The Block Diagonalization of Circulant Matrices over the Quaternion Field

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Abstract. This paper has studied some properties of circulant matrices, and makes use of the complex expression of quaternion to obtain that the circulant matrices over the quaternion field can be transformed into block-diagonal matrices under the unitary similarity. At the end of the article, we give a specific numerical example.

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1. Introduction

Circulant matrix is a class of very important special matrix, widely applied in modern technology field, such as in codes theory, mathematical statistics, theoretical physics, solid-state physics, structural computing, digital image process, control theory, optimization, matrix decomposition etc. With the proposition of the quaternion definition and its application in many engineering fields, how to popularize the properties and conclusions of circulant matrices from the domain to the quaternion field is becoming a hot subject of recent research. In the [4], Yu has discussed the contemporary up-triangulation of two quaternion matrices. In the [5], Jiang and Wei have given out the equivalent conditions of diagonalization of quaternion matrix by using the unitary vector. In the [6], Zhou has studied the conditions of diagonalization of the g-circulant matrices. In the [7], Zhang has studied the block k-circulant matrix and its diagonalization. This paper mainly studies the block diagonalization of the quaternion circulant matrices, and gives a simplified method to determine circulant matrices whether invertible or not.

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2. Preliminaries

Let $R$ be the real field, $C$ be the complex field and $Q$ be the real quaternion field. $A(a_{ij})$ denotes the matrix on the quaternion field, where $a_{ij} \in Q$. Denote $\alpha = a + bi + cj + dk \in Q$, where $a, b, c, d \in R$, and $\overline{\alpha} = a - bi - cj - dk$ is called conjugate quaternion of $\alpha$. $Q^{n \times n}$ and $CQ$ respectively represent the set of $n$-order quaternion matrices and the set of the quaternion circulant matrices.

**Definition 1** Let $A$ be a $n$-order matrix over the quaternion field.

If $A = \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_0 \end{pmatrix}$ $(a_i \in Q, i = 0, 1, \ldots, n-1)$, we call it the quaternion circulant matrix. Note that $A$ only has relationship with the elements of its first line, we denote $A$ with $CQ(a_0, a_1, \ldots, a_{n-1}) \in CQ$.

$J = CQ(0, 1, \ldots, 0)$, that is $J = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix}$ is called the basic circulant matrix. Obviously, all the $J, J^2, \ldots, J^n = I$ (n-order identity matrix) are circulant matrices, and we have $A = a_0I + a_1J + \ldots + a_{n-1}J^{n-1}$. Let $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$, then $A = f(J)$, so we call $f(x)$ as the generated polynomial of circulant matrix $A$.

**Definition 2** Let $A \in Q^{n \times n}$, if $AA^* = A^*A = I$, then $A$ is called the $n$-order quaternion unitary matrix, and the set of all the quaternion unitary matrices is denoted as $Q(H, u)$.

**Definition 3** Let $\Omega = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \end{pmatrix}$, where $\omega$ is the $n$-primitive unit root of 1. It can be easily proved that $\Omega^* = \overline{\Omega}, \Omega\Omega^* = \Omega^*\Omega = I$ and $(\Omega^*)^2 = \Omega^2 = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \end{pmatrix}$.

3. Analysis of the block diagonal form of quaternion circulant matrices

**Lemma 1** Let $H$ be a $n$-order matrix over the complex field, then $jH = \overline{H}j$ and $jHj = -\overline{H}$.

**Lemma 2** Let $H \in Q^{n \times n}$, then $H$ can be expressed as $H = H_1 + H_2j$, where $H_1, H_2 \in C$. 

Theorem. Let $\Omega$ defined in Definition 3, such that $\Omega = \frac{1}{2}(\Omega^* - j\Omega^*) = \frac{1}{2}(\Omega + j\Omega)^*$, where $\Omega$ as defined in the Definition 3.

Lemma 4 If $A = CQ(a_0, a_1, \ldots, a_{n-1})$, then the eigenvalues of $A$ are $\lambda_j = f(\omega^j) = a_0 + a_1\omega^j + a_2(\omega^j)^2 + \cdots + a_{n-1}(\omega^j)^{n-1} (j = 1, 2, \ldots, n)$.

Lemma 5 $A(a_{ij})$ (where $a_{ij} \in C$) is a circulant matrix if and only if $A = \Omega D \Omega^*$, where $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and $\lambda_j (j = 1, 2, \ldots, n)$ as obtained in the Lemma 4.

Theorem. Let $H = CQ(a_0, a_1, \ldots, a_{n-1}) \in CQ$, then exists an $U \in Q(H, u)$, such that $H$ is similar to a block diagonal matrix, that is $UHU^* = diag(S_1, S_2, \ldots, S_t)$, where $S_i$ are the one or two order matrix and have the specific forms as below,

$$(1) S_1 = (u_1 + v_1), (2) S_i = \begin{pmatrix} u_i & v_i \\ v_{n-i+2} & u_{n-i+2} \end{pmatrix} (i = 2, 3; n > 4),$$

$$(3) S_3 = (u_4 + v_3)(n = 4), (4) S_i = \begin{pmatrix} u_{2(t-i)+6} & v_i \\ v_{2t+1-i} & u_{2(t-i)+5} \end{pmatrix}, 4 \leq i \leq t-1(n = 2t > 4),$$

$$(5) S_t = (u_4 + v_t), (n = 2t > 4), (6) S_i = \begin{pmatrix} u_{2(t-i)+5} & v_i \\ v_{2t+1-i} & u_{2(t-i)+4} \end{pmatrix}, 4 \leq i \leq t(n = 2t-1).$$

Proof. From Lemma 2, $H = H_1 + H_2j$, where $H_1, H_2 \in C$. Owing to Lemma 5, put $\Omega$ defined in Definition 3, such that $\Omega H_1 \Omega^* = D_1, \Omega H_2 \Omega^* = D_2$, where $D_1 = diag(f(\omega), f(\omega^2), \ldots, f(\omega^n)), D_2 = diag(g(\omega), \ldots, g(\omega^n))$, where $f(x), g(x)$ are respectively the generated polynomials of $H_1$ and $H_2$. Then we obtain

$$(\Omega + j\Omega)(H_1 + H_2j)(\Omega + j\Omega)^{-1} = \frac{1}{2}(\Omega + j\Omega)(H_1 + H_2j)(\Omega^* - j\Omega^*)$$

$$= \frac{1}{2}(D_1 + D_2 \Omega j \Omega^* + \Omega j \Omega^*D_1 - D_2 - D_1 \Omega j \Omega^* + D_2 + \overline{D_1} + \overline{D_2} \Omega j \Omega^*).D_2$$

$$= \frac{1}{2}[2ReD_1 + 2ImD_2 + D_2 \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 0 \end{pmatrix} j + \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 0 \end{pmatrix} \overline{D_1} \cdot j]$$

$$- D_1 \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 0 \end{pmatrix} j + \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 0 \end{pmatrix} \overline{D_2} \cdot j]$$

$$= \begin{pmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_n \end{pmatrix} + \begin{pmatrix} v_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & v_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & v_n \end{pmatrix} j = M + Nj,$$
Now we discuss the n.

When \( n \) is odd, select \( G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix},
\)
evidently , \( GG^T = I \), visa calculating, we obtain
\[
G(\Omega + \Omega j)(H_1 + H_2 j)(\Omega + \Omega j)^{-1}G^T
= G\left[\frac{\sqrt{2}}{2}(\Omega + \Omega j)\right] (H_1 + H_2 j) \left[\frac{\sqrt{2}}{2}(\Omega + \Omega j)^*\right] G^T = G(M + Nj)G^T
\]

\[
= \begin{pmatrix}
u_1 + v_1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & u_2 & v_2 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & u_n^* & v_n & u_n & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & u_5^* & \frac{v_{n-1} + 1}{2} \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{v_{n+1} + 1}{2} & u_4
\end{pmatrix}
\]

When \( n \) is even, select \( G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix},
\)
evidently , \( GG^T = I \), by computing, we can get
\[
G(\Omega + \Omega j)(H_1 + H_2 j)(\Omega + \Omega j)^{-1}G^T
= G\left[\frac{\sqrt{2}}{2}(\Omega + \Omega j)\right] (H_1 + H_2 j) \left[\frac{\sqrt{2}}{2}(\Omega + \Omega j)^*\right] G^T = G(M + Nj)G^T
\]
The block diagonalization of circulant matrices

\[
\begin{pmatrix}
  u_1 + v_1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
  0 & u_2 & v_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\
  0 & v_n & u_n & 0 & 0 & \ldots & 0 & 0 & 0 \\
  0 & 0 & 0 & u_3 & v_3 & \ldots & 0 & 0 & 0 \\
  0 & 0 & 0 & v_{n-1} & u_{n-1} & \ldots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & 0 & \ldots & u_6 & v_2 & 0 \\
  0 & 0 & 0 & 0 & 0 & \ldots & v_2 + v_4 & u_5 & 0 \\
  0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & u_4 + v_2 + 1 \\
\end{pmatrix}
\]

Summarily, we set \( U = \sqrt{2} G(\Omega + \Omega j) \), which is an unitary matrix, then \( UHU^* = \text{diag}(S_1, S_2, \ldots, S_t) \), and the \( S_i \) have the forms as shown in the title.

**Remark.**

It is easy to see that this paper give one more help to judge a circulant matrix whether invertible or not.

(1) Accordig to the paper, \( H \) is invertible as long as \( \text{diag}(S_1, S_2, \ldots, S_t) \) is invertible, and as long as \( S_i \) obtained in the theorem are invertible. Then we can transform the inverse matrices problem of high-order circulant matrices into that of two-order matrices.

(2) With the proof of the theorem, we can find that the article [2] has some mistakes in Theorem 4 and give the appropriate expression in the theorem.

**4. Example**

For \( n=4 \), we give an example. Let

\[
H = \begin{pmatrix}
  1 + j + k & 2 + k & 3 + 2j & 4 + j - k \\
  4 + j - k & 1 + j + k & 2 + k & 3 + 2j \\
  3 + 2j & 4 + j - k & 1 + j + k & 2 + k \\
  2 + k & 3 + 2j & 4 + j - k & 1 + j + k \\
\end{pmatrix} = H_1 + H_2j,
\]

where \( H_1 = \begin{pmatrix}
  1 & 2 & 3 & 4 \\
  4 & 1 & 2 & 3 \\
  3 & 4 & 1 & 2 \\
  2 & 3 & 4 & 1 \\
\end{pmatrix} \), \( H_2 = \begin{pmatrix}
  1 + i & i & 2 & 1 - i \\
  1 - i & 1 + i & i & 2 \\
  2 & 1 - i & 1 + i & i \\
  i & 2 & 1 - i & 1 + i \\
\end{pmatrix} \).

According to the theorem above, we can select \( U = \sqrt{2} G(\Omega + \Omega j) \), where

\[
G = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad \Omega + \Omega j = \begin{pmatrix}
  1 + j & 1 + j & 1 + j & 1 + j \\
  1 + j & -i - k & -1 - j & i + k \\
  1 + j & -1 - j & 1 + j & -1 - j \\
  1 + j & i + k & -1 - j & -i - k \\
\end{pmatrix},
\]

such that \( UHU^* = \begin{pmatrix}
  10 + 2i + 4j & 0 & 0 & 0 \\
  0 & -2 & -j - k & 0 \\
  0 & -j + k & -2 + 2i & 0 \\
  0 & 0 & 0 & -2 + i + 2j \\
\end{pmatrix} \).
And according to the Remark, we only need to judge the matrix
\[ S = \begin{pmatrix} -2 & -j - k \\ -j + k & -2 + 2i \end{pmatrix} \]
whether invertible or not, and we easily prove that \( S \) is invertible so the matrix \( H \) is invertible.

References


[3] Likuan Zhao, Xiaopeng Yue, Xuezhi Du, The promotion of several theorems about circulant matrices [J], Journal of Qufu Normal University, 2006, 32(2).


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