Dimension and Continuity on $T_0$-Alexandroff Spaces

Hisham Mahdi

Islamic University of Gaza
Department of Mathematics
P.O. Box 108, Gaza, Palestine
hmahdi@iugaza.edu.ps

Ahmed EL-Mabhouh

Islamic University of Gaza
Department of Mathematics
P.O. Box 108, Gaza, Palestine
mabhouh@iugaza.edu.ps

Nader Said

Islamic University of Gaza
Department of Mathematics
P.O. Box 108, Gaza, Palestine
a.n.89@hotmail.com

Abstract

In this paper, dimension, continuity and multifunctions are studied on $T_0$-Alexandroff spaces. The concept of posets is used to characterize continuity, pre-continuity and semi-continuity. The main result of this paper is to show that

$$\dim(X) = \ell(X)$$

where $\ell(X)$ is the length of the poset $(X, \leq)$.

Mathematics Subject Classification: Primary 54F05, 54C08, 54F45; Secondary 54F65

Keywords: Alexandroff spaces, generalized continuity, dimension theory, multi-functions, upper and lower semicontinuous
1 Introduction

Let \((P, \leq)\) be a poset. The set of all maximal (resp. minimal) elements is denoted by \(M\) (resp \(m\)). If \(A\) is a subset of \(P\), Then the order of \(P\) induces an order in \(A\). In this case, we define \(M(A)\) (resp. \(m(A)\)) to be the set of all maximal (resp. minimal) elements of \(A\) under the induced order.

The length of the poset \(P\) (denoted by \(\ell(P)\)) is the length of the longest chain in \(P\). We say that \(P\) satisfies the ascending chain condition (briefly, \(ACC\)), if for each increasing chain \(x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots\) in \(P\), there exists \(k \in \mathbb{N}\) such that \(x_k = x_{k+1} = \cdots\). We say that \(P\) satisfies the descending chain condition (briefly, \(DCC\)), if for each decreasing chain \(x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots\) in \(P\), there exists \(k \in \mathbb{N}\) such that \(x_k = x_{k+1} = \cdots\). And we say that \(P\) is of finite chain condition (briefly, \(FCC\)), if it satisfies both \(ACC\) and \(DCC\).

It should be noted that, in the case when \(P\) satisfies \(ACC\) (resp. \(DCC\)), the set \(M\) (resp. \(m\)) is nonempty set. Moreover, if \(\ell(P) < \infty\) then \(P\) satisfies both \(ACC\) and \(DCC\), but the converse is not true.

Alexandroff spaces were first studied in 1937 by P. Alexandroff \[3\]. It is a topological space in which arbitrary intersection of open sets is open. Equivalently, each singleton has a minimal neighborhood base. So, any discrete space is Alexandroff, and any finite space is also Alexandroff.

For each \(T_0\)-Alexandroff space \((X, \tau)\), there is a corresponding poset \((X, \leq_\tau)\), in one to one and onto way, where each one of them is completely determined by the other. Given a poset \((P, \leq)\), then \(\mathbb{B} = \{\uparrow x : x \in X\}\) is a base for a topology on \(P\). This topology, denoted by \(\tau_\leq\), is \(T_0\)-Alexandroff topology. On the other hand, if \((X, \tau)\) is Alexandroff space, we define the a pre-order \(\leq_\tau\), called (Alexandroff) specialization pre-order, by \(a \leq_\tau b\) if and only if \(a \in \{b\}\) or equivalently \(b \in U\) whenever \(U\) is a neighborhood of \(a\). This pre-order is a partial order if and only if \(X\) is \(T_0\). Moreover, if \((X, \leq)\) is a poset and \(\tau(\leq)\) its induced \(T_0\)-Alexandroff topology, then the specialization order of \(\tau(\leq)\) is the order \(\leq\) itself; that is, \(\leq_{\tau(\leq)}=\leq\). If \((X, \tau)\) is a \(T_0\)-Alexandroff space with specialization order \(\leq_\tau\), then the induced topology by the specialization order is the topology \(\tau\) itself; that is, \(\tau(\leq_\tau) = \tau\). So, we will consider \((X, \tau(\leq))\) to be a \(T_0\)-Alexandroff space \((X, \tau)\) together with its specialization order \(\leq\), and where the corresponding poset is \((X, \leq)\)[7].

For a \(T_0\)-Alexandroff space \((X, \tau(\leq))\), and for \(x \in X\), the collection consisting of one set \(\{\uparrow x\}\) is the minimal neighborhood base for \(x\) and sometimes denoted by \(\{V(x)\}\).

Recall that a topological space \((X, \tau)\) is called locally finite if each element \(x\) of \(X\) is contained in a finite open set and a finite closed set. It is clearly that finite spaces are locally finite and locally finite spaces are Alexandroff spaces.
2 Preliminary Notes

Let \((X, \tau)\) be a topological space and \(A \subseteq X\). Then \(A^c, A^o, A', \bar{A}\) and \(bd(A)\) will denote the complement, the interior, the limit points, the closure and the boundary of \(A\) respectively.

**Definition 2.1.** [7] Let \((X, \tau(\leq))\) be a \(T_0-\)Alexandroff Space. If the corresponding poset \((X, \leq)\) satisfies

(i) ACC, then \(X\) is called Artinian space.

(ii) DCC, then \(X\) is called Noetherian space.

(iii) FCC, then \(X\) is called generalized locally finite space (g-locally finite space).

**Definition 2.2.** [7]

(i) Let \((X, \tau(\leq))\) be an Artinian space. If \(x \in X\), then we define \(\hat{x} = \uparrow x \cap M\); the set of maximal elements of \(X\) greater than or equal to \(x\).

(ii) Let \((X, \tau(\leq))\) be a Noetherian space. If \(x \in X\), we define \(\check{x} = \downarrow x \cap m\); the set of minimal elements of \(X\) less than or equal to \(x\).

**Definition 2.3.** A subset \(A\) of a space \((X, \tau)\) is called

(i) semi-open [12] if \(A \subseteq A^c\), and semi-closed set [14] if \(A^c\) is semi-open. Thus \(A\) is semi-closed if and only if \(\bar{A} \subseteq A\). If \(A\) is both semi-open and semi-closed then \(A\) is called semi-regular [5].

(ii) preopen [4] if \(A \subseteq \overline{A^c}\), and preclosed set [11] if \(A^c\) is preopen. Thus \(A\) is preclosed if and only if \(\overline{A^c} \subseteq A\).

(iii) \(\alpha\)-open [13] if \(A \subseteq \overline{A^c}\), and \(\alpha\)-closed set [9] if \(A^c\) is \(\alpha\)-open. Thus \(A\) is \(\alpha\)-closed if and only if \(\overline{A^c} \subseteq A\).

(iv) \(\gamma\)-open [2] if \(A\) is a union of semi-open and preopen sets.

The class of all semi-open (resp. preopen, \(\alpha\)-open, \(\gamma\)-open) sets is denoted by \(SO(X)\) (resp. \(PO(X)\), \(\tau_\alpha\), \(\gamma(X)\)).

The basic results in the following proposition proved in [7]:

**Proposition 2.4.** Let \((X, \tau(\leq))\) be a \(T_0-\)Alexandroff space, \(x \in X\), and \(A \subseteq X\). Then

1. \(bd(\uparrow x) = \downarrow x - \{x\}\).
2. $\{x\} = \downarrow x$.

3. $A$ is open (closed) set iff $A$ is up (down) set in the corresponding poset. Moreover, if $X$ is an Artinian space, then

4. $A^\circ = \emptyset$ if and only if $A \cap M = \emptyset$.

5. $A = \cup \{\downarrow x : x \in M(A)\} = \downarrow M(A)$.

6. $A' = \cup \{\downarrow x \setminus \{x\} : x \in M(A)\}$.

7. $A$ is preopen if and only if $\downarrow x \cap A = \emptyset$ for all $x \in A^c \cap M$. Equivalently, $A$ is preopen if and only if $\hat{x} \subseteq A$ for all $x \in A$.

8. $A$ is semi-open if and only if $M(A) \subseteq M$.

**Corollary 2.5.** [7]

Let $(X, \tau(\leq))$ be an Artinian $T_0$–Alexandroff space. Then

1. $PO(X) \subseteq SO(X)$.

2. $PO(X) = \tau_\alpha$.

**Definition 2.6.** [8] We say that a topological space $X$ is submaximal if each dense subset is open.

**Theorem 2.7.** [7] Let $(X, \tau(\leq))$ be a $T_0$–Alexandroff space. Then the following are equivalent:

(1) $X$ is submaximal.

(2) Each element of $X$ is either maximal or minimal.

3 The Dimension In Artinian $T_0$–Alexandroff Spaces

**Definition 3.1.** [6] Let $X$ be a topological space. We say $\dim X = -1$ if and only if $X = \emptyset$. Let $n$ be a non-negative integer such that the dimension is defined for each $k \leq n - 1$. Then $\dim X \leq n$ if $X$ has a base $\beta$ such that $\dim(\text{bd}(B)) \leq n - 1$ for all $B \in \beta$.

We can say that $\dim X = n$ if $\dim X \leq n$ and $\dim X \nleq n - 1$, in this case, there is no base $\beta$ such that $\dim(\text{bd}(B)) \leq n - 2$ for all $B \in \beta$.

Recall that in any topological space, $\text{bd}(B) = \emptyset$ if and only if $B$ is clopen set. So we have the following proposition:
**Proposition 3.2.** A topological space \( X \) has dimension zero if and only if \( X \) has a base of clopen sets.

Note that \( \dim(X) = 0 \) for any discrete space \( X \) but the converse is not true. For example, \( X = \{1,2,3\} \), with topology \( \tau = \{\emptyset, X, \{1\}, \{2,3\}\} \) has \( \dim X = 0 \) but not discrete.

**Theorem 3.3.** Let \( X \) be a \( T_0 - \)Alexandroff Space. Then \( X \) has zero-dimension if and only if \( X \) is discrete.

**Proof.** \( \dim X = 0 \) if and only if \( X \) has a base of clopen sets. Now, if \( X \) is a \( T_0 - \)Alexandroff, then \( X \) has a base of clopen sets if and only if \( X \) is discrete.

**Remark 3.4.** If \( X \) is an Alexandroff space, and if \( B \) is the minimal base, then for any base \( K \) of \( X \) we have \( B \subseteq K \). So, we can use the minimal base \( B \) to find the dimension of \( X \).

**Proposition 3.5.** Let \( (X, \tau(\leq)) \) be a \( T_0 - \)Alexandroff Space. Then \( \dim X = n \) iff the minimal base \( B \) has the property that \( \dim(\text{bd}(\uparrow x)) \leq n - 1 \) for each \( x \in X \) and if there exists \( x_0 \in X \) with \( \dim(\text{bd}(\uparrow x_0)) = n - 1 \).

**Proof.** \((\Rightarrow)\) Straightforward from Remark 3.4.

\((\Leftarrow)\) If there exists \( x_0 \in X \) with \( \dim(\text{bd}(\uparrow x_0)) = n - 1 \), then there is no base \( B \) of \( X \) with \( \dim(\text{bd}(B)) \leq n - 2 \) for \( B \in \beta \).

**Theorem 3.6.** Let \( (X, \tau(\leq)) \) be a \( T_0 - \)Alexandroff space. Then \( \dim X \leq 1 \) if and only if \( X \) is submaximal. Moreover, if there exist two distinct points \( x, y \in X \) such that \( x \leq y \), then \( \dim X = 1 \).

**Proof.** \((\Rightarrow)\) Suppose to the contrary that \( X \) is not submaximal, so \( X \) has three distinct points \( x < y < z \) (\( x < y \) means \( x \leq y \) and \( x \neq y \)). By proposition 2.4 part (1), \( \text{bd}(\uparrow z) = \downarrow z - \{z\} \). So \( x, y \in \text{bd}(\uparrow z) \). Since \( \dim X \leq 1 \), we get either \( \dim(\text{bd}(\uparrow z)) = 0 \) or \( \text{bd}(\uparrow z) = \phi \). But \( x, y \in \text{bd}(\uparrow z) \), so \( \text{bd}(\uparrow z) = \phi \) and hence \( \dim(\text{bd}(\uparrow z)) = 0 \). Then \( \text{bd}(\uparrow z) \) is a discrete subspace. Therefore, the induced order on \( \text{bd}(\uparrow z) \) is the anti chain. But \( x, y \in \text{bd}(\uparrow z) \) and \( x < y \) which is a contradiction.

\((\Leftarrow)\) If \( X \) is submaximal, then by Proposition 2.7 each singleton is either maximal or minimal. Hence an elements \( x \) is either maximal in \( X \) (and hence \( \text{bd}(\uparrow x) = \downarrow x - \{x\} \)) or \( \text{bd}(\uparrow x) \) is empty set. So either \( \dim(\text{bd}(\uparrow x)) = 0 \) or \( \dim(\text{bd}(\uparrow x)) = -1 \). Thus, \( \dim X \leq 1 \).

Now, if there exist two distinct points \( x \leq y \) in \( X \), then \( \text{bd}(\uparrow y) = \downarrow y - \{y\} \) is a minimal point with discrete topology. So, \( \dim(\text{bd}(\uparrow y)) = 0 \). Thus by Proposition 3.5, \( \dim X = 1 \).
Theorem 3.7. Let $((X, \tau(\leq)))$ be a $T_0$-Alexandroff space such that $\ell(X) < \infty$ with respect to the corresponding poset $(X, \leq)$. Then $\dim X$ in the topological sense equals $\ell(X)$ in the sense of posets.

Proof. If $\ell(X) = 0$, then the order on $X$ is anti-chain and the Alexandroff topology is discrete, so by Theorem 3.3 $\dim X = 0$. If $\ell(X) = 1$, then $X$ is submaximal, so by Theorem 3.6 $\dim X = 1$.

Now using strong form of induction, suppose that for any $T_0$-Alexandroff space $Z$ with $\ell(Z) < k$, we have that $\dim Z = \ell(Z)$ for all $k \geq 2$. Let $X$ be a $T_0$-Alexandroff space with $\ell(X) = k$. So $X$ is both Artinian and Noetherian space. So for any $x \in X$, $\ell(\downarrow x) \leq \ell(X) = k$. By Proposition 2.4 part (1), $bd(\uparrow x) = \downarrow x - \{x\}$, and so $\ell(bd(\uparrow x)) = \ell(\downarrow x) - 1 \leq k - 1 < k$. Thus, by assumption, $\dim bd(\uparrow x) \leq k - 1$ and hence, $\dim X \leq k$. Now since $\ell(X) = k$, the longest chain in $X$ has length $k$. Let $C = \{c_0, c_1, \cdots, c_k\}$ be the longest chain. We may assume $c_0 < c_1 < \cdots < c_k$. This implies that $\ell(bd(\uparrow c_k)) = k - 1$ and then $\dim bd(\uparrow c_k) = k - 1$. Therefore, $\dim X = k$. \hfill \Box

4 Continuity in Artinian $T_0$-Alexandroff Spaces

Definition 4.1. If $f : (X, \leq_X) \longrightarrow (Y, \leq_Y)$ be a function from a poset $X$ into a poset $Y$, then $f$ is called order-preserving if $x_1 \leq x_2$ in $X$ then $f(x_1) \leq f(x_2)$ in $Y$.

Definition 4.2. Let $f : X \rightarrow Y$ be a function from a topological space $X$ to a topological space $Y$, and let $\mathcal{A}$ be a collection of subsets of $X$. We say that $f$ is $\mathcal{A}$-continuous if $f^{-1}(V) \in \mathcal{A}$, for every open set $V$ in $Y$.

Remark 4.3. As special cases of the above definition, we have that

(i) $f$ is continuous if $\mathcal{A} = \tau$.

(ii) $f$ is precontinuous [10] if $\mathcal{A} = PO(X)$.

(iii) $f$ is semi-continuous [12] if $\mathcal{A} = SO(X)$.

(iv) $f$ is $\gamma$-continuous [1] if $\mathcal{A} = \gamma(X)$.

(v) $f$ is $\alpha$-continuous [1] if $\mathcal{A} = \tau_\alpha$.

In general, there is no relation between different types of continuity. In the case, when $X$ is Artinian space, we proved that $PO(X) = \tau_\alpha \subseteq SO(X) = \gamma(X)$. Hence the following implications hold on Artinian spaces.

continuous $\implies$ precontinuous
\[
\text{semi-continuous} \iff \alpha\text{-continuous} \\
\iff \gamma\text{-continuous.}
\]

**Corollary 4.4.** Let \((X, \tau(\leq))\) be an Artinian \(T_0\)-Alexandroff space. Then \(\gamma(X) = SO(X)\); that is, a set \(A\) is \(\gamma\)-open if and only if \(A\) is semi-open.

*Proof.* Firstly, recall that a union of semi-open sets is semi-open. Now if \(\mathcal{A}\) is open neighborhood of \(x\), we have \(f(\uparrow x) \subseteq \uparrow f(a)\); that is, for all \(x \in X\), if \(x \geq a\) then \(f(x) \geq f(a)\).

\(-\) Let \(f: X \to Y\) be a continuous function, and let \(x \in X\). Since \(\uparrow f(x)\) is a neighborhood of \(f(x)\), then by continuity of \(f\), we have \(f^{-1}(\uparrow f(x))\) is open neighborhood of \(x\). Since \(\uparrow x\) is the smallest neighborhood of \(x\), so we have \(\uparrow x \subseteq f^{-1}(\uparrow f(x))\) and hence \(f(\uparrow x) \subseteq \uparrow f(x)\).

\(\Rightarrow\) Let \(x \in X\) be arbitrary, and let \(W\) be an open neighborhood of \(f(x)\). Then \(\uparrow f(x) \subseteq W\), so by assumption \(f(\uparrow x) \subseteq \uparrow f(x) \subseteq W\). Hence \(\uparrow x \subseteq f^{-1}(W)\), which implies that \(f\) is continuous.

**Corollary 4.6.** Let \(f: (X, \leq_X) \to (Y, \leq_Y)\) be a function from a poset \(X\) into a poset \(Y\), and let \(\tau(\leq_X)\) and \(\tau(\leq_Y)\) be the corresponding \(T_0\)-Alexandroff spaces. Then \(f\) is continuous in the sense of topology if and only if \(f\) is order-preserving in the sense of poset.

**Theorem 4.7.** Let \(X\) be an Artinian space and let \(Y\) be a \(T_0\)-Alexandroff space. The function \(f: X \to Y\) is precontinuous if and only if for all \(x \in X\), \(f(\hat{x}) \subseteq \uparrow f(x)\).

*Proof.* \(\Rightarrow\) Suppose \(f\) is precontinuous. Since for all \(x \in X\), \(\uparrow f(x)\) is open in \(Y\), then by precontinuity of \(f\) we have that \(f^{-1}(\uparrow f(x))\) is preopen set in \(X\) for all \(x \in X\), and by proposition 2.4 part (7) we have \(\hat{x} \subseteq f^{-1}(\uparrow f(x))\) for all \(x \in X\). Therefore \(f(\hat{x}) \subseteq \uparrow f(x)\) for all \(x \in X\).

\(\Leftarrow\) Suppose that \(f(\hat{x}) \subseteq \uparrow f(x)\) for all \(x \in X\) and suppose that \(W\) is an open set in \(Y\). Let \(x \in f^{-1}(W)\), so \(f(x) \in W\) and hence \(\uparrow f(x) \subseteq W\). Therefore \(f(\hat{x}) \subseteq \uparrow f(x) \subseteq W\) and hence \(\hat{x} \subseteq f^{-1}(W)\). Since \(x \in f^{-1}(W)\) is arbitrary, by Proposition 2.4 part (7), \(f^{-1}(W)\) is preopen set in \(X\).
Theorem 4.8. Let $X$ be an Artinian space and let $Y$ be a $T_0$–Alexandroff space. The function $f : X \to Y$ is semi-continuous if and only if for all $x \in X$, there is $y \in \hat{x}$ such that $f(y) \in \uparrow f(x)$. In equivalent form, for all $x \in X$, $f(\hat{x}) \cap \uparrow f(x) \neq \phi$.

Proof. ($\Rightarrow$) Let $x \in X$ be arbitrary. Since $\uparrow f(x)$ is open in $Y$, $f^{-1}(\uparrow f(x))$ is semi-open set in $X$ containing $x$. Since $X$ is Artinian, there exists $y \in M(f^{-1}(\uparrow f(x)))$ such that $x \leq y$. By Proposition 2.4 part (8), $M(f^{-1}(\uparrow f(x))) \subseteq M$. Thus $y \in M$ and $x \leq y$. Therefore $y \in \hat{x}$, and $f(y) \in \uparrow f(x)$.

($\Leftarrow$) Let $W$ be an open set in $Y$, and let $x \in M(f^{-1}(W))$. By assumption, there exists $y \in \hat{x}$ such that $f(y) \in \uparrow f(x)$. Since $x \in f^{-1}(W)$, we get $f(x) \in W$, and hence $\uparrow f(x) \subseteq W$. So, we get that $f(y) \in W$, and so $y \in f^{-1}(W)$. Now, $x \leq y$ in $f^{-1}(W)$ and $x$ is maximal, so $x = y$. Moreover, $y \in \hat{x} \subseteq M$ yields to $x \in M$. Hence $M(f^{-1}(\uparrow f(x))) \subseteq M$. Thus $f^{-1}(W)$ is semi-open set in $X$. \qed

5 Continuity of Multifunction in Artinian $T_0$–Alexandroff Spaces

Definition 5.1. [6] Let $X$ and $Y$ be two nonempty sets and $\mathcal{P}(Y)$ be the power set of $Y$. A multifunction is a function $F : X \to \mathcal{P}(Y)$.

Definition 5.2. [2] A multifunction $F : X \to \mathcal{P}(Y)$ from a topological space $X$ into a topological space $Y$ is called:

1. upper semicontinuous multifunction at a point $a \in X$ if for every open set $V$ in $Y$ such that $F(a) \subseteq V$, there exists an open set $U$ containing $a$ such that $F(U) = \bigcup_{x_i \in U} F(x_i) \subseteq V$.

2. lower semicontinuous multifunction at a point $a \in X$ if for every open set $V$ in $Y$ such that $F(a) \cap V \neq \phi$, there exists an open set $U$ containing $a$ such that $F(x) \cap V \neq \phi$ for each $x \in U$.

3. continuous multifunction at a point $a \in X$ if it is both upper semicontinuous and lower semicontinuous multifunction at $a$.

Theorem 5.3. Let $X$ and $Y$ be $T_0$–Alexandroff spaces, and let $F : X \to \mathcal{P}(Y)$ be a multifunction. Then

(i) $F$ is upper semicontinuous multifunction at $x$ if and only if for all $y \geq x$, $F(y) \subseteq \uparrow F(x)$.

(ii) $F$ is lower semicontinuous multifunction at $x$ if and only if for all $y \in \uparrow x$ and all $z \in F(x)$, $F(y) \cap \uparrow z \neq \phi$.
Proof. (i) (⇒) Let $F$ be upper semicontinuous multifunction at $x \in X$. Let $V = \bigcup_{y \in F(x)} \uparrow y = \uparrow F(x)$, which is open set in $Y$. Since $F(x) \subseteq V$, by definition of upper semicontinuous multifunction, there exists an open set $U$ containing $x$ such that $\bigcup_{x, i \in U} F(x_i) \subseteq V$. Now $x \in U$, $\uparrow x \subseteq U$ and hence if $y \geq x$ then $y \in U$. Therefore $F(y) \subseteq V = \uparrow F(x)$.

(⇒) Suppose that for all $y \geq x$, $F(y) \subseteq \uparrow F(x)$. Now, let $V$ be any open set in $Y$ such that $F(x) \subseteq V$, so $\uparrow F(x) \subseteq V$. Take $U = \uparrow x$, we get $x \in U$ and for all $y \in U$, $y \geq x$ and hence $F(y) \subseteq \uparrow F(x) \subseteq V$. Hence $F$ is upper semicontinuous multifunction.

(ii) (⇒) Let $F$ be lower semicontinuous multifunction at $x \in X$. For each $z \in F(x)$, take for a special case $V = \uparrow z$. Since $F(x) \cap V \neq \emptyset$, there is an open set $U$ in $X$ with $x \in U$ and $F(y) \cap \uparrow z \neq \emptyset$ for all $y \in U$. Since $\uparrow x \subseteq U$, and $y \in \uparrow x \Rightarrow y \in U$, the proof of this direction is complete.

(iii) (⇒) Let $V$ be an arbitrary open set in $Y$ with $F(x) \cap V \neq \emptyset$. Take $z \in F(x) \cap V$. Then $\uparrow z \subseteq V$. Choose $U = \uparrow x$. So from the given, for all $y \in \uparrow x$, $\emptyset \neq F(y) \cap \uparrow z \subseteq F(y) \cap V$. Then $F(y) \cap V \neq \emptyset$ for all $y \in U$. Hence $F$ is lower semicontinuous multifunction.

**Corollary 5.4.** Let $X$ and $Y$ be $T_0$-Alexandroff spaces, and let $F : X \to \mathcal{P}(Y)$ be a multifunction. Then $F$ is continuous multifunction at $x \in X$ if and only if for all $y \geq x$ we get that

(i) for all $a \in F(y)$, there exists $z \in F(x)$ such that $a \in \uparrow z$.

(ii) for all $z \in F(x)$, there exists $a \in F(y)$ such that $a \in \uparrow z$.

**Theorem 5.5.** Let $\pi : X \to Y$ be a continuous function from a topological space $X$ into a one-dimensional $T_0$-Alexandroff space $Y$. If for any minimal point $y$ in $Y$, $\pi^{-1}(y)$ is only one point and for any maximal point $z \in \hat{y}$, $\pi^{-1}(\pi^{-1}(z)) \subseteq cl(\pi^{-1}(z))$, then $\pi$ is an open function.

**Proof.** Let $W$ be an open set in $X$. We want to prove that $\pi(W) = \bigcup_{y \in \pi(W)} \uparrow y$.

It is clear that $\pi(W) \subseteq \bigcup_{y \in \pi(W)} \uparrow y$. We want to prove that $\uparrow y \subseteq \pi(W)$ for each $y \in \pi(W)$. By Theorem 3.6 we have two cases:

**case 1:** If $y$ is a maximal point, then $\uparrow y = \{y\} \subseteq \pi(W)$ and we are done.

**case 2:** If $y$ is a minimal point, then by the assumption, $\pi^{-1}(y) = x$ is one point. Let $z$ be a maximal point such that $z \in \hat{y}$. By hypothesis, $x \in cl(\pi^{-1}(z))$. Since $W$ is an open neighborhood of $x$, then $\pi^{-1}(z) \cap W \neq \emptyset$. For a point $v \in \pi^{-1}(z) \cap W$, we have $z = \pi(v) \in \pi(W)$. Since $\uparrow y = \{y\} \cup \hat{y}$, and $z \in \hat{y}$ is arbitrary, then $\uparrow y \subseteq \pi(W)$. This completes the proof. \(\square\)
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Received: December, 2009