-1} u_k.$$  \hspace{1cm} (6)

Let $(N_{n_0}^m, d)$ be a metric space where $d : N_{n_0}^m \times N_{n_0}^m \rightarrow \mathbb{R}^+$ is defined by

$$d(M_1, M_2) = \sum_{i=1}^{m} \zeta_i | M_1(p_i) - M_2(p_i) |; \quad \zeta_i > 0$$

and consider the matrix difference equation which describes the dynamical behavior of the discrete event system modeled by a $PN$

$$M' = M + A^T u, \quad u = \sum_{k=0}^{d-1} u_k$$  \hspace{1cm} (7)

where, $M \in N^m$, denotes the marking (state) of the $PN$, $A \in \mathbb{Z}^{n \times m}$, its incidence matrix and $u \in N^n$, is a sequence of firing vectors. Then, the following results concerns in what to the stability problem means.

**Proposition 9** Let $PN$ be a Petri net. $PN$ is uniform practical stable if there exists a $\Phi$ strictly positive $m$ vector such that

$$\Delta v = u^T A \Phi \leq 0$$  \hspace{1cm} (8)

**Lemma 10** Let suppose that Proposition (9) holds then,

$$\Delta v = u^T A \Phi \leq 0 \iff A \Phi \leq 0$$  \hspace{1cm} (9)

**Remark 11** Notice that since the state space of a TPN is contained in the state space of the same now not timed $PN$, stability of $PN$ implies stability of the TPN.
2.1 Lyapunov Stabilization

Notice, that in the solution of the stability problem, the \( u \) vector does not play any role, so why not to take advantage of it in order to get some specific behavior. Consider the matrix difference equation which describes the dynamical behavior of the discrete event system modeled by a Petri net

\[
M' = M + A^T u
\]

We are interested in finding a firing sequence vector, control law, such that system (7) remains bounded.

**Definition 12** Let \( PN \) be a Petri net. \( PN \) is said to be stabilizable if there exists a firing transition sequence with transition count vector \( u \) such that system (7) remains bounded.

**Proposition 13** Let \( PN \) be a Petri net. \( PN \) is stabilizable if there exists a firing transition sequence with transition count vector \( u \) such that the following equation holds

\[
\Delta v = A^T u \leq 0 \quad (10)
\]

**Remark 14** It is important to underline that by fixing a particular \( u \), which satisfies (10), we restrict the state space to those markings (states) that are finite. The technique can be utilized to get some type of regulation and/or eliminate some undesirable events (transitions). Notice that in general (8) \( \Rightarrow \) (10) and that the opposite is also true (this is illustrated with the following two examples).

(8) \( \Rightarrow \) (10) Consider the Petri net model shown in fig.1.

![Figure 1](image-url)
The incidence matrix which represents the model is

\[
A = \begin{bmatrix}
-1 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\] (11)

Then, picking \( \Phi = [1, 1, 1] \) uniform practical stability is concluded. However, there is no \( u \) such that \( A^T u \leq 0 \). (10) \( \not\Rightarrow \) (8). Consider the Petri net model depicted in fig. 2.

![Image](image_url)

Figure 2.

The structure is typical of an unbounded Petri net model in which the marking in \( p_1 \) can grow indefinitely due to the repeated firing of \( t_1 \). However, by taking \( u = [k, k], \ k > 0 \) equation (10) is satisfied therefore, the system becomes bounded i.e., is stabilizable.

**Remark 15** Notice that by firing all the transitions in the same proportion i.e., \( u = [k, k], \ k > 0 \) an unbounded PN becomes stable. This guarantees that there is no possibility that the marking will grow without bound at any place between two transitions. This basic idea motivates the definition of stability for TPN which will be given in section (4).

3 Max-Plus Algebra [1, 2, 4]

In this section the concept of max-plus algebra is defined. Its algebraic structure is described. Matrices and graphs are presented. The spectral theory of matrices is discussed. The problem of solving linear equations is addressed. Finally, max-plus recurrence equations for timed Petri nets are introduced.

3.1 Basic Definitions

**NOTATION:** \( \mathbb{N} \) is the set of natural numbers, \( \mathbb{R} \) is the set of real numbers, \( \epsilon = -\infty \), \( e = 0 \), \( \mathbb{R}_{max} = \mathbb{R} \cup \{\epsilon\} \), \( n = 1, 2, \ldots, n \)

Let \( a, b \in \mathbb{R}_{max} \) and define the operations \( \oplus \) and \( \otimes \) by: \( a \oplus b = \max(a, b) \) and \( a \otimes b = a + b \).
**Definition 16** The set $\mathbb{R}_{\text{max}}$ with the two operations $\oplus$ and $\otimes$ is called a max-plus algebra and is denoted by $\mathbb{R}_{\text{max}} = (\mathbb{R}_{\text{max}}, \oplus, \otimes, \epsilon, e)$.

**Definition 17** A semiring is a nonempty set $R$ endowed with two operations $\oplus_R$, $\otimes_R$, and two elements $\epsilon_R$ and $e_R$ such that: $\oplus_R$ is associative and commutative with zero element $\epsilon_R$, $\otimes_R$ is associative, distributes over $\oplus_R$, and has unit element $e_R$, $\in_R$ is absorbing for $\otimes_R$ i.e., $a \otimes_R \epsilon = \epsilon \otimes a = a, \forall a \in R$. Such a semiring is denoted by $\mathbb{R} = (R, \oplus_R, \otimes_R, \epsilon, e)$. In addition if $\otimes_R$ is commutative then $R$ is called a commutative semiring, and if $\oplus_R$ is such that $a \oplus_R a = a, \forall a \in R$ then it is called idempotent.

**Theorem 18** The max-plus algebra $\mathbb{R}_{\text{max}} = (\mathbb{R}_{\text{max}}, \oplus, \otimes, \epsilon, e)$ has the algebraic structure of a commutative and idempotent semiring.

### 3.2 Matrices and Graphs

Let $\mathbb{R}_{\text{max}}^{n \times n}$ be the set of $n \times n$ matrices with coefficients in $\mathbb{R}_{\text{max}}$ with the following operations: The sum of matrices $A, B \in \mathbb{R}_{\text{max}}^{n \times n}$, denoted $A \oplus B$ is defined by: $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max (a_{ij}, b_{ij})$ for $i$ and $j \in \mathbb{N}$. The product of matrices $A \in \mathbb{R}_{\text{max}}^{n \times l}$, $B \in \mathbb{R}_{\text{max}}^{l \times n}$, denoted $A \otimes B$ is defined by: $(A \otimes B)_{ik} = \bigotimes_{j=1}^{l} a_{ij} \otimes b_{jk} = \max \{a_{ij} + b_{jk}\}$ for $i$ and $k \in \mathbb{N}$. Let $E \in \mathbb{R}_{\text{max}}^{n \times n}$ denote the matrix with all its elements equal to $\epsilon$ and denote by $E \in \mathbb{R}_{\text{max}}^{n \times n}$ the matrix which has its diagonal elements equal to $e$ and all the other elements equal to $\epsilon$. Then, the following result can be stated.

**Theorem 19** The 5-tuple $\mathbb{R}_{\text{max}}^{n \times n} = (\mathbb{R}_{\text{max}}^{n \times n}, \oplus, \otimes, E, E)$ has the algebraic structure of a noncommutative idempotent semiring.

**Definition 20** Let $A \in \mathbb{R}_{\text{max}}^{n \times n}$ and $k \in \mathbb{N}$ then the $k$-th power of $A$ denoted by $A^{\otimes k}$ is defined by: $A^{\otimes k} = A \otimes A \otimes \cdots \otimes A$, where $A^{\otimes 0}$ is set equal to $E$.

**Definition 21** A matrix $A \in \mathbb{R}_{\text{max}}^{n \times n}$ is said to be regular if $A$ contains at least one element distinct from $\epsilon$ in each row.

**Definition 22** Let $\mathcal{N}$ be a finite and non-empty set and consider $\mathcal{D} \subseteq \mathcal{N} \times \mathcal{N}$. The pair $G = (\mathcal{N}, \mathcal{D})$ is called a directed graph, where $\mathcal{N}$ is the set of elements called nodes and $\mathcal{D}$ is the set of ordered pairs of nodes called arcs. A directed graph $G = (\mathcal{N}, \mathcal{D})$ is called a weighted graph if a weight $w(i, j) \in \mathbb{R}$ is associated with any arc $(i, j) \in \mathcal{D}$.
Let \( A \in \mathbb{R}^{n \times n}_{\text{max}} \) be any matrix, a graph \( G(A) \), called the communication graph of \( A \), can be associated as follows. Define \( \mathcal{N}(A) = n \) and a pair \((i, j) \in n \times n\) will be a member of \( \mathcal{D}(A) \iff a_{ji} \neq \epsilon \), where \( \mathcal{D}(A) \) denotes the set of arcs of \( G(A) \).

**Definition 23** A path from node \( i \) to node \( j \) is a sequence of arcs \( p = \{(i_k, j_k) \in \mathcal{D}(A)\}_{k=1}^{\infty} \) such that \( i = i_1, j_k = i_{k+1} \), for \( k < m \) and \( j_m = j \). The path \( p \) consists of the nodes \( i = i_1, i_2, \ldots, i_m, j_m = j \) with length \( m \) denoted by \( |p|_1 = m \). In the case when \( i = j \) the path is said to be a circuit. A circuit is said to be elementary if nodes \( i_k \) and \( i_1 \) are different for \( k \neq l \). A circuit consisting of one arc is called a self-loop.

Let us denote by \( P(i, j; m) \) the set of all paths from node \( i \) to node \( j \) of length \( m \geq 1 \) and for any arc \((i, j) \in \mathcal{D}(A)\) let its weight be given by \( a_{ij} \) then the weight of a path \( p \in P(i, j; m) \) denoted by \( |p|_w \) is defined to be the sum of the weights of all the arcs that belong to the path. The average weight of a path \( p \) is given by \( |p|_w / |p|_1 \). Given two paths, as for example, \( p = ((i_1, i_2), (i_2, i_3)) \) and \( q = ((i_3, i_4), (i_4, i_5)) \) in \( G(A) \) the concatenation of paths \( \circ : G(A) \times G(A) \rightarrow G(A) \) is defined as \( p \circ q = ((i_1, i_2), (i_2, i_3), (i_3, i_4), (i_4, i_5)) \). The communication graph \( G(A) \) and powers of matrix \( A \) are closely related as it is shown in the next theorem.

**Theorem 24** Let \( A \in \mathbb{R}^{n \times n}_{\text{max}} \), then \( \forall k \geq 1 : [A^\otimes k]_{ji} = \max\{|p|_w : p \in P(i, j; k)\} \), where \([A^\otimes k]_{ji} = \epsilon\) in the case when \( P(i, j; k) \) is empty i.e., no path of length \( k \) from node \( i \) to node \( j \) exists in \( G(A) \).

**Definition 25** Let \( A \in \mathbb{R}^{n \times n}_{\text{max}} \) then define the matrix \( A^+ \in \mathbb{R}^{n \times n}_{\text{max}} \) as: \( A^+ = \bigoplus_{k=1}^{\infty} A^\otimes k \). Where the element \([A^+]_{ji}\) gives the maximal weight of any path from \( j \) to \( i \). If in addition one wants to add the possibility of staying at a node then one must include matrix \( E \) in the definition of matrix \( A^+ \) giving rise to its Kleene star representation defined by:

\[
A^* = \bigoplus_{k=0}^{\infty} A^\otimes k.
\] (12)

**Lemma 26** Let \( A \in \mathbb{R}^{n \times n}_{\text{max}} \) be such that any circuit in \( G(A) \) has average circuit weight less than or equal to \( \epsilon \). Then it holds that:

\[
A^* = \bigoplus_{k=0}^{n-1} A^\otimes k.
\] (13)
Definition 27 Let $G = (\mathcal{N}, \mathcal{D})$ be a graph and $i, j \in \mathcal{N}$, node $j$ is reachable from node $i$, denoted as $i \mathcal{R} j$, if there exists a path from $i$ to $j$. A graph $G$ is said to be strongly connected if $\forall i, j \in \mathcal{N}, j \mathcal{R} i$. A matrix $A \in \mathbb{R}^{n \times n}_{\max}$ is called irreducible if its communication graph is strongly connected, when this is not the case matrix $A$ is called reducible.

Remark 28 In this paper irreducible matrices are just considered. It is possible to treat the reducible case by transforming it into its normal form and computing its generalized eigenmode see [4].

3.2.1 Spectral Theory

Definition 29 Let $A \in \mathbb{R}^{n \times n}_{\max}$ be a matrix. If $\mu \in \mathbb{R}^{\max}$ is a scalar and $v \in \mathbb{R}^{n}_{\max}$ is a vector that contains at least one finite element such that:

$$A \otimes v = \mu \otimes v \quad (14)$$

then, $\mu$ is called an eigenvalue and $v$ an eigenvector.

Remark 30 Notice that the eigenvalue can be equal to $\epsilon$ and is not necessarily unique. Eigenvectors are certainly not unique indeed, if $v$ is an eigenvector then $\alpha \otimes v$ is also an eigenvector for all $\alpha \in \mathbb{R}$.

Let $C(A)$ denote the set of all elementary circuits in $G(A)$ and write:

$$\lambda = \max_{p \in C(A)} \frac{|p|_{w}}{|p|_{1}} \quad (15)$$

for the maximal average circuit weight. Notice that since $C(A)$ is a finite set, the maximum of (15) is attained (which is always the case when matrix $A$ is irreducible). In case $C(A) = \emptyset$ define $\lambda = \epsilon$.

Definition 31 A circuit $p \in G(A)$ is said to be critical if its average weight is maximal. The critical graph of $A$, denoted by $G^{c}(A) = (\mathcal{N}^{c}(A), \mathcal{D}^{c}(A))$, is the graph consisting of those nodes and arcs that belong to critical circuits in $G(A)$.

Lemma 32 Let assume that $G(A)$ contains at least one circuit then, any circuit in $G^{c}(A)$ is critical.

Definition 33 Let $A \in \mathbb{R}^{n \times n}_{\max}$ be a matrix and $\mu$ an eigenvalue of $A$ with associated eigenvector $v$ then, the support of $v$ consists of the set of nodes of $G(A)$ which correspond to finite entries of $v$. 
Lemma 34 Let $A \in \mathbb{R}_{\text{max}}^{n \times n}$ be an irreducible matrix then any $v \in \mathbb{R}_{\text{max}}^{n}$ which satisfies (14) has all components different from $\epsilon$.

Next, the most important result of this sub-section is given.

Theorem 35 If $A \in \mathbb{R}_{\text{max}}^{n \times n}$ is irreducible, then there exists one and only one finite eigenvalue (with possible several eigenvectors). This eigenvalue is equal to the maximal average weight of circuits in $G(A)$:

$$\lambda(A) = \max_{p \in C(A)} \frac{|p|_w}{|p|_1}$$

(16)

3.2.2 Linear Equations

Theorem 36 Let $A \in \mathbb{R}_{\text{max}}^{n \times n}$ and $b \in \mathbb{R}_{\text{max}}^{n}$. If the communication graph $G(A)$ has maximal average circuit weight less than or equal to $\epsilon$, then $x = A^* \otimes b$ solves the equation $x = (A \otimes x) \oplus b$. Moreover, if the circuit weights in $G(a)$ are negative then the solution is unique.

3.3 Max-Plus recurrence equations for timed event Petri Nets

Definition 37 Let $A_m \in \mathbb{R}_{\text{max}}^{n \times n}$ for $0 \leq m \leq M$ and $x(m) \in \mathbb{R}_{\text{max}}^{n}$ for $-M \leq m \leq -1; M \geq 0$. Then, the recurrence equation: $x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k - m); k \geq 0$ is called an $M$th order recurrence equation.

Theorem 38 The $M$th order recurrence equation, given by equation $x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k - m); k \geq 0$, can be transformed into a first order recurrence equation $x(k + 1) = A \otimes x(k); k \geq 0$ provided that $A_0$ has circuit weights less than or equal to zero.

With any timed event Petri net, matrices $A_0, A_1, ..., A_M \in \mathbb{N}^n \times \mathbb{N}^n$ can be defined by setting $[A_m]_{ij} = a_{ij}$, where $a_{ij}$ is the largest of the holding times with respect to all places between transitions $t_i$ and $t_j$ with $m$ tokens, for $m = 0, 1, ..., M$, with $M$ equal to the maximum number of tokens with respect to all places. Let $x_i(k)$ denote the $k$th time that transition $t_i$ fires, then the vector $x(k) = (x_1(k), x_2(k), ... , x_m(k))^T$, called the state of the system, satisfies the $M$th order recurrence equation: $x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k - m); k \geq 0$ Now, assuming that all the hypothesis of theorem (38) are satisfied, and setting
\[
\hat{x}(k) = (x^T(k), x^T(k-1), ..., x^T(k-M+1))^T, \quad \text{equation } x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m); \quad k \geq 0 \text{ can be expressed as: } \hat{x}(k + 1) = \hat{A} \otimes \hat{x}(k); \quad k \geq 0, \text{ which is known as the standard autonomous equation.}
\]

4 The Solution to the Stability Problem for Discrete Event Dynamical Systems Modeled with timed Petri Nets

This section defines what it means for a TPN to be stable, then gathering the results previously presented in the papers past sections the solution to the problem is obtained.

**Definition 39** A TPN is said to be stable if all the transitions fire with the same proportion i.e., if there exists \( q \in \mathbb{N} \) such that

\[
\lim_{k \to \infty} \frac{x_i(k)}{k} = q, \forall i = 1, ..., n \quad (17)
\]

This last definition tell us that in order to obtain a stable TPN all the transitions have to be fired \( q \) times. However, it will be desirable to be more precise and know exactly how many times. The answer to this question is given next.

**Lemma 40** Consider the recurrence relation \( x(k + 1) = A \otimes x(k), k \geq 0, x(0) = x_0 \in \mathbb{R}^n \text{ arbitrary.} \) A an irreducible matrix and \( \lambda \in \mathbb{R} \) its eigenvalue then,

\[
\lim_{k \to \infty} \frac{x_i(k)}{k} = \lambda, \forall i = 1, ..., n \quad (18)
\]

**Proof.** Let \( v \) be an eigenvector of \( A \) such that \( x_0 = v \) then,

\[
x(k) = \lambda^k \otimes v \Rightarrow x(k) = k\lambda + v \Rightarrow \frac{x(k)}{k} = \lambda + \frac{v}{k} \Rightarrow \lim_{k \to \infty} \frac{x_i(k)}{k} = \lambda
\]

Now starting with an unstable TPN, collecting the results given by: proposition (13), what has just been discussed about recurrence equations for TPN at the end of subsection (3.3) and the previous lemma (40) plus theorem (35), the solution to the problem is obtained.
5 Predator-Prey Dynamical Systems

Consider the interaction of populations, in which there are exactly two species, one of which the predators eat the other the prey thereby affecting each other’s growth rates. Such pairs exist throughout nature: fish and sharks, lions and gazelles, birds and insects, to mention some. It is assumed that, the predator species is totally dependent on a single prey species as its only food supply, the prey has unlimited food supply, and that there is no threat to the prey other than the specific predator. The reader is referred to [5] for a discussion of the predator-prey’s evolution interaction process.

Consider the TPN model (Fig 3.) of a predator-prey system. Where the events (transitions) that drive the system are: t: prays at threat , s: the predator starts attacking the prey, d: the predator departs. The places (that represent the states of the system) are: R: preys resting, P: the preys are in danger, B: the preys are being eaten, I: the predator is idle. The holding times associated to the places R and I are $Cr$ and $Cd$ respectively, (with $Cr > Cd$).

The incidence matrix that represents the $PN$ model is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Therefore since there does not exists a $\Phi$ strictly positive $m$ vector such that $A\Phi \leq 0$ the sufficient condition for stability is not satisfied. Moreover, the $PN$
(TPN) is unbounded since by the repeated firing of t, the marking in P grows indefinitely, which means that there are not enough predators to balance the growing condition of the preys. However, by taking \( u = [k, k, k]; k > 0 \), we get that \( A^T u \leq 0 \). Therefore, the PN is stabilizable which implies that the TPN is stable i.e., by firing all the transitions at the same speed an unknown oscillating behavior of both populations is achieved. Now, from the TPN model we obtain that matrix \( \hat{A} \), which defines the standard autonomous equation, is equal to:

\[
\hat{A} = \begin{pmatrix}
Cr & \varepsilon & \varepsilon \\
Cr & \varepsilon & Cd \\
\varepsilon & \varepsilon & Cd \\
\end{pmatrix}.
\]

(19)

Therefore, \( \lambda(A) = \max_{p \in C(A)} \frac{|p|}{|p|_1} = \max\{Cr, Cd\} = Cr \). This tells that in order for the TPN to be stable and oscillate at a fixed known frequency, all the transitions must fire at the same speed as the preys arrive i.e., they have to be eaten as soon as they are at disposition. This is consistent with the observed oscillatory behavior in real life.

6 Conclusions

The main contribution of the paper consists in combining Lyapunov theory with max-plus algebra to study the stability problem for predator-pray systems treated as discrete event dynamical systems modeled with timed Petri nets. The results obtained are consistent with how the predator-prey system behaves in real life.

References


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