

# Generalized Pebbling Number

**A. Lourdusamy**

Department of Mathematics  
St. Xavier's College (Autonomous)  
Palayamkottai - 627 002, India  
lourdugnanam@hotmail.com

**C. Muthulakshmi**

Department of Mathematics  
Sri Paramakalyani College  
Alwarkurichi - 627 412, India

## Abstract

The generalized pebbling number of a graph  $G$ ,  $f_{gl}(G)$ , is the least positive integer  $t$  such that however  $t$  pebbles are placed on the vertices of  $G$ , we can move a pebble to any vertex by a sequence of moves, each move taking  $p$  pebbles off one vertex and placing one on an adjacent vertex. In this paper, we determine the generalized pebbling number of complete graph  $K_n$ , path  $P_n$ , cycle  $C_n$  and  $K_{1,n}$ . Also, we obtain the bounds for  $f_{gl}(K_{m,n})$ .

**Keywords:** Graph, Complete graph, Complete bipartite graph

## 1 Introduction

Let  $G$  be a simple connected graph. The pebbling number of  $G$  is the smallest number  $f(G)$  such that however these  $f(G)$  pebbles are placed on the vertices of  $G$ , We can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex [2].

Suppose  $t$  pebbles are distributed onto the vertices of a graph  $G$ , a generalized  $p$  pebbling step  $[u, v]$  consists of removing  $p$  pebbles from a vertex  $u$ , and then placing one pebble at an adjacent vertex  $v$ , for any  $p \geq 2$ . Is it possible to move a pebble to a root vertex  $r$ , if we can repeatedly apply generalized  $p$  pebbling steps? It is answered in the affirmative by Chung in [1].

The generalized pebbling number of a vertex  $v$  in a graph  $G$  is the smallest number  $f_{gl}(v, G)$  with the property that from every placement of  $f_{gl}(v, G)$  pebbles on  $G$ , it is possible to move a pebble to  $v$  by a sequence of pebbling moves where a pebbling move consists of the removal of  $p$  pebbles from a vertex and the placement of one pebble on an adjacent vertex. The generalized pebbling number of the graph  $G$ , denoted by  $f_{gl}(G)$ , is the maximum  $f_{gl}(v, G)$  over all vertices  $v$  in  $G$ .

$\lfloor x \rfloor$  stands for the largest integer less than or equal to  $x$  and  $\lceil x \rceil$  stands for the smallest integer  $\geq x$ .

## 2 Generalized pebbling number for complete graph, path, cycle and $K_{1,n}$

**Theorem 2.1.** For any graph  $G$ ,  $f_{gl}(G) \geq (p-1)|V(G)| - (p-2)$  where  $p \geq 2$ .

**Proof.** Let  $G$  be a connected graph on  $|V(G)|$  vertices. Let  $r$  be a root vertex. Place zero pebbles on  $r$  and place  $p-1$  pebbles at every vertex of  $G \setminus \{r\}$ . We cannot move a pebble to  $r$ . So,  $f_{gl}(r, G) \geq (p-1)|V(G)| - (p-2)$ .

Therefore  $f_{gl}(G) \geq (p-1)|V(G)| - (p-2)$ . ■

**Theorem 2.2.** For a complete graph  $K_n$ ,  $f_{gl}(K_n) = (p-1)n - (p-2)$  where  $p \geq 2$

**Proof.** For a complete graph  $K_n$ , by Theorem 2.1,  $f_{gl}(K_n) \geq (p-1)n - (p-2)$ . Let  $r$  be a root vertex in  $K_n$ . Let  $(p-1)n - (p-2)$  pebbles be placed such that vertex  $r$  in  $K_n$  has no pebble. There exists at least one vertex  $u$  in  $K_n$ ,  $u \neq r$  such that  $u$  has at least  $p$  pebbles. Since  $\deg(r) = n-1$ ,  $r$  is adjacent to all other vertices, in particular to  $u$ . So, we can move a pebble from  $u$  to  $r$ . So,  $f_{gl}(r, K_n) \leq (p-1)n - (p-2)$ .

Therefore,  $f_{gl}(K_n) = (p-1)n - (p-2)$  since  $r$  is arbitrary. ■

**Theorem 2.3.** *If  $G$  contains a cut vertex, then  $f_{gl}(G) \geq (p-1)n + (p^2 - 3p + 3)$  where  $n = |V(G)|$  and  $p \geq 2$ .*

**Proof.** Let  $u$  be a cut vertex of  $G$  and let  $C_1$  and  $C_2$  be two distinct components of  $G \setminus \{u\}$ .

Take a vertex  $r$  in  $C_1$  and a vertex  $y$  in  $C_2$  ( $r$  is our root vertex). Place  $p - 1$  pebbles at each vertex of  $V(G) \setminus \{u, y, r\}$  and place  $p^2 - 1$  pebbles at  $y$ . We cannot move a pebble to  $r$ . So,  $f_{gl}(G) \geq (p - 1)n + (p^2 - 3p + 3)$ . ■

**Theorem 2.4.** *For a path of length  $n$ ,  $f_{gl}(P_n) = p^n$ , where  $p \geq 2$ .*

**Proof.** We prove the result by induction on  $n$ .

When  $n = 0$ , the result is true. When  $n = 1$ , let  $(u, v)$  be a path. If  $p$  pebbles are given assume that they are placed at  $u$  (or  $v$ ), then we can move a pebble to  $v$  (or  $u$ ). If we place at least one pebble at  $u$  and the remaining at  $v$ , then the question of moving does not arise. Hence the result is true for  $n = 1$ .

Now, let us assume that the result is true for all paths of length less than  $n$ . Consider a path  $P_n = v_0, v_1, v_2, \dots, v_n$ . Suppose  $p^n$  pebbles are placed. Consider the case when no pebble is placed on  $v_n$ . Consider the path  $P_{n-1} = v_0, v_1, v_2, \dots, v_{n-1}$ . All the  $p^n$  pebbles have been placed in  $P_{n-1}$ . The  $p^n$  pebbles can be arbitrarily grouped into  $p$  disjoint groups of  $p^{n-1}$  pebbles. Then by induction we can move  $p$  pebbles to  $v_{n-1}$ . Hence one pebble can be moved to  $v_n$ . So,  $f_{gl}(v_n, P_n) \leq p^n$ . By placing  $p^n - 1$  pebbles at  $v_0$ , we cannot move a pebble to  $v_n$ . So,  $f_{gl}(v_n, P_n) = p^n$ . A similar argument shows that  $f_{gl}(v_0, P_n) = p^n$ .

Now consider an internal vertex  $v_k$  ( $0 < k < n$ ). Suppose  $p^n$  pebbles are placed on vertices other than  $v_k$ . By removing  $v_k$  from the path,  $P_n$  is decomposed into two paths. Let  $P_1 = v_0, v_1, v_2, \dots, v_{k-1}$  and  $P_2 = v_{k+1}, v_{k+2}, \dots, v_n$ .

At least one of  $P_1$  or  $P_2$  contains not less than  $p^{n-1}$  pebbles, Without loss of generality, assume that  $P_1$  has not less than  $p^{n-1}$  pebbles.  $P_1$  is of length at most  $n - 2$ . But it contains  $p^{n-1}$  pebbles, by splitting these  $p^{n-1}$  pebbles into  $p$  arbitrary disjoint sets of  $p^{n-2}$  pebbles each, we can bring  $p$  pebbles to  $v_{k-1}$ . So,  $f_{gl}(v_k, P_n) \leq p^n$  for any internal vertex  $v_k$ .

Hence  $f_{gl}(P_n) = p^n$ . ■

**Theorem 2.5.** Let  $C_n$  denote a cycle with  $n$  vertices.

$$\text{Then, } f_{gl}(C_n) = \begin{cases} p^{n/2}, & \text{if } n \text{ is even,} \\ 1 + 2\lfloor (p/p + 1)(p^{\lfloor n/2 \rfloor} - 1) \rfloor, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** Let  $C_n = (v_0, v_1, v_2, \dots, v_{n-1})$ .

**Case 1:**  $n$  is even.

The edges of  $C_n$  can be partitioned into two disjoint paths say  $P_1$  and  $P_2$  each from  $v_0$  to  $v_{n/2}$ . By placing  $p^{n/2} - 1$  pebbles at  $v_{n/2}$  which is the farthest vertex from  $v_0$ , we cannot move a pebble to  $v_0$ . So  $f_{gl}(v_0, C_n) \geq p^{n/2}$ . Let  $K$  be the number of pebbles placed at  $v_{n/2}$ . If  $K = p^{n/2}$  then we can move a pebble to  $v_0$ .

Suppose  $K < p^{n/2}$ , we place  $p^{n/2} - K$  pebbles on other vertices. If we place all pebbles in one path say  $P_1$ , then  $P_1$  has  $p^{n/2}$  pebbles. So, we can move a pebble to  $v_0$  (Theorem 2.4). If we distribute  $p^{n/2} - K$  pebbles at the internal vertices of both paths, we have to distribute at least  $\lfloor (p^{n/2} - K)/2 \rfloor$  pebbles at the internal vertices of any one of paths,  $P_1$  and  $P_2$ . Without loss of generality, assume that we distribute at least  $\lfloor (p^{n/2} - K)/2 \rfloor$  pebbles at the internal vertices of the path  $P_1$ . Placing one pebble at  $v_{n/2-i}$  is equivalent to placing  $p^i$  pebbles at  $v_{n/2}$ . So, placing  $\lfloor (p^{n/2} - K)/2 \rfloor$  pebbles in  $v_1, v_2, \dots, v_{n/2-1}$  is equivalent to placing at least  $p^{n/2}$  pebbles at  $v_{n/2}$ . So, we can move a pebble to  $v_0$ . So,  $f_{gl}(v_0, C_n) = p^{n/2}$ .

Hence  $f_{gl}(C_n) = p^{n/2}$  since  $v_0$  is arbitrary.

**Case 2:**  $n$  is odd.

Take  $K = 2\lfloor (p/p + 1)(p^{\lfloor n/2 \rfloor} - 1) \rfloor$ .

Suppose  $K$  pebbles are placed as follows:  $K/2$  pebbles are placed at each of the vertices  $v_{\lfloor n/2 \rfloor}$  and  $v_{\lfloor n/2 \rfloor + 1}$ . We can move  $\lfloor K/2p \rfloor$  pebbles from the either of the vertex to the other vertex. So, one of these two vertices has got  $p^{\lfloor n/2 \rfloor} - 1$  pebbles. But these two vertices are at a distance  $\lfloor n/2 \rfloor$  from  $v_0$ . Therefore a pebble cannot be moved to  $v_0$ . So,  $f_{gl}(v_0, C_n) \geq 1 + K$ . Let us prove that  $f_{gl}(v_0, C_n) = 1 + K$ . Suppose that  $K + 1$  pebbles are placed at the vertices of  $C_n$ . Consider the paths  $P_1 = (v_0, v_1, v_2, \dots, v_{\lfloor n/2 \rfloor})$  and  $P_2 = (v_{\lfloor n/2 \rfloor + 1}, \dots, v_{n-1}, v_0)$ . In the positioning of  $K + 1$  pebbles if  $P_1$  or  $P_2$

has at least  $p^{\lfloor n/2 \rfloor}$  pebbles then we are done. When both  $P_1$  and  $P_2$  have less than  $p^{\lfloor n/2 \rfloor}$  pebbles each, the special case is placing  $K/2$  pebbles at either of the vertices  $v_{\lfloor n/2 \rfloor}$  and  $v_{\lfloor n/2 \rfloor + 1}$  and  $K/2 + 1$  at the other vertex. Now, we can move  $\lfloor K/2p \rfloor$  pebbles from the vertex with  $K/2$  pebbles to the vertex with  $K/2 + 1$  pebbles. After this move one of the vertices  $v_{\lfloor n/2 \rfloor}$  and  $v_{\lfloor n/2 \rfloor + 1}$  has  $p^{\lfloor n/2 \rfloor}$  pebbles. Since these two vertices are at a distance  $\lfloor n/2 \rfloor$  from  $v_0$ , a pebble can be moved to  $v_0$ .

In all other positioning, it is equivalent to placing at least  $K/2$  pebbles at either of the vertices  $v_{\lfloor n/2 \rfloor}$  and  $v_{\lfloor n/2 \rfloor + 1}$  and at least  $1 + K/2$  pebbles at the other vertex because placing one pebble at  $v_{\lfloor n/2 \rfloor + 1}$  is equivalent to placing  $p$  pebbles at  $v_{\lfloor n/2 \rfloor}$ . So, one of the vertices  $v_{\lfloor n/2 \rfloor}$  and  $v_{\lfloor n/2 \rfloor + 1}$  has got at least  $p^{\lfloor n/2 \rfloor}$  pebbles. Hence, a pebble can be moved to vertex  $v_0$ . So,  $f_{gl}(v_0, C_n) = 1 + K$ .

Therefore,  $f_{gl}(C_n) = 1 + K$  since  $v_0$  is arbitrary.  $\blacksquare$

**Theorem 2.6.**  $f_{gl}(K_{1,n}) = (p - 1)n + (p^2 - 2p + 2)$  if  $n > 1$  and  $p \geq 2$ .

**Proof.** Let  $V(K_{1,n}) = V_1 \cup V_2$  where  $V_1 = \{r\}$  and  $V_2 = \{s_1, s_2, \dots, s_n\}$ . By Theorem 2.2,  $f_{gl}(r, K_{1,n}) = (p - 1)n + 1$ .

To find  $f_{gl}(s_1, K_{1,n})$ .

We place zero pebbles on  $r$  and  $s_1$ . Place  $p^2 - 1$  pebbles at  $s_2$  and  $p - 1$  pebbles at each of the vertices  $s_3, s_4, \dots, s_n$ . We cannot move a pebble to  $s_1$ . So  $f_{gl}(s_1, K_{1,n}) \geq (p - 1)n + (p^2 - 2p + 2)$ .

**Claim:**  $f_{gl}(s_1, K_{1,n}) = (p - 1)n + (p^2 - 2p + 2)$ .

**Case i:** If  $r$  has only one pebble, the other  $(p - 1)(n - 1 + p)$  pebbles are placed in the  $n - 1$  vertices of  $V_2 \setminus \{s_1\}$ . So at least  $(p - 1)$  of them will have at least  $2(p - 1)$  pebbles and hence  $p - 1$  pebbles can be moved to  $r$ . Now  $r$  has  $p$  pebbles and so one pebble can be moved to  $s_1$ .

**Case ii:** If  $r$  has  $K + 2$  pebbles where  $K = 0, 1, 2, \dots, (p - 3)$ , the other  $(p - 1)n + (p^2 - 2p - K)$  ( $p \geq 3$ ) pebbles are placed in the  $n - 1$  vertices of  $V_2 \setminus \{s_1\}$ , at least  $p - K - 1$  of them will have at least  $2p - 3$  pebbles, and one of them will have at least  $3p - 5$  pebbles, and hence  $p - K$  pebbles can be moved to  $r$ . Now  $r$  has  $p + 2$  pebbles and so one pebble can be moved to  $s_1$  (By placing  $(p - 1)n + (p^2 - 2p - K)$  pebbles onto the  $n - 1$  vertices of  $V_2 \setminus \{s_1\}$ , one vertex has got  $3p - 5$  pebbles, and  $K - 1$  vertices have got  $2p - 4$  pebbles,

and  $p - K$  vertices have got  $2p - 3$  pebbles, and  $n - p - 1$  vertices have got  $p - 1$  pebbles).

**Case iii:** If  $r$  has  $p$  pebbles, then we are done.

**Case iv:** Let  $r$  have zero pebbles. If there is a vertex in  $V_2 \setminus \{s_1\}$  having  $p^2$  pebbles then we are done. Otherwise, We claim that there are  $p$  vertices of  $V_2 \setminus \{s_1\}$  having at least  $p$  pebbles. This is because if  $p - 1$  vertices has  $2p - 2$  pebbles and all others have  $p - 1$  pebbles, then the total count comes to only  $(p - 1)n + (p^2 - 3p + 2)$  which contradicts the fact that we have placed  $(p - 1)n + (p^2 - 2p + 2)$  pebbles. So  $p$  pebbles can be moved to  $r$  and then one to  $s_1$ . So  $f_{gl}(s_1, K_{1,n}) = (p - 1)n + (p^2 - 2p + 2)$ .

Therefore,  $f_{gl}(K_{1,n}) = (p - 1)n + (p^2 - 2p + 2)$  since  $s_1$  is arbitrary. ■

**Theorem 2.7.** For  $m, n \geq 3$ ,  $(p - 1)(m + n - 1) + 1 \leq f_{gl}(K_{m,n}) \leq (m + n - 1)p$  where  $p \geq 3$ .

**Proof.** Let  $V(K_{m,n}) = V_1 \cup V_2$  where each vertex of  $V_1$  is adjacent to each vertex of  $V_2$ .  $V_1$  has  $m$  elements and  $V_2$  has  $n$  elements. Let  $r$  be an element of  $V_1$ . By placing  $p - 1$  pebbles on all the  $n$  vertices of  $V_2$  and all the  $m - 1$  vertices of  $V_1 \setminus \{r\}$ , we cannot move a pebble to  $r$ . So  $f_{gl}(K_{m,n}) \geq (p - 1)(m + n - 1) + 1$ .

By placing  $(m + n - 1)p$  pebbles on all the vertices of  $V_1 \setminus \{r\}$ , there exists at least one vertex  $u_1$  in  $V_1 \setminus \{r\}$  which has  $p^2 - 1$  pebbles and there exists at least one vertex  $u_2$  in  $V_1 \setminus \{u_1, r\}$  which has at least  $p$  pebbles. We can move  $p - 1$  pebbles from  $u_1$  and we can move one pebble from  $u_2$  to any vertex  $w_1$  in  $V_2$ . From  $w_1$ , we can move one pebble to  $r$  which is adjacent to  $w_1$ .  $f_{gl}(K_{m,n}) \leq (m + n - 1)p$ .

Hence  $(p - 1)(m + n - 1) + 1 \leq f_{gl}(K_{m,n}) \leq (m + n - 1)p$  where  $p \geq 3$ . ■

### 3 Conclusion

We have determined the generalized pebbling number of complete graph  $K_n$ , path  $P_n$ , cycle  $C_n$  and  $K_{1,n}$ . Also, We have obtained the bounds for  $f_{gl}(K_{m,n})$ .

## References

- [1] Fan R.K. Chung, *Pebbling in hypercubes*, SIAM J. Discrete Math. **2** (1989), 467–472.
- [2] C. Xavier and A. Lourdusamy, *Pebbling number in graphs*, Pure and Applied Matematika Sciences **XLIII** (March 1996), no. 1-2, 73–79.

**Received: November, 2009**