On Imaginable $T$-Fuzzy Ideals of $\Gamma$-Rings

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Abstract

Using $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy ideals in $\Gamma$-rings, and some related properties are investigated. We show that the family of $T$-fuzzy ideals is a completely distributive lattice.

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1 Introduction

N. Nobusawa ([4]) introduced the notion of a $\Gamma$-ring, as more general than a ring. W. E. Barnes ([2]) weakened slightly the conditions in the definition of the $\Gamma$-ring in the sense of Nobusawa. The notion of fuzzy set in a set was introduced by L. A. Zadeh ([7]), and since then this concept have been applied to various algebraic structures. On the other hand, Schweizer and Sklar ([5]) introduced the notion of triangular norm ($t$-norm) and triangular conorm ($t$-conorm). Triangular norm and triangular conorm are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively. Thus, the $t$-norm generalizes the conjunctive operator and the $t$-conorm generalizes the disjunctive operator. In application, $t$-norm $T$ and $t$-conorm $S$ are two functions that map the unit square into the unit interval. In this paper, we inquired further into the properties on fuzzy ideals. Using $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy ideal of an $\Gamma$-ring $M$, and obtain some related results. We show that the family of $T$-fuzzy ideals is a completely distributive lattice.
2 Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

If \( M = \{x, y, z, \cdots\} \) and \( \Gamma = \{\alpha, \beta, \gamma, \cdots\} \) are additive abelian groups, and for all \( x, y, z \) in \( M \) and all \( \alpha, \beta \) in \( \Gamma \), the following conditions are satisfied
- \( x\alpha y \) is an element of \( M \),
- \( (x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)y = x\alpha y + x\beta y, x\alpha(y + z) = x\alpha y + x\alpha z \),

then \( M \) is called a \( \Gamma \)-ring.

A subset \( A \) of the \( \Gamma \)-ring \( M \) is a left (resp. right) ideal of \( M \) if \( A \) is an additive subgroup of \( M \) and \( M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\} \) is contained in \( A \). If \( A \) is both a left and a right ideal, then \( A \) is a two-sided ideal, or simply an ideal of \( M \).

A fuzzy set in \( M \) is a function \( \mu : M \to [0, 1] \). Let \( \mu \) be a fuzzy set in \( \in M \). For \( \alpha \in [0, 1], \) the set \( U(\mu, \alpha) = \{x \in M | \mu(x) \geq \alpha\} \) is called level set of \( \mu \). A fuzzy set \( \mu \) in a \( \Gamma \)-ring \( M \) is called a left (resp. right) ideal of \( M \) if it satisfies:
- \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \),
- \( \mu(x\alpha y) \geq \mu(y) \) (resp. \( \mu(x\alpha y) \geq \mu(x) \)), for all \( x, y \in M \) and all \( \alpha \in \Gamma \).

A fuzzy set \( \mu \) in a \( \Gamma \)-ring \( M \) is called a fuzzy ideal of \( M \) if \( \mu \) is a both a fuzzy left and a fuzzy right ideal of \( M \).

**Definition 2.1.** ([5]) By a \( t \)-norm \( T \), we mean a function \( T : [0, 1] \times [0, 1] \to [0, 1] \) satisfying the following conditions:
- (T1) \( T(x, 1) = x \),
- (T2) \( T(x, y) \leq T(x, z) \) if \( y \leq z \),
- (T3) \( T(x, y) = T(y, x) \),
- (T4) \( T(x, T(y, z)) = T(T(x, y), z) \),

for all \( x, y, z \in [0, 1] \).

For a \( t \)-norm \( T \) on \([0, 1] \), denote by \( \Delta_T \) the set of element \( \alpha \in [0, 1] \) such that \( T(\alpha, \alpha) = \alpha \), i.e., \( \Delta_T := \{\alpha \in [0, 1] | T(\alpha, \alpha) = \alpha\} \).

**Proposition 2.2.** Every \( t \)-norm \( T \) has a useful property:

\[
T(\alpha, \beta) \leq \min(\alpha, \beta)
\]

for all \( \alpha, \beta \in [0, 1] \).
Throughout this paper, all proofs are going to proceed the only left cases, because the right cases are obtained from similar method. In what follows, the terms “fuzzy ideal” and “Noetherian Γ-ring” mean “fuzzy left ideal” and “left Noetherian Γ-ring”, respectively. We denote $0_M$ the zero element of a Γ-ring $M$.

3 $T$-fuzzy ideals in Γ-rings

**Definition 3.1.** A fuzzy set $\mu$ in a Γ-ring $M$ is called a fuzzy left (resp. right) ideal of $M$ with respect to a $t$-norm $T$ (briefly, a $T$-fuzzy ideal of $M$) if it satisfies:

(TF1) $\mu(x - y) \geq T\{\mu(x), \mu(y)\}$,

(TF2) $\mu(x \alpha y) \geq \mu(y)$ (resp. $\mu(x \alpha y) \geq \mu(x)$), for all $x, y \in M$ and all $\alpha \in \Gamma$.

**Example 3.2.** If $G$ and $H$ are additive abelian groups and $M = \text{Hom}(G, H)$, $\Gamma = \text{Hom}(H, G)$, then $M$ is a Γ-ring with the operations pointwise addition and composition of homomorphisms ([1]). Let $T$ be a $t$-norm defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

for all $\alpha, \beta \in [0, 1]$. Define a fuzzy set $\mu : M \rightarrow [0, 1]$ by $\mu(0_M) = 0.5, \mu(f) = 0.3$ where $f$ is any member of $M$ with $f \neq 0_M$. Then $\mu$ is a $T$-fuzzy ideal of $M$.

**Definition 3.3.** Let $T$ be a $t$-norm. A fuzzy set $\mu$ in $M$ is said to satisfy imaginable property if $\text{Im}(\mu) \subseteq \Delta_T$.

**Proposition 3.4.** Let $I$ be an ideal of a Γ-ring and let $\mu$ be a fuzzy set in $M$ defined by

$$\mu(x) := \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise} \end{cases},$$

for all $x \in M$, where $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Then $\mu$ is a $T$-fuzzy ideal of $M$, where $T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0, 1]$.

**Proof.** Let $x, y \in M$. If $x, y \in I$, then

$$T(\mu(x), \mu(y)) = T(\alpha, \alpha) = \max(2\alpha - 1, 0)$$

$$= \begin{cases} 2\alpha & \text{if } \alpha \geq \frac{1}{2}, \\ \beta & \text{if } \alpha < \frac{1}{2}, \end{cases}$$

$$\leq \alpha = \mu(x - y),$$

and for all $\alpha \in \Gamma$, we have $\mu(x \alpha y) = \mu(y) = \alpha$. If $x \in I$ and $y \notin I$ (or, $x \notin I$ and $y \in I$), then
\[ T(\mu(x), \mu(y)) = T(\alpha, \beta) = \max(\alpha + \beta - 1, 0) \]

\[ = \begin{cases} 
\alpha + \beta - 1 & \text{if } \alpha + \beta \geq \frac{1}{2}, \\
\beta & \text{otherwise,} 
\end{cases} \leq \beta = \mu(x - y), \]

and for all \(\alpha \in \Gamma\), we have \(\mu(x\alpha y) \geq \beta = \mu(y)\). If \(x \notin I\) and \(y \notin I\), then

\[ T(\mu(x), \mu(y)) = T(\alpha, \beta) = \max(2\beta - 1, 0) \]

\[ = \begin{cases} 
2\beta - 1 & \text{if } \beta \geq \frac{1}{2}, \\
0 & \text{otherwise}, 
\end{cases} \leq \beta = \mu(x - y), \]

and for all \(\alpha \in \Gamma\), we have \(\mu(x\alpha y) \geq \beta = \mu(y)\). Hence \(\mu\) is a \(T\)-fuzzy ideal of a \(\Gamma\)-ring \(M\).

**Theorem 3.5.** Let \(\mu\) be a \(T\)-fuzzy ideal of a \(\Gamma\)-ring \(M\) and let \(\alpha \in \Gamma\) be such that that \(T(\alpha, \alpha) = \alpha\). Then \(U(\mu; \alpha)\) is either empty or an ideal of \(M\) for all \(x \in M\).

**Proof.** Let \(x, y \in U(\mu; \alpha)\). Then we have \(\mu(x) \geq \alpha\) and \(\mu(y) \geq \alpha\), and so

\[ \mu(x - y) \geq T(\mu(x), \mu(y)) \geq T(\alpha, \alpha) = \alpha, \]

which implies that \(x - y \in U(\mu; \alpha)\). Now let \(x \in M, y \in U(\mu; \alpha)\) and \(\gamma \in \Gamma\). Then we have \(\mu(\alpha\gamma y) \geq \mu(y) \geq \alpha\), so \(x\gamma y \in U(\mu; \alpha)\). Hence \(U(\mu; \alpha)\) is an ideal of \(M\). ending the proof.

Since \(T(1, 1) = 1\), we have the following corollary.

**Corollary 3.6.** If \(\mu\) is a \(T\)-fuzzy ideal of a \(\Gamma\)-ring \(M\), then \(U(\mu; 1)\) is either empty or an ideal of \(M\).

**Proposition 3.7.** Let \(T\) be a \(t\)-norm on \([0, 1]\) and let \(\mu\) be a fuzzy set in a \(\Gamma\)-ring \(M\) with \(\text{Im}(\mu) = \{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n\}\) where \(\alpha_i < \alpha_j\) whenever \(i > j\). Suppose that there exists a chain of ideals of \(M\):

\[ G_0 \subset G_1 \subset \cdots \subset G_n = M \]

such that \(\mu(\bar{G}_k) = \alpha_k\), where \(\bar{G}_k = G_k \setminus G_{k-1}\) and \(G_1 = 0\) for \(k = 0, 1, \cdots, n\). Then \(\mu\) is a \(T\)-fuzzy ideal of \(M\).

**Proof.** Let \(x, y \in M\). If \(x\) and \(y\) belong to the same \(\bar{G}_k\), then \(\mu(x) = \mu(y) = \alpha_k\) and \(x - y \in G_k\). Hence

\[ \mu_A(x - y) \geq \alpha_k = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y)). \]
Let \( x \in \tilde{G}_i \) and \( y \in \tilde{G}_j \) for every \( i \neq j \). Without loss of generality we may assume that \( i > j \). Then \( \mu(x) = \alpha_i < \alpha_j = \mu(y) \) and \( x - y \in G_i \). It follows that
\[
\mu(x - y) \geq \alpha_i = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y)).
\]

Now let \( x \in M, y \in G_k \) and \( \alpha \in \Gamma \). Then there exists \( G_k \) such that \( y \in \tilde{G}_k \) for some \( k \in \{0, 1, 2, \cdots \} \). Hence we have \( x\alpha y \in G_j \) which implies that \( \mu(x\alpha y) \geq \alpha_j = \mu(y) \). Hence \( \mu \) is an \( T \)-fuzzy ideal of a \( \Gamma \)-ring. \( \Box \)

**Definition 3.8.** An \( \Gamma \)-ring \( M \) is said to satisfy the ascending (resp. descending) chain condition if for every ascending (resp. descending) sequence \( A_1 \subseteq A_2 \subseteq A_3 \cdots \) (resp. \( A_1 \supseteq A_2 \supseteq A_3 \cdots \)) of ideals of \( M \), there exists a natural number \( n \) such that \( A_n = A_k \) for all \( n \geq k \). If \( M \) satisfies the ascending chain condition, we say \( M \) is a Noetherian \( \Gamma \)-ring.

**Theorem 3.9.** Let \( \{A_k \mid k \in N\} \) be a family of ideals of a \( \Gamma \)-ring \( M \) which is nested, i.e., \( A_1 \supset A_2 \supset A_3 \supset \cdots \). Let \( \mu \) be a fuzzy set in \( M \) defined by
\[
\mu(x) = \begin{cases} 
\frac{k}{k+1} & \text{if } x \in A_k \setminus A_{k+1}, k = 0, 1, 2, \ldots, \\
1 & \text{if } x \in \bigcap_{k=0}^{\infty} A_k,
\end{cases}
\]
for all \( x \in M \) where \( A_0 \) stands for \( M \). Let \( \alpha \) be such that \( T(\alpha, \alpha) = \alpha \). Then \( \mu \) is an \( T \)-fuzzy ideal of a \( \Gamma \)-ring \( M \).

**Proof.** Let \( x \) and \( y \in M \). Assume that \( x \in A_k \setminus A_{k+1} \) and \( y \in A_r \setminus A_{r+1} \) for \( k = 0, 1, 2, \ldots; r = 0, 1, 2, \ldots \). Without loss of generality we may assume that \( k \leq r \). Then obviously \( y \in A_k \). Since \( A_k \) is an ideal of a \( \Gamma \)-ring \( M \), it follows that \( x - y \in A_k \) and \( x\alpha y \in A_k \) which implies that \( \mu(x - y) \geq \frac{k}{k+1} = T(\mu(x), \mu(y)) \)
and \( \mu(x\alpha y) \geq \frac{k}{k+1} = \mu(y) \) for all \( \alpha \in \Gamma \). If \( x \in \bigcap_{k=0}^{\infty} A_k \) and \( y \in \bigcap_{k=0}^{\infty} A_k \), then \( x - y \in \bigcap_{k=0}^{\infty} A_k \). Hence \( \mu(x - y) = 1 = T(\mu(x), \mu(y)) \) and \( \mu(x\alpha y) = 1 \geq \mu(y) \).

If \( x \notin \bigcap_{k=0}^{\infty} A_k \) and \( y \in \bigcap_{k=0}^{\infty} A_k \), then there exists \( n \in N \) such that \( x \in A_n \setminus A_{n+1} \). It follows that \( x - y \in A_n \) which implies that \( \mu(x - y) \geq \frac{n}{n+1} = T(\mu(x), \mu(y)) \)
and \( \mu(x\alpha y) \geq \frac{n}{n+1} = \mu(y) \) for all \( \alpha \in \Gamma \). Finally assume that \( x \in \bigcap_{k=0}^{\infty} A_k \) and \( y \notin \bigcap_{k=0}^{\infty} A_k \), then \( y \in A_n \setminus A_{n+1} \) for some \( n \in N \). Hence \( x - y \in A_n \), and thus \( \mu(x - y) \geq \frac{n}{n+1} = T(\mu(x), \mu(y)) \) and \( \mu(x\alpha y) \geq \frac{n}{n+1} = \mu(y) \) for all \( \alpha \in \Gamma \). Therefore \( \mu \) is an \( T \)-fuzzy ideal of a \( \Gamma \)-ring \( M \). \( \Box \)

**Theorem 3.10.** Let \( M \) be an \( \Gamma \)-ring satisfying descending chain condition and let \( \mu \) be an \( T \)-fuzzy ideal of \( M \). Let \( \alpha \in [0, 1] \) be such that \( T(\alpha, \alpha) = \alpha \). If
a sequence of elements of $\text{Im} \mu$ is strictly increasing, then $\mu$ has finite number of values.

Proof. Let $\{t_k\}$ be a strictly increasing sequence of elements of $\text{Im} \mu$. Then $0 \leq t_1 \leq t_2 \leq \cdots \leq 1$. Then $U(\mu; t_r)$ is an ideal of $M$ for all $r = 2, 3, \ldots$ from Theorem 3.12. Let $x \in U(\mu; t_r)$. Then $\mu(x) \geq t_r \geq t_{r-1}$, and so $x \in U(\mu; t_{r-1})$. Hence $U(\mu; t_r) \subseteq U(\mu; t_{r-1})$. Since $t_{r-1} \in \text{Im} \mu$, there exists $x_{r-1} \in M$ such that $\mu(x_{r-1}) = t_{r-1}$. It follows that $x_{r-1} \in U(\mu; t_{r-1})$ but $x_{r-1} \notin U(\mu; t_r)$. Thus $U(\mu; t_r) \subset U(\mu; t_{r-1})$, and so we obtain a strictly descending sequence $U(\mu; t_1) \supset U(\mu; t_2) \supset U(\mu; t_3) \supset \cdots$ of ideals of $M$ which is not terminating. This contradicts the assumption that $M$ satisfies the descending chain condition. Consequently, $\mu$ has finite number of values. \hfill \Box

Theorem 3.11. Let $M$ be a $\Gamma$-ring and let $\alpha \in [0, 1]$ be such that $T(\alpha, \alpha) = \alpha$. Then the following are equivalent:

(i) $M$ is a Noetherian $\Gamma$-ring

(ii) The set of values of any ideal of $M$ is a well-ordered subset of $[0, 1]$.

Proof. (i) $\implies$ (ii). Let $\mu$ be an $T$-fuzzy ideal of $M$. Suppose that the set of values of $\mu$ is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequences $\{t_k\}$ such that $\mu(x_k) = t_k$. It follows that $U(\mu; t_1) \subset U(\mu; t_2) \subset \cdots$ is a strictly ascending chain of ideals of $M$, where $U(\mu; t_r) = \{x \in M \mid \mu(x) \geq t_r\}$ for every $r = 1, 2, \ldots$. This contradicts the assumption that $M$ is a Noetherian $\Gamma$-ring.

(ii) $\implies$ (i). Suppose that the condition (ii) is satisfied and $M$ is not a Noetherian $\Gamma$-ring. Then there exists a strictly ascending chain

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

of ideals of $M$. Note that $A := \bigcup_{k \in \mathbb{N}} A_k$ is an ideal of $M$. Define a fuzzy set in $M$ by

$$\mu(x) := \begin{cases} \alpha & \text{if } x \notin A, \\ \frac{1}{r} & \text{where } r = \min\{k \in \mathbb{N} \mid x \in A_k\} \end{cases}$$

We claim that $\mu$ is an $T$-fuzzy ideal of $M$. Let $x, y \in M$. If $x \in A_k \setminus A_{k-1}$ and $y \in A_k \setminus A_{k-1}$, then $x - y \in A_k$ and $x \alpha y \in A_k$. It follows that $\mu(x - y) \geq \frac{1}{k} = T(\mu(x), \mu(y))$ and $\mu(x \beta y) \geq \frac{1}{k} \geq \mu(y)$ for all $\beta \in \Gamma$. Suppose that $x \in A_k$ and $y \in A_k \setminus A_r$ for all $r < k$. Since $A_k$ is an ideal of $M$, it follows that $x - y \in A_k$. Hence $\mu(x) \geq \frac{1}{k} \geq \frac{1}{k+1} \geq \mu(y)$, and so $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$. Similarly, for the case $x \in A_k \setminus A_r$ and $y \in A_k$, we have $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$ and $\mu(x \beta y) \geq \mu(y)$ for all $\beta \in \Gamma$. Thus $\mu$ is a $T$-fuzzy ideal of $M$. Since the chain is not terminating, $\mu$ has a strictly descending sequence of values. This contradicts the assumption that the value set of any ideal is well-ordered. Therefore $M$ is a Noetherian $\Gamma$-ring. \hfill \Box
For a family \( \{ \mu_\alpha \mid \alpha \in \Lambda \} \) fuzzy sets in \( M \), define the join \( \bigvee_{\alpha \in \Lambda} \mu_\alpha \) and the meet \( \bigwedge_{\alpha \in \Lambda} \mu_\alpha \) as follows:

\[
\bigvee_{\alpha \in \Lambda} \mu_\alpha(x) = \sup \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}, \quad \bigwedge_{\alpha \in \Lambda} \mu_\alpha(x) = \inf \{ \mu_\alpha(x) \mid \alpha \in \Lambda \},
\]

for all \( x \in M \), where \( \Lambda \) is any index set.

**Theorem 3.12.** The family of \( T \)-fuzzy ideals in \( M \) is a completely distributive lattice with respect to meet “\( \wedge \)” and join “\( \vee \)”.

**Proof.** Since \([0,1]\) is a completely distributive lattice with respect to the usual ordering in \([0,1]\), it is sufficient to show that \( \bigvee_{\alpha \in \Lambda} \mu_\alpha \) and \( \bigwedge_{\alpha \in \Lambda} \mu_\alpha \) are \( T \)-fuzzy ideals of \( M \) for a family of \( T \)-fuzzy ideals \( \{ \mu_\alpha \mid \alpha \in \Lambda \} \).

For any \( x, y \in M \), we have

\[
\bigvee_{\alpha \in \Lambda} \mu_\alpha(x - y) = \sup \{ \mu_\alpha(x - y) \mid \alpha \in \Lambda \}
\]

\[
\geq \sup \{ T(\mu_\alpha(x), \mu_\alpha(y)) \mid \alpha \in \Lambda \}
\]

\[
\geq T(\sup \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}, \sup \{ \mu_\alpha(y) \mid \alpha \in \Lambda \})
\]

\[
= T((\bigvee_{\alpha \in \Lambda} \mu_\alpha)(x), (\bigvee_{\alpha \in \Lambda} \mu_\alpha)(y)).
\]

Similarly,

\[
\bigwedge_{\alpha \in \Lambda} \mu_\alpha(x - y) = \inf \{ \mu_\alpha(x - y) \mid \alpha \in \Lambda \}
\]

\[
\geq \inf \{ T(\mu_\alpha(x), \mu_\alpha(y)) \mid \alpha \in \Lambda \}
\]

\[
\geq T(\inf \{ \mu_\alpha(x) \mid \alpha \in \Lambda \}, \inf \{ \mu_\alpha(y) \mid \alpha \in \Lambda \})
\]

\[
= T((\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(x), (\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(y)).
\]

Now let \( x, y \in M \) and \( \alpha \in \Gamma \). Then

\[
\bigvee_{\alpha \in \Lambda} \mu_\alpha(x\alpha y) = \sup \{ \mu_\alpha(x\alpha y) \mid \alpha \in \Lambda \}
\]

\[
\geq \sup \{ \mu_\alpha(y) \mid \alpha \in \Lambda \}
\]

\[
= (\bigvee_{\alpha \in \Lambda} \mu_\alpha)(y),
\]

\[
\bigwedge_{\alpha \in \Lambda} \mu_\alpha(x\alpha y) = \inf \{ \mu_\alpha(x\alpha y) \mid \alpha \in \Lambda \}
\]

\[
\geq \inf \{ \mu_\alpha(y) \mid \alpha \in \Lambda \}
\]

\[
= (\bigwedge_{\alpha \in \Lambda} \mu_\alpha)(y),
\]

Hence \( \bigvee_{\alpha \in \Lambda} \mu_\alpha \) and \( \bigwedge_{\alpha \in \Lambda} \mu_\alpha \) are \( T \)-fuzzy ideals of \( M \), completing the proof. \( \square \)
**Theorem 3.13.** Let $T$ be a $t$-norm and let $\mu$ be an imaginable fuzzy set in $M$. If each non-empty upper level set $U(\mu; \alpha)$ of $\mu$ is an ideal of a $\Gamma$-ring $M$, then $\mu$ is an imaginable $T$-fuzzy ideal of a $\Gamma$-ring $M$.

**Proof.** Suppose each non-empty upper level set $U(\mu; \alpha)$ of $\mu$ is an ideal of $M$. Then we first show that $\mu(x - y) \geq \min(\mu(x), \mu(y))$ for all $x, y \in M$.

In fact, if not then there exist $x_0, y_0 \in M$ such that $\mu(x_0 - y_0) < \min(\mu(x_0), \mu(y_0))$. Taking $\alpha_0 := \frac{1}{2}(\mu(x_0 - y_0) + \min(\mu(x_0), \mu(y_0)))$, we get $\mu(x_0 - y_0) < \alpha_0 < \min(\mu(x_0), \mu(y_0))$ and thus $x_0, y_0 \in U(\mu; \alpha)$ and $x_0 - y_0 \notin U(\mu; \alpha)$. This is a contradiction. Hence $\mu(x - y) \geq \min(\mu(x), \mu(y)) \geq T(\mu(x), \mu(y))$ for all $x, y \in M$. Now if (TF2) is not true, then $\mu(x_0 \gamma y_0) < \mu(y_0)$ for some $x_0, y_0 \in M$ and $\gamma \in \Gamma$. Taking $s_1 := \frac{1}{2}(\mu(x_0 \gamma y_0) + \mu(y_0))$, then $0 \leq s_1 < \mu(y_0)$ and $\mu(x_0 \gamma y_0) < s_1$. Hence $y_0 \in U(\mu; s_1)$ and $x_0 \gamma y_0 \notin U(\mu; s_1)$, a contradiction. This completes the proof. 

**References**


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