On a Result of Datta and Jha

Sanjib Kumar Datta
Department of Mathematics
University of North Bengal
Darjeeling, Pin-734013, West Bengal, India
sanjib.kr.datta@yahoo.co.in

Eleja Jerin
Department of Mathematics
Kalitala Diar R.J.K. High School
Kalital Diar, P.O.-Berhampore
Dist.-Murshidabad, PIN-742101, West Bengal, India

Abstract
In this paper we generalise a result of Datta and Jha [1].

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1 Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$ with an infinite number of zeros $z_1, z_2, z_3, \ldots$ such that $0 < |z_1| \leq |z_2| \leq |z_3| \leq \cdots$ and $|z_n| = r_n \to \infty$ as $n \to \infty$. Then the exponent of convergence of the zeros of $f$ denoted by $\rho_1$ is defined as

$$\rho_1 = \limsup_{n \to \infty} \frac{\log n}{\log r_n}.$$ 

In 2007 Lahiri and Banerjee [2] proved the following result:

**Theorem A.**[2] Let $f$ be an entire function, with zeros $z_1, z_2, z_3, \ldots$ where $0 < |z_1| \leq |z_2| \leq |z_3| \cdots$ and $|z_n| = r_n \to \infty$ as $n \to \infty$. Let $\rho_1 > 0$ be the exponent of convergence of the zeros of $f$. If $0 \leq \alpha < \rho_1$, then there
exists a continuum number of entire functions each having \( \alpha \) as the exponent of convergence of its zeros.

Introducing the following definition Datta and Jha [1] extended Theorem A to \( L^*(p,q) \) th exponent of convergence of zeros of an entire function \( f \):

**Definition 1.** [1] The \( L^* - (p,q) \) th exponent of convergence of the zeros of an entire function \( f \) denoted by \( \rho_1^{L^*}(p,q) \) is defined as:

\[
\rho_1^{L^*}(p,q) = \limsup_{n \to \infty} \frac{\log [p] n}{\log [q] \left\{ r_n e^{L(r_n)} \right\}},
\]

where \( p, q \) are any two positive integers with \( p > q \).

In this paper we generalise the notion of Datta and Jha [1] and give the following definition:

**Definition 2.** The \( L^m - (p,q) \) th exponent of convergence of the zeros of an entire function \( f \) denoted by \( \rho_1^{L^m}(p,q) \) is defined as:

\[
\rho_1^{L^m}(p,q) = \limsup_{n \to \infty} \frac{\log [p] n}{\log [q] \left\{ r_n \exp^m[L(r_n)] \right\}},
\]

where \( p, q \) are any two positive integers with \( p > q \) and \( \exp^m x = \exp(\exp^{m-1} x) \) for \( m = 1, 2, 3, \ldots; \exp^0 x = x \).

We do not explain the standard notations and definitions of the value distribution theory of entire functions as those are available in [4].

## 2 Lemma.

In this section we present a lemma which will be needed in the sequel.

**Lemma 1.** [3] Let \( \{g_n\} \) be a non decreasing sequence of real numbers with \( g_n \to \infty \) as \( n \to \infty \). Let \( b = \limsup_{n \to \infty} \frac{\log n}{g_n} > 0 \) (\( b \) may be +\( \infty \)) and let \( 0 < t < b \). Then there exists a subsequence \( \{g_{j_n}\} \) of \( \{g_n\} \) such that

\[
\limsup_{n \to \infty} \frac{\log n}{g_{j_n}} = t.
\]
3 Theorem.

In this section we present the main result of the paper.

**Theorem 1.** Let $f$ be an entire function with zeros $z_1, z_2, z_3, \ldots$ such that $0 < |z_1| \leq |z_2| \leq |z_3| \leq \cdots$ and $|z_n| = r_n \exp^m |L(r_n)| \to \infty$ as $n \to \infty$. Let $\rho_1^{m*}(p, q) > 0$ be the $L^{m*}-(p, q)$ th exponent of convergence of the zeros of $f$ where $p$ and $q$ are positive integers with $p < q$. If $0 \leq \gamma_0 < \rho_1^{m*}(p, q)$, then there exists a continuum number of entire functions each having $\gamma_0$ as the $L^{m*}-(p, q)$ th exponent of convergence of its zeros.

**Proof.** We first show that there exists an entire function which has $\gamma_0$ as the $L^{m*}-(p, q)$ th exponent of convergence of its zeros.

**Case I.** $\gamma_0 = 0$.

Since $r_n \exp^m |L(r_n)| \to \infty$ as $n \to \infty$, there exists a sequence $\{k_n\}$ of positive integers with $k_1 < k_2 < k_3 < \cdots$, such that

$$r_{k_n} \exp^m |L(r_{k_n})| > \exp^q \left( n \log^p n \right)$$

for $n = 1, 2, 3, \cdots$

i.e.,

$$\log^q \{r_{k_n} \exp^m |L(r_{k_n})|\} > n \log^p n$$

for $n = 1, 2, 3, \cdots$

which gives that

$$\limsup_{n \to \infty} \frac{\log^p n}{\log^q \{r_{k_n} \exp^m |L(r_{k_n})|\}} = 0 .$$

Let $w_{k_n}$ be a point in the complex plane such that $|w_{k_n}| = r_{k_n} \exp^m |L(r_{k_n})|$ for $n = 1, 2, 3, \cdots$. $w_{k_n}$ may be $z_{k_n}$ also.

So in view of Weierstrass’s theorem it follows that there exists an entire function $\kappa(z)$ which has zeros only at the points $w_{k_n}$ for $n = 1, 2, 3, \cdots$ and therefore $\kappa(z)$ is the desired function.

**Case II.** $0 < \gamma_0 < \rho_1^{m*}(p, q)$.

Let us choose $\tau_n = \log^q \{r_{k_n} \exp^m |L(r_{k_n})|\}$ and $s_0 = \log^{p-1} n$. Then the sequence $\{\tau_n\}$ satisfies the conditions of Lemma 1.

Hence there exists a subsequence $\{\tau_{k_n}\}$ of $\{\tau_n\}$ such that

$$\beta_0 = \limsup_{n \to \infty} \frac{\log s_0}{\tau_{k_n}} .$$

Thus there exists an entire function $\kappa(z)$ which serves the purpose.

Now it is easy to verify that the set

$$\{ w_{k_n} : w_{k_n} \in \mathbb{C} \& |w_{k_n}| = r_{k_n} \exp^m |L(r_{k_n})| \to \infty \text{ as } n \to \infty \}$$
has the cardinality $c$. Also we know that if an entire function $f(z)$ has zeros at $z_n$ for $n = 1, 2, 3, \cdots$ then for any entire function $\kappa(z)$, the function of the form $e^{\kappa(z)}f(z)$ is an entire function. Thus we obtain a family of entire functions having the cardinality $c$, where each member of the family has $\gamma_0$ as the $L^{m*} - (p, q)\text{th}$ exponent of convergence of its zeros where $p, q$ are positive integers and $p > q$. This proves the theorem.

References


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