Homotopy Perturbation Method for Solving
the Second Kind of Non-Linear Integral Equations

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Abstract

In this paper, a numerical solution for solving the second kind of non-linear integral equation is presented. An application of homotopy perturbation method is applied to solve the second kind of non-linear integral equation such that Voltra and Fredholm integral equation. the results reveal that the homotopy perturbation method is very effective and simple and gives the exact solution.

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1 Introduction and Preliminary Notes

Non-linear phenomena, that appear in many application in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modeled by PDEs and by integral equations as well. In recent years, a large amount of literature developed concerning the modified decomposition method introduced by Wazwaz [8, 9] by applying it to a large size of applications in applied sciences. A new perturbation method called homotopy perturbation method (HPM) was proposed by He in 1997 and systematical description in 2000 which is, in fact, coupling of the traditional perturbation method and homotopy in topology [4]. Until recently, the application of the HPM [8, 9] in non-linear problems has been developed by scientists and engineers, because this method is the most effective and convenient ones for both weakly and strongly non-linear equations. In this paper, this method is applied for the second kind of non-linear integral equations such that Voltra and Fredholm integral equation.
Definition 1.1 For ”\( \varepsilon \)" we have

\[ P_n(x) = \varepsilon P_m(x), \]

where \( x \) and \( \varepsilon \) are dimensionless, \( P_n \) and \( P_m \) are polynomials in \( x \) of order \( n \) and \( m \), respectively, and \( n \geq m \). When \( \varepsilon = 0 \), the problem reduces to

\[ P_n(x) = 0, \]

which is called the reduced or unperturbed equation.

Consider a non-linear equation in the form

\[ Lu + Nu = 0, \]

where \( L \) and \( N \) are linear operator and non-linear operator, respectively. In order to use the homotopy perturbation, a suitable construction of a homotopy equation is of vital importance. Generally, a homotopy can be constructed in the form

\[ Lu + p(Nu + Nu - Lu) = 0, \]

where \( L \) can be a linear operator or a simple non-linear operator, and the solution of \( Lu = 0 \) with possible some unknown parameter can best describe the original non-linear system. For example, for a non-linear oscillator we can choose \( Lu = u + \omega^2 u \), where \( \omega \) is the frequency of the non-linear oscillator.

The non-linear Fredholm integral equations are given by

\[ u(x) = f(x) + \int_0^1 K(x, y)\{R(u(y)) + N(u(y))\} \, dy, \quad (1) \]

and the non-linear Voltra integral equations are given by

\[ u(x) = f(x) + \int_0^x K(x, y)\{R(u(y)) + N(u(y))\} \, dy, \quad (2) \]

\( u(x) \) is a unknown function that will be determined, \( K(x, y) \) is the kernel of the integral equation, \( f(x) \) is an analytic function, \( R(u) \) and \( N(u) \) are linear and non-linear functions of \( u \), respectively [5, 6].

2 Main Results

To illustrate the HPM, we consider (1) as

\[ L(u) = u(x) - f(x) - \int_0^1 K(x, y)\{R(u(y)) + N(u(y))\} \, dy = 0. \quad (3) \]
As a possible remedy, we can define $H(u, p)$ by

$$H(u, 0) = F(u), \quad H(u, 1) = L(u),$$

where $F(u)$ is an integral operator with known solution $u_0$, which can be obtained easily. Typically, we may choose a convex homotopy by

$$H(u, p) = (1 - p)F(u) + pL(u), \quad (4)$$

and continuously trace an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(U, 1)$. The embedding parameter $p$ monotonically increase from zero to unit as the trivial problem $L(u) = 0$. The embedding parameter $p \in [0, 1]$ can be considered as an expanding parameter.

The HPM uses the homotopy parameter $p$ as expanding parameter to obtain

$$u = u_0 + pu_1 + p^2u_2 + \cdots. \quad (5)$$

When $p \to 1,(5)$ corresponds to (4) becomes the approximate solution of (3), i.e.,

$$U = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \cdots. \quad (6)$$

The series (6) is convergent for most cases, and also the rate of convergent depends on $L(u)$.

### 3 Numerical Example

This section contained four example of non-linear Fredholm and Voltra integral equation of the second kind.

**Example 3.1** Consider the non-linear Fredholm integral equation whit exact solution $u(x) = \sinh(x)$,

$$u(x) = \sinh(x) - 1 + \int_0^1 \left(\cosh(y)^2 - u(y)^2\right) dy. \quad (7)$$

We define

$$F(u) = u(x) - \sinh(x),$$

$$L(u) = u(x) - \sinh(x) + 1 - \int_0^1 \left(\cosh(y)^2 - u(y)^2\right) dy = 0,$$

and substituting $F(u)$ and $L(u)$ in (4) and equating the terms whit identical power of $p$, we obtain
\[ p^0 : u_0(x) = \sinh(x), \]
\[ p^1 : u_1(x) = -1 + \int_0^1 \left( \cosh(y)^2 - u_0(y)^2 \right) dy = 0, \]
\[ p^{k+2} : u_{k+2}(x) = -1 + \int_0^1 \left( \cosh(y)^2 - u_{k+1}(y)^2 \right) dy = 0, \]

such that \( k \geq 0. \) With using (6) we have

\[ U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + \cdots = \sinh(x). \]

**Example 3.2** Consider the non-linear Fredholm integral equation with exact solution \( u(x) = \cos x, \)

\[ u(x) = \cos(x) - x + \int_0^1 x \left( u(y)^2 - \sin(y)^2 \right) dy. \]  \hspace{1cm} (8)

We define

\[ F(u) = u(x) - \cos(x), \]
\[ L(u) = u(x) - \cos(x) + x - \int_0^1 x \left( u(y)^2 - \sin(y)^2 \right) dy = 0, \]

and substituting \( F(u) \) and \( L(u) \) in (4) and equating the terms with identical power of \( p, \) we obtain

\[ p^0 : u_0(x) = \cos(x), \]
\[ p^1 : u_1(x) = -x + \int_0^1 x \left( u_0(y)^2 - \sin(y)^2 \right) dy = 0, \]
\[ p^{k+2} : u_{k+2}(x) = \int_0^1 x \left( u_{k+1}(y)^2 - \sin(y)^2 \right) dy = 0, \]

such that \( k \geq 0. \) With using (6) we have

\[ U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + \cdots = \cos(x). \]

**Example 3.3** Consider the non-linear Voltea integral equation with exact solution \( u(x) = \sec(x), \)

\[ u(x) = \sec(x) + \tan(x) + x - \int_0^x \left( 1 + u(y)^2 \right) dy. \]  \hspace{1cm} (9)
We define

\[ F(u) = u(x) - \sec(x), \]

\[ L(u) = u(x) - \sec(x) - \tan(x) - x + \int_0^x \left( 1 + u(y)^2 \right) dy = 0, \]

and substituting \( F(u) \) and \( L(u) \) in (4) and equating the terms with identical power of \( p \), we obtain

\[ p^0 : \quad u_0(x) = \sec(x), \]

\[ p^1 : \quad u_1(x) = \tan(x) + x - \int_0^x \left( 1 + u_0(y)^2 \right) dy = 0, \]

\[ p^{k+2} : \quad u_{k+2}(x) = - \int_0^x \left( 1 + u_{k+1}(y)^2 \right) dy = 0, \]

such that \( k \geq 0 \). With using (6) we have

\[ U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + \cdots = \sec(x). \]

**Example 3.4** Consider the non-linear Volta integral equation with exact solution \( u(x) = e^x \),

\[ u(x) = e^x + \frac{1}{2} x \left( e^{2x} - 1 \right) - \int_0^x \left( xu(y)^2 \right) dy. \]

We define

\[ F(u) = u(x) - e^x, \]

\[ L(u) = u(x) - e^x - \frac{1}{2} x \left( e^{2x} - 1 \right) + \int_0^x \left( xu(y)^2 \right) dy, \]

and substituting \( F(u) \) and \( L(u) \) in (4) and equating the terms with identical power of \( p \), we obtain

\[ p^0 : \quad u_0(x) = e^x, \]

\[ p^1 : \quad u_1(x) = \frac{1}{2} x \left( e^{2x} - 1 \right) - \int_0^x \left( xu_0(y)^2 \right) dy = 0, \]

\[ p^{k+2} : \quad u_{k+2}(x) = - \int_0^x \left( xu_{k+1}(y)^2 \right) dy = 0, \]

such that \( k \geq 0 \). With using (6) we have

\[ U(x) = u(x) = \lim_{p \to 1} u = u_0(x) + u_1(x) + u_2(x) + \cdots = e^x. \]
4 Conclusion

In this paper, we use an application of homotopy perturbation method for solving the second kind of non-linear integral equation such as Fredholm and Voltra integral equations.

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References


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