An ADI Finite Element Method for the Sine-Gordon Equation in Two Dimensions

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Abstract

An ADI Finite Element Method and error estimations for Sine-Gordon Equation are studied in this paper, by using this method, a multidimensional problem can be solved as a series of one dimensional problems, with the help of theory and skill of prior estimates of differential equations optimal order error estimate is derived. At last, we give the numerical results of the scheme.

Mathematics Subject Classification: 65M60

Keywords: Sine-Gordon Equation; Alternating-Direction; Finite Element Method; Error Estimates

1 Introduction

we shall consider the numerical solution of the following Sine-Gordon equation:

\[ u_{tt} + \alpha u_t - \gamma \Delta u + \beta \sin u = f, \quad (x, t) \in \Omega \times (0, T], \quad (1-1) \]

\[ u|_{\partial \Omega} = 0, \quad t \in (0, T], \quad (1-2) \]

\[ u(x, 0) = u_0(x), \quad u_t|_{t=0} = u_1(x). \quad (1-3) \]

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where $\Omega = [a, b] \times [c, d]$, $u = u(x, t) \in R, \gamma > 0, \beta > 0$. Zhou shengfan[1] has proved the existence and uniqueness of the solution of equation, Liang zongqi[2] has researched the global solution and made the numerical computation in one dimension, Xu qubin[3] has made numerical simulation using finite difference method. In this paper, we will give an ADI finite element Method in two dimensions.

We assume the following meshes to cover the computational domain $[a, b] \times [c, d]$, consisting of cells $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, for $1 \leq i, j \leq N$, where $a = x_0 < x_1 < \cdots < x_N = b$, $c = y_0 < y_1 < \cdots < y_N = d$, we again denote $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $h = \max_{1 \leq i, j \leq N} \{\Delta x_i, \Delta y_j\}$, and the partition is regular, namely there is a constant $C$ independent of $h$, such as:

$$\Delta x_i \geq Ch, \Delta y_j \geq Ch$$

We define a finite element space $H^1_0(\Omega) = \{u \in H^1(\Omega), u |_{\partial \Omega} = 0\}$. $S_h(\Omega) = S_h[a, b] \times S_h[c, d]$, $S_h[a, b], S_h[c, d]$ are subspaces of $H^1_0[a, b], H^1_0[c, d]$ respectively. We then denote the tensor product basis as $\{\alpha_p(x)\beta_q(y)\}_{1 \leq p, q \leq N}$, where $\{\alpha_p(x)\}_{p=1}^{N}$ and $\{\beta_q(y)\}_{q=1}^{N}$ are the bases for one dimensional spaces $S_h[a, b], S_h[c, d]$ respectively. So for any $U \in S_h(\Omega)$ can be written as:

$$U(t, x, y) = \sum_p \sum_q \xi_{pq}(t) \alpha_p(x) \beta_q(y).$$

so $S_h(\Omega)$ is a subspace of $H^1_0(\Omega)$ that has the approximation properties [5]: for some integer $r \geq 2$ and any $\phi \in H^{m+1}(\Omega) \cap H^1_0(\Omega)$, there is a constant $K$ independent of $h$ [4]:

$$\inf_{v \in S_h(\Omega)} \{||\phi - v|| + h||\phi - v||_1\} \leq Kh^{m+1}||\phi||_{m+1}, \quad 1 \leq m + 1 \leq r + 1$$

We denote: $M = \left[\frac{T}{\tau}\right], t_n = n\tau, n = 0, 1, 2, 3 \cdots M$,

$$a_{ip} = \int_a^b \alpha_i(x)\alpha_p(x)dx, \quad a'_{ip} = \int_a^b \frac{d\alpha_i}{dx} \frac{d\alpha_p}{dx} dx$$

$$b_{jq} = \int_c^d \beta_j(y)\beta_q(y)dy, \quad b'_{jq} = \int_c^d \frac{d\beta_j}{dy} \frac{d\beta_q}{dy} dy$$

$$w^n = w(x, t_n), \quad \partial_t w^n = \frac{w^{n+1} - w^n}{\tau}.$$

2 ADI Finite Element Schemes

Lemma 2.1. [5] If the partition is regular, for any $v \in H^{m+1}(\Omega) \cap h, v$ is piecewise interpolation of $v$ in $S_h(\Omega)$, there is a constant $C$ independent of $h$:

$$\|v - \nabla_h v\|_{r, \Omega} \leq Ch^{m+1-r}\|v\|_{m+1, \Omega}, \quad r = 0, 1. \quad (2-1)$$
When $t = t_n$, for any $v \in H_0^1(\Omega)$:

$$(u^n_t, v) + \alpha(u^n, v) + \gamma(\nabla u^n, \nabla v) + \beta(sin u^n, v) = (f^n, v). \quad (2-2)$$

We define the ADI finite element scheme as:

Find $U^{n+1} \in S_h(\Omega)$, such that for $1 \leq n \leq M - 1$

$$
\begin{align*}
(U^{n+1} - 2U^n + U^{n-1})/\tau^2, v) + \theta(\partial/\partial x(U^{n+1} - 2U^n + U^{n-1}), \partial v/\partial x) \\
+ \theta(\partial^2/\partial x\partial y(U^{n+1} - 2U^n + U^{n-1}), \partial^2 v/\partial x\partial y) \\
+ \alpha(U^n - U^{n-1})/\tau, v) + \gamma(\partial U^n/\partial x, \partial v/\partial x) + \gamma(\partial U^n/\partial y, \partial v/\partial y) + \beta(sin U^n, v) \\
= (f^n, v), \quad \forall v \in S_h(\Omega) \quad (2-3)
\end{align*}
$$

$$
U^0 = \cap_h u_0(x), \quad (2-4)
$$

$$
U^1 = \cap_h [U^0 + (\Delta t)u_0 + (\Delta t)^2/2 u^{0}_{tt}]
= \cap_h [U^0 + (\Delta t)u_1 + (\Delta t)^2/2 (-\alpha u_1 + \gamma \partial^2 u_0/\partial x_1 \gamma \partial^2 u_0/\partial x_2 - \beta sin u_0 + f)]. \quad (2-5)
$$

Let $v = \alpha_i \beta_j$, (2-3) can be written as:

$$
\sum_p \sum_q (a_{ip} + \theta \tau^2 a'_{ip})(b_{jq} + \theta \tau^2 b'_{jq})\xi^n_{pq} + R^n_{ij} = R^{n-1}_{ij} + R^n_{ij}, \quad (2-6)
$$

If we use transition variable $Z^n_{ij}$, the last formula can be written as:

$$
\sum_p (a_{ip} + \theta \tau^2 a'_{ip})Z^n_{pj} = R^{n-1}_{ij} + R^n_{ij}, \quad (2-6)
$$

$$
\sum_q (b_{jq} + \theta \tau^2 b'_{jq})\xi^n_{pq} = Z^n_{pj}. \quad (2-7)
$$

We compute (2-6) in x-direction and (2-7) in y-direction, and obviously, the coefficient matrixes of (2-6) and (2-7) are all symmetric and positive, so the solution is existent and unique.

3 error estimate

Theorem 3.1. Let $u$ be the generalized solution of equation (1.1)-(1.3), $u \in H^{m+1}(\Omega) \cap H_0^1(\Omega)$, $U^n$ is the solution of scheme (2.3)-(2.5), $e^n = u^n - U^n$, then there is a constant $C$ independent of $\Delta t$ and $h$, such that,

$$
\|\partial e^n\|^2 + |e^n|^2 + \tau^4 \|\partial^2(\partial e^n)/\partial x\partial y\|^2 \leq C(\tau^2 + h^{2m}), n\tau \leq T.
$$
proof: (2-2) can be written as:

\[
\left(\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2}, v\right) + \theta\left(\frac{\partial}{\partial x} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x}\right) + \theta\left(\frac{\partial}{\partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y}\right) + \theta^2 \tau^2 \left(\frac{\partial^2}{\partial x \partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial^2 v}{\partial x \partial y}\right) + \alpha \left(\frac{u^n - u^{n-1}}{\tau}, v\right) + \gamma \left(\frac{\partial u^n}{\partial x}, \frac{\partial v}{\partial x}\right) + \gamma \left(\frac{\partial u^n}{\partial y}, \frac{\partial v}{\partial y}\right) + \beta (\sin u^n, v)
\]

\[
= (f^n, v) + (\mu_n, v) + \alpha (\rho_n, v) + \theta \left(\frac{\partial}{\partial x} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x}\right) + \theta \left(\frac{\partial}{\partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y}\right) + \theta^2 \tau^2 \left(\frac{\partial^2}{\partial x \partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial^2 v}{\partial x \partial y}\right).
\]

(3-1)

Where: \( \mu_n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} - u^n = O(\tau^2), \quad \rho_n = \frac{u^n - u^{n-1}}{\tau} - u^n = O(\tau) \).

Error equation can be derived by (3-1) minus (2-3):

\[
\left(\frac{e^{n+1} - 2e^n + e^{n-1}}{\tau^2}, v\right) + \theta \left(\frac{\partial}{\partial x} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial v}{\partial x}\right) + \theta \left(\frac{\partial}{\partial y} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial v}{\partial y}\right) + \theta^2 \tau^2 \left(\frac{\partial^2}{\partial x \partial y} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial^2 v}{\partial x \partial y}\right) + \alpha \left(\frac{e^n - e^{n-1}}{\tau}, v\right) + \gamma \left(\frac{\partial e^n}{\partial x}, \frac{\partial v}{\partial x}\right) + \gamma \left(\frac{\partial e^n}{\partial y}, \frac{\partial v}{\partial y}\right) = -\beta (\sin u^n - \sin u^n, v) + (\mu_n, v) + \alpha (\rho_n, v) + \theta \left(\frac{\partial}{\partial x} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x}\right) + \theta \left(\frac{\partial}{\partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y}\right) + \theta^2 \tau^2 \left(\frac{\partial^2}{\partial x \partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial^2 v}{\partial x \partial y}\right).
\]

(3-2)

We can write (3-2) as:

\[
\left(\frac{e^{n+1} - 2e^n + e^{n-1}}{\tau^2}, v\right) + \theta \left(\frac{\partial}{\partial x} (e^{n+1} + e^{n-1}), \frac{\partial v}{\partial x}\right) + \theta \left(\frac{\partial}{\partial y} (e^{n+1} + e^{n-1}), \frac{\partial v}{\partial y}\right) + \theta^2 \tau^2 \left(\frac{\partial^2}{\partial x \partial y} (e^{n+1} - 2e^n + e^{n-1}), \frac{\partial^2 v}{\partial x \partial y}\right) = -\beta (\sin u^n - \sin u^n, v) + (\mu_n, v) + \alpha (\rho_n, v) + \theta \left(\frac{\partial}{\partial x} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x}\right) + \theta \left(\frac{\partial}{\partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y}\right) + \theta^2 \tau^2 \left(\frac{\partial^2}{\partial x \partial y} (u^{n+1} - 2u^n + u^{n-1}), \frac{\partial^2 v}{\partial x \partial y}\right) - \alpha \left(\frac{e^n - e^{n-1}}{\tau}, v\right) + (2\theta - \gamma) \left(\frac{\partial e^n}{\partial x}, \frac{\partial v}{\partial x}\right) + (2\theta - \gamma) \left(\frac{\partial e^n}{\partial y}, \frac{\partial v}{\partial y}\right).
\]

(3-3)
let \( v = e^{n+1} - e^{n-1} \):

\[
\frac{e^{n+1} - 2e^n + e^{n-1}}{\tau^2}, e^{n+1} - e^{n-1} = (\partial_t e^n - \partial_t e^{n-1}, \partial_t e^n + \partial_t e^{n-1})
\]

\[
= \|\partial_t e^n\|^2 - \|\partial_t e^{n-1}\|^2,
\]

\[
(\frac{\partial}{\partial x}(e^{n+1} - e^{n-1}), \frac{\partial}{\partial y}(e^{n+1} - e^{n-1})) + (\frac{\partial}{\partial y}(e^{n+1} + e^{n-1}), \frac{\partial}{\partial y}(e^{n+1} - e^{n-1}))
\]

\[
= |e^{n+1}|_1^2 - |e^{n-1}|_1^2,
\]

\[
(\frac{\partial^2}{\partial x \partial y}(e^{n+1} - 2e^n + e^{n-1}), \frac{\partial^2}{\partial x \partial y}(e^{n+1} - e^{n-1})) = \tau^2(\|\partial^2(\partial_t e^n)\|_1^2 - \|\partial^2(\partial_t e^{n-1})\|_1^2).
\]

Based on the fact that \( |\sin x| \leq |x|, x \in R \) we can obtain:

\[
-\beta(\sin u^n - \sin u^{n-1}, e^{n+1} - e^{n-1}) = -\beta \int_{\Omega} 2 \cos \frac{u^n + U^n}{2} \sin \frac{e^{n+1} - e^{n-1}}{2} d\Omega
\]

\[
\leq \beta \tau \|e^n\|^2 + \frac{\beta \tau}{2}(\|\partial_t e^n\|^2 + \|\partial_t e^{n-1}\|^2).
\]

and

\[
(\mu_n, v) + \alpha(\rho_n, v) \leq C(\tau^2 + \|e^{n+1}\|_1^2 + \|e^n\|_1^2).
\]

From the Taylor expansion we can get:

\[
\theta(\frac{\partial}{\partial x}(u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial x}) + \theta(\frac{\partial}{\partial y}(u^{n+1} - 2u^n + u^{n-1}), \frac{\partial v}{\partial y})
\]

\[
\leq C(\tau^4 + \|e^{n+1}\|_1^2 + \|e^n\|_1^2).
\]

In respect that:

\[
\frac{\partial^2}{\partial x \partial y}(u^{n+1} - 2u^n + u^{n-1}) = \int_{-\tau}^{\tau} (\tau - |z|) \frac{\partial^4 u(t_n + z)}{\partial x \partial y \partial t^2} dz = O(\tau^2),
\]

we can easily obtain:

\[
\theta^2 \tau^2(\frac{\partial^2}{\partial x \partial y}(u^{n+1} - 2u^n + u^{n-1}), \frac{\partial^2 v}{\partial x \partial y}) \leq \theta^2 \tau^3(\tau^4 + \|\partial^2(\partial_t e^n)\|_1^2 + \|\partial^2(\partial_t e^{n-1})\|_1^2).
\]

\[
-\frac{\alpha}{\tau}(e^n - e^{n-1}, e^{n+1} - e^{n-1}) \leq C\tau(\|\partial_t e^n\|^2 + \|\partial_t e^{n-1}\|^2).
\]

\[
(2\theta - \gamma)(\frac{\partial e^n}{\partial x}, \frac{\partial v}{\partial x}) + (2\theta - \gamma)(\frac{\partial e^n}{\partial y}, \frac{\partial v}{\partial y}) \leq C(|e^n|_1^2 + |e^{n+1}|_1^2 + |e^{n-1}|_1^2)
\]

Take the above results into (3-3), and we can get:

\[
\|\partial_t e^n\|^2 - \|\partial_t e^{n-1}\|^2 + \theta(\|e^{n+1}\|_1^2 - \|e^n\|_1^2) + \theta^2 \tau^4(\|\partial^2(\partial_t e^n)\|_1^2 - \|\partial^2(\partial_t e^{n-1})\|_1^2).
\]
\[ \leq \beta \tau \|e^n\|^2 + \frac{\beta \tau}{2} (\|\partial_t e^n\|^2 + \|\partial_t e^{n-1}\|^2) + C(\tau^2 + \|e^{n+1}\|_1^2 + \|e^n\|_1^2) \\
+ C(\tau^4 + \|e^{n+1}\|_1^2 + \|e^n\|_1^2 + \|e^{n-1}\|_1^2) + C\tau^3 (\tau^4 + \|\frac{\partial^2 (\partial_t e^n)}{\partial x \partial y}\|_1^2 + \|\frac{\partial^2 (\partial_t e^{n-1})}{\partial x \partial y}\|_1^2) \\
+ C\tau (\|\partial_t e^n\|^2 + \|\partial_t e^{n-1}\|^2). \]

By reason of the fact: \( e^n = e^0 + \tau \sum_{l=0}^{n-1} \partial_t e^l \),

We know: \( \|e^n\|_1^2 \leq C \|e^0\|_1^2 + 2n \tau \sum_{l=0}^{n-1} \|\partial_t e^l\|^2. \)

On the two ends, from 1 to N sum about n:

\[ \|\partial_t e^N\|^2 + \theta (\|e^{N+1}\|_1^2 + \|e^N\|_1^2) + \theta^2 \tau^4 \|\frac{\partial^2 (\partial_t e^N)}{\partial x \partial y}\|_1^2 \]

\[ \leq \|\partial_t e^0\|^2 + \theta (\|e^1\|_1^2 + \|e^0\|_1^2) + \theta^2 \tau^4 \|\frac{\partial^2 (\partial_t e^0)}{\partial x \partial y}\|_1^2 \]

\[ + C\tau^2 + C\tau \sum_{n=1}^{N-1} (\|e^n\|_1^2 + \|\partial_t e^n\|_1^2 + \|\frac{\partial^2 (\partial_t e^n)}{\partial x \partial y}\|_1^2). \]

Obviously,

\[ |\partial_t e^0\|_1^2 + \theta (\|e^1\|_1^2 + \|e^0\|_1^2) + \theta^2 \tau^4 \|\frac{\partial^2 (\partial_t e^0)}{\partial x \partial y}\|_1^2 \leq C(\tau^4 + h^2)^m. \]

so the Theorem is proved from Gronwall inequality.

4 numerical experiments

We make the following numerical experiments using the schemes in this paper:

\[ u_{tt} + u_t - \Delta u + 2\sin u = f, \]

\[ u_0(x, y) = (1 - \cos \pi x)(1 - \cos \pi y), \]

\[ u_1(x, y) = -(1 - \cos \pi x)(1 - \cos \pi y), \]

\[ f(x, y, t) = -\pi^2 e^{-t}(\cos \pi x + \cos \pi y - 2\cos \pi x \cos \pi y) + 2\sin(e^{-t}(1 - \cos \pi x)(1 - \cos \pi y)). \]

we can easily get \( u(x, y, t) = e^{-t}(1 - \cos \pi x)(1 - \cos \pi y). \)

We consider the numerical solution on domain \([0, 2] \times [0, 2]\), let \( cfl = 0.2, \theta = 1.0 \), the result is as follows:
An ADI FEM for the Sine-Gordon equation

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