Upper Semi-Continuous
Interval-Valued Multihomomorphisms

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Abstract

It has been known since Cauchy’s time that if \( f \) is a continuous homomorphism on \((\mathbb{R}, +)\), then there is a constant \( c \in \mathbb{R} \) such that \( f(x) = cx \) for all \( x \in \mathbb{R} \). The purpose of this paper is to extend this result to interval-valued multifunctions on \( \mathbb{R} \) as follows: For an interval-valued multifunction \( f \) on \( \mathbb{R} \), \( f \) is an upper semi-continuous multihomomorphism on \((\mathbb{R}, +)\) if and only if \( f \) has one of the following forms:

\[
\begin{align*}
f(x) &= \{cx\}, \\
f(x) &= \mathbb{R}, \\
f(x) &= (0, \infty), \\
f(x) &= (-\infty, 0), \\
f(x) &= [cx, \infty) \\
\end{align*}
\]

and \( f(x) = (-\infty, cx] \) where \( c \) is a constant.

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1 Introduction

The cardinality of a set \( X \) will be denoted by \(|X|\).

A multifunction from a nonempty set \( X \) into a nonempty set \( Y \) is a function \( f : X \to \mathcal{P}(Y) \) where \( \mathcal{P}(Y) \) is the power set of \( Y \) and \( \mathcal{P}^*(Y) = \mathcal{P}(Y) \setminus \{\emptyset\} \).

By a multifunction on \( X \) we mean a multifunction from \( X \) into itself.

A multifunction \( f \) from a group \( G \) into a group \( G' \) is called a multihomomorphism if

\[
f(xy) = f(x)f(y) = \{st \mid s \in f(x) \text{ and } t \in f(y)\}
\]

for all \( x, y \in G \).

We note here that the concept of multi-valued endomorphisms of a hypergroup appeared in [1, page 176] is more general than ours. Multihomomorphisms between cyclic groups were characterized in [6]. These characterizations were used in [3] to determine a multihomomorphism from a cyclic group \( G \) into a
cyclic group $G'$ whose image covers $G'$. In [8], the authors gave some remarkable necessary conditions of multihomomorphisms from any group into groups of real numbers under the usual addition and multiplication.

A multifunction $f$ from a topological space $X$ into a topological space $Y$ is said to be upper [lower] semi-continuous at $a \in X$ if for any open set $V$ in $Y$ such that $f(a) \subseteq V$ if $f(a) \cap V \neq \emptyset$, there exists an open set $U$ in $X$ containing $a$ such that $f(U) \subseteq V$ if $f(x) \cap V \neq \emptyset$ for all $x \in U$. See [4], page 261. If $f$ is upper [lower] semi-continuous at every point in $X$, then $f$ is said to be upper [lower] semi-continuous on $X$. Evidently, the upper [lower] semi-continuity coincides with the continuity of a single-valued function at $a \in X$. Some results of upper and lower semicontinuity of multifunctions between two topological spaces can be found in [7], [5] and [2].

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{N}$ the set of natural numbers (positive integers). By an interval-valued multifunction on $\mathbb{R}$ we mean a multifunction $f$ on $\mathbb{R}$ such that $f(x)$ is an interval in $\mathbb{R}$ for every $x \in \mathbb{R}$. Hence the concept of interval-valued multihomomorphism on $(\mathbb{R}, +)$ is a generalization of the concept of homomorphism on $(\mathbb{R}, +)$.

It has been known since Cauchy's time that if $f$ is a continuous homomorphism on $(\mathbb{R}, +)$ with the usual topology, then there exists a constant $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. This result is extended in this paper to interval-valued multifunctions on $\mathbb{R}$. We characterize when an interval-valued multifunction $f$ on $\mathbb{R}$ is an upper semi-continuous multihomomorphism on $(\mathbb{R}, +)$.

## 2 Main Results

To obtain our main result, the following series of lemmas is needed.

**Lemma 2.1.** If $f$ is an interval-valued multihomomorphism on $(\mathbb{R}, +)$, then $f(0)$ is one of the followings: $\{0\}, \mathbb{R}, (0, \infty), (-\infty, 0), [0, \infty), (-\infty, 0]$.

**Proof.** If $f(0)$ is bounded below, then

$$\inf(f(0)) = \inf(f(0 + 0)) = \inf(f(0) + f(0)) = \inf(f(0)) + \inf(f(0)),$$

so $\inf(f(0)) = 0$. Similarly, if $f(0)$ is bounded above, then $\sup(f(0)) = 0$. Since $f(0)$ is an interval in $\mathbb{R}$, the desired result follows. \qed
Lemma 2.2. If \( f \) is an interval-valued multihomomorphism on \((\mathbb{R}, +)\), then for every \( x \in \mathbb{R}, f(x) \) and \( f(0) \) are intervals in \( \mathbb{R} \) of the same form, that is,

\[
    f(x) = \begin{cases} \{y\} & \text{if } f(0) = \{0\}, \\ \mathbb{R} & \text{if } f(0) = \mathbb{R}, \\ (y, \infty) & \text{if } f(0) = (0, \infty), \\ (-\infty, y) & \text{if } f(0) = (-\infty, 0), \\ [y, \infty) & \text{if } f(0) = [0, \infty), \\ (-\infty, y] & \text{if } f(0) = (-\infty, 0] \\
\end{cases}
\]

for some \( y \in \mathbb{R} \).

Proof. Let \( x \in \mathbb{R}, a \in f(-x) \) and \( b \in f(x) \). Then

\[
    f(0) = f(x - x) = f(x) + f(-x) \supseteq f(x) + a, \\
    f(x) = f(x + 0) = f(x) + f(0) \supseteq b + f(0)
\]

(1)

From (1), we have that if \( f(0) = \{0\} \), then \( |f(x)| = 1 \), and

\[
    f(0) \text{ is bounded above [bounded below] if and only if } f(x) \text{ is bounded above [bounded below]}. 
\]

(2)

It follows directly from (2) that \( f(x) = \mathbb{R} \) if \( f(0) = \mathbb{R} \). Assume that \( f(0) = (0, \infty) \). By (2), \( f(x) \) is either \((y, \infty)\) or \([y, \infty)\) for some \( y \in \mathbb{R} \). Since \( f(x) + f(0) = f(x + 0) = f(x) \) and \([y, \infty) + (0, \infty) = (y, \infty) \neq [y, \infty)\), we deduce that \( f(x) = (y, \infty) \). By a similar argument, we have that if \( f(0) = (-\infty, 0) \), then \( f(x) = (-\infty, y) \) for some \( y \in \mathbb{R} \).

Next, assume that \( f(0) = [0, \infty) \). It follows from (2) that \( f(x) \) is either \((y, \infty)\) or \([y, \infty)\) for some \( y \in \mathbb{R} \). From (2), we also have that \( f(-x) \) is either \((z, \infty)\) or \([z, \infty)\) for some \( z \in \mathbb{R} \). But since \([0, \infty) = f(0) = f(x) + f(-x)\), it is immediate that \( f(x) = [y, \infty) \) and \( f(-x) = [z, \infty) \). It can be proved similarly that if \( f(0) = (-\infty, 0] \), then \( f(x) = (-\infty, y]\) for some \( y \in \mathbb{R} \). \( \square \)

Lemma 2.3. Let \( f \) be an interval-valued multihomomorphism on \((\mathbb{R}, +)\). If \( x, y \in \mathbb{R} \) are such that \( f(x) \) is \((y, \infty), (-\infty, y], [y, \infty) \) or \((-\infty, y]\), then \( f(-x) \) is \((-y, \infty), (-\infty, -y), [-y, \infty) \) or \((-\infty, -y]\), respectively.
Proof. It is directly obtained from Lemma 2.2 and the fact that \( f(x) + f(-x) = f(x - x) = f(0) \). \qed

Lemma 2.4. If \( f \) is an interval-valued multihomomorphism on \((\mathbb{R}, +)\), then for all \( x \in \mathbb{R} \) and \( m, n \in \mathbb{N} \), \( f \left( \frac{m}{n} x \right) = \frac{m}{n} f(x) \).

Proof. By Lemma 2.2, for all \( x \in \mathbb{R} \), \( f(x) \) is one of the followings : \( \{y\}, \mathbb{R}, (y, \infty), (-\infty, y), [y, \infty), (-\infty, y] \) for some \( y \in \mathbb{R} \). This implies that

\[
\text{for all } x \in \mathbb{R} \text{ and } m \in \mathbb{N}, f(\frac{m}{n} x) = f(x) + \cdots + f(x) = m f(x),
\]

and hence

\[
\text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N}, \quad \frac{1}{n} f(x) = \frac{1}{n} \left( f \left( \frac{x}{n} + \cdots + \frac{x}{n} \right) \right) = \frac{1}{n} \left( nf \left( \frac{x}{n} \right) \right) = f \left( \frac{x}{n} \right).
\]

Thus (1) and (2) yield the result that

\[
\text{for all } x \in \mathbb{R} \text{ and } m, n \in \mathbb{N}, f \left( \frac{m}{n} x \right) = mf \left( \frac{x}{n} \right) = \frac{m}{n} f(x).
\]

Lemma 2.5. Let \( f \) be an interval-valued multihomomorphism on \((\mathbb{R}, +)\) which is upper semi-continuous at 0.

(i) If \( f(0) = (0, \infty) \), then \( f(x) = (0, \infty) \) for all \( x \in \mathbb{R} \).

(ii) If \( f(0) = (-\infty, 0) \), then \( f(x) = (-\infty, 0) \) for all \( x \in \mathbb{R} \).

Proof. (i) Since \((0, \infty)\) is an open set in \(\mathbb{R}\) such that \((0, \infty) \supseteq f(0)\) and \( f \) is upper semi-continuous at 0, there is a \( \delta > 0 \) such that \( f((-\delta, \delta)) \subseteq (0, \infty) \). Let \( x \in (-\delta, \delta) \). By Lemma 2.2, \( f(x) = (y, \infty) \) for some \( y \in \mathbb{R} \). Then \( -x \in (-\delta, \delta) \) and by Lemma 2.3, \( f(-x) = (-y, \infty) \). Thus \((y, \infty) \cup (-y, \infty) \subseteq (0, \infty)\) which implies that \( y = 0 \). This proves that

\[
\text{for all } x \in (-\delta, \delta), f(x) = (0, \infty).
\]
If $z \in \mathbb{R}$, then there is an $n \in \mathbb{N}$ such that $\frac{z}{n} \in (-\delta, \delta)$, so by Lemma 2.4 and the above fact, we have

$$f(z) = f\left(n \left(\frac{z}{n}\right)\right) = nf\left(\frac{z}{n}\right) = n(0, \infty) = (0, \infty).$$

Hence (i) is proved.

(ii) can be proved analogously to (i). \qed

Lemma 2.6. Let $f$ be an interval-valued multihomomorphism on $(\mathbb{R}, +)$ and $c \in \mathbb{R}$.

(i) If $f(1) = [c, \infty)$, then for all $q \in \mathbb{Q}$, $f(q) = [cq, \infty)$.

(ii) If $f(1) = (-\infty, c]$, then for all $q \in \mathbb{Q}$, $f(q) = (-\infty, cq]$.

Proof. (i) By Lemma 2.2, $f(0) = [0, \infty) = [c0, \infty)$. If $q \in \mathbb{Q}$ is such that $q > 0$, then by Lemma 2.3 and Lemma 2.4,

$$f(q) = f(q1) = qf(1) = cq[0, \infty) = [cq, \infty),$$

$$f(-q) = f(q(-1)) = qf(-1) = q[-c, \infty) = (-c)q, \infty) = [(-c)q, \infty) = [c(-q), \infty).$$

Hence (i) is proved.

(ii) can be proved analogously to (i). \qed

Lemma 2.7. Let $f$ be an interval-valued multihomomorphism on $(\mathbb{R}, +)$, $x, y \in \mathbb{R}, \epsilon > 0$ and $\delta > 0$.

(i) If $f(x) = [y, \infty)$ and $f((x - \delta, x + \delta)) \subseteq (y - \epsilon, \infty)$, then $\min f(z) \in (y - \epsilon, y + \epsilon)$ for all $z \in (x - \delta, x + \delta)$.

(ii) If $f(x) = (-\infty, y]$ and $f((x - \delta, x + \delta)) \subseteq (-\infty, y + \epsilon)$, then $\max f(z) \in (y - \epsilon, y + \epsilon)$ for all $z \in (x - \delta, x + \delta)$.

Proof. (i) Let $z \in (x - \delta, x + \delta)$. Then by Lemma 2.2, $f(z) = [t, \infty)$ for some $t \in \mathbb{R}$. Thus $f(z) = [t, \infty) \subseteq (y - \epsilon, \infty)$. To show that $t \in (y - \epsilon, y + \epsilon)$, it remains to show that $t < y + \epsilon$. Since $z \in (x - \delta, x + \delta)$, we have $-\delta < x - z < \delta$, so $2x - z \in (x - \delta, x + \delta)$. Hence

$$f(2x - z) = 2f(x) + f(-z) \quad \text{from Lemma 2.4}$$

$$= 2[y, \infty) + [-t, \infty) \quad \text{from Lemma 2.3}$$

$$= [2y - t, \infty) \subseteq (y - \epsilon, \infty).$$

It follows that $2y - t > y - \epsilon$. Thus $y + \epsilon > t$. Therefore (i) is proved.

(ii) can be proved similarly to (i). \qed
Lemma 2.8. Let $f$ be an interval-valued multihomomorphism on $(\mathbb{R}, +)$. If $f$ is upper semi-continuous at 0, then $f$ is upper semi-continuous on $\mathbb{R}$.

Proof. If $f(0) = \{0\}$, then by Lemma 2.2, $|f(x)| = 1$ for all $x \in \mathbb{R}$, so $f$ is a homomorphism on $(\mathbb{R}, +)$ which is continuous at 0. It follows obviously that $f$ is continuous on $\mathbb{R}$.

If $f(0) = \mathbb{R}$, then by Lemma 2.2, $f(x) = \mathbb{R}$ for all $x \in \mathbb{R}$, thus $f$ is upper semi-continuous on $\mathbb{R}$. If $f(0) = (0, \infty)$ or $(-\infty, 0)$, then by Lemma 2.5, $f(x) = f(0)$ for all $x \in \mathbb{R}$, and hence $f$ is upper semi-continuous on $\mathbb{R}$.

Next, assume that $f(0) = [0, \infty)$ and let $x \in \mathbb{R}$. Then by Lemma 2.2, $f(x) = [y, \infty)$ for some $y \in \mathbb{R}$. Let $V$ be an open set in $\mathbb{R}$ containing $[y, \infty)$. Then $(y - \epsilon, y + \epsilon) \subseteq V$ for some $\epsilon > 0$, and so $(y - \epsilon, \infty) \subseteq V$. Thus $V - y$ is an open set in $\mathbb{R}$ and $(-\epsilon, \infty) \subseteq V - y$. Since $f$ is upper semi-continuous at 0, there is an open set $U$ in $\mathbb{R}$ such that $0 \in U$ and $f(U) \subseteq (-\epsilon, \infty)$. Hence $U + x$ is an open set in $\mathbb{R}$ containing $x$ and

$$f(U + x) = f(U) + f(x) \subseteq (-\epsilon, \infty) + [y, \infty) = (y - \epsilon, \infty) \subseteq V.$$ 

This proves that $f$ is upper semi-continuous at every point $x \in \mathbb{R}$. By a similar argument, the desired result follows for the case that $f(0) = (-\infty, 0]$.

Hence by Lemma 2.1, the lemma is proved. \qed

Lemma 2.9. Let $c \in \mathbb{R}$ and define the multifunctions $g$ and $h$ on $\mathbb{R}$ by

$$g(x) = [cx, \infty) \text{ and } h(x) = (-\infty, cx] \text{ for all } x \in \mathbb{R}.$$ 

Then $g$ and $h$ are upper semi-continuous multihomomorphisms on $(\mathbb{R}, +)$.

Proof. It is evident that $g$ and $h$ are multihomomorphisms on $(\mathbb{R}, +)$. We have that $g(0) = [0, \infty)$ and $h(0) = (-\infty, 0]$. If $c = 0$, then $g(x) = [0, \infty)$ and $h(x) = (-\infty, 0]$ for all $x \in \mathbb{R}$, so $g$ and $h$ are upper semi-continuous on $\mathbb{R}$. Next, assume that $c \neq 0$. To show that $g$ is upper semi-continuous at 0, let $V$ be an open set containing $[0, \infty)$. Then there is an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq V$. Thus $(-\epsilon, \infty) \subseteq V$. Let $x \in \left(\frac{-\epsilon}{|c|}, \frac{\epsilon}{|c|}\right)$. Then $|c|x \in (-\epsilon, \epsilon)$, so $cx \in (-\epsilon, \epsilon)$. Hence $g(x) = [cx, \infty) \subseteq (-\epsilon, \infty) \subseteq V$. This shows that $g\left(\left(\frac{-\epsilon}{|c|}, \frac{\epsilon}{|c|}\right)\right) \subseteq V$. Then $g$ is upper semi-continuous at 0. It can be shown similarly that $h$ is upper semi-continuous at 0. Hence by Lemma 2.8, $g$ and $h$ are upper semi-continuous on $\mathbb{R}$. \qed
Theorem 2.10. Let $f$ be an interval-valued multifunction on $\mathbb{R}$. Then $f$ is an upper semi-continuous multihomomorphism on $(\mathbb{R}, +)$ if and only if $f$ is one of the followings:

(i) There is a constant $c \in \mathbb{R}$ such that $f(x) = \{cx\}$ for all $x \in \mathbb{R}$.
(ii) $f(x) = \mathbb{R}$ for all $x \in \mathbb{R}$.
(iii) $f(x) = (0, \infty)$ for all $x \in \mathbb{R}$.
(iv) $f(x) = (-\infty, 0)$ for all $x \in \mathbb{R}$.
(v) There is a constant $c \in \mathbb{R}$ such that $f(x) = [cx, \infty)$ for all $x \in \mathbb{R}$.
(vi) There is a constant $c \in \mathbb{R}$ such that $f(x) = (-\infty, cx]$ for all $x \in \mathbb{R}$.

Proof. Assume that $f$ is an upper semi-continuous multihomomorphism on $(\mathbb{R}, +)$. By Lemma 2.1, $f(0)$ is one of $\{0\}, \mathbb{R}, (0, \infty), (-\infty, 0), [0, \infty)$ and $(-\infty, 0]$.

Case 1: $f(0) = \{0\}$. By Lemma 2.2, $f$ is a function on $\mathbb{R}$. Then $f$ is a continuous homomorphism on $(\mathbb{R}, +)$. By the known result mentioned previously, $f$ satisfies (i).

Case 2: $f(0) = \mathbb{R}$. By Lemma 2.2, $f(x) = \mathbb{R}$ for all $x \in \mathbb{R}$.

Case 3: $f(0) = (0, \infty)$. Then by Lemma 2.5(i), $f$ satisfies (iii).

Case 4: $f(0) = (-\infty, 0)$. By Lemma 2.5(ii), $f$ satisfies (iv).

Case 5: $f(0) = [0, \infty)$. It follows from Lemma 2.2 that $f(1) = [c, \infty)$ for some $c \in \mathbb{R}$. By Lemma 2.6(i),

$$f(q) = [cq, \infty) \text{ for all } q \in \mathbb{Q}.$$ (3)

Let $x \in \mathbb{R}$. Then from Lemma 2.2, $f(x) = [y, \infty)$ for some $y \in \mathbb{R}$. Note that for every $n \in \mathbb{N}$, $(y - \frac{1}{n}, \infty)$ is an open set containing $f(x)$. Since $f$ is upper semi-continuous at $x$, we deduce that

for every $n \in \mathbb{N}$, there is a $\delta_n > 0$ such that

$$\delta_n < \frac{1}{n} \text{ and } f(x - \delta_n, x + \delta_n) \subseteq (y - \frac{1}{n}, \infty).$$ (4)

For each $n \in \mathbb{N}$, let $q_n \in \mathbb{Q}$ be such that $q_n \in (x - \delta_n, x + \delta_n)$. From (1), $f(q_n) = [cq_n, \infty)$ for all $n \in \mathbb{N}$. From (2) and Lemma 2.7(i), we have that for
every $n \in \mathbb{N}, cq_n \in (y - \frac{1}{n}, y + \frac{1}{n})$. This implies that $\lim_{n \to \infty} cq_n = y$. But since $q_n \in (x - \delta_n, x + \delta_n)$ and $\delta_n < \frac{1}{n}$, it is immediate that $\lim_{n \to \infty} q_n = x$. Then
\[
cx = c \lim_{n \to \infty} q_n = \lim_{n \to \infty} cq_n = y,
\]
and hence $f(x) = [cx, \infty)$. This shows that $f$ satisfies (v).

**Case 6:** $f(0) = (-\infty, 0]$. An analogous proof using Lemma 2.2, Lemma 2.6(ii) and Lemma 2.7(ii) yields the result that $f$ satisfies (vi).

For the converse, assume that one of (i)-(vi) holds. If $f$ satisfies (i), then $f$ is a continuous homomorphism on $(\mathbb{R}, +)$. If $f$ satisfies (ii), (iii) or (iv), then $f$ is obviously an upper semi-continuous multihomomorphism on $(\mathbb{R}, +)$. If $f$ satisfies (v) or (vi), then by Lemma 2.9, $f$ is an upper semi-continuous multihomomorphism on $(\mathbb{R}, +)$.

The proof is thereby completed. \(\square\)

**References**


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