Precise Asymptotics for the First Moment of the Error Variance Estimator in Linear Models

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Abstract

In this paper we investigate the precise asymptotics in the law of the logarithm for the first moment of the error variance estimator of a linear model, where the estimation is based on residual sum of square.

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1 Introduction and main result

Let \( \{X, X_n, n \geq 1\} \) be i.i.d. random variables with \( P(X \neq 0) > 0 \) and partial sums \( \{S_n, n \geq 1\} \), and consider series of the type

\[
f(\epsilon) = \sum_n a_n P(|S_n| \geq \epsilon b_n), \quad \epsilon > 0,
\]

where \( a_n, b_n > 0 \) and \( \sum_n a_n = \infty \). Then there exist a threshold \( \alpha \), such that \( f(\epsilon) = \infty \) for \( \epsilon < \alpha \), while \( f(\epsilon) < \infty \) for \( \epsilon > \alpha \). The so-called precise asymptotics problem consists in finding, under appropriate moment conditions, an elementary function \( g(\epsilon) > 0, \epsilon > \alpha \), such that \( \lim_{\epsilon \downarrow \alpha} g(\epsilon) = 0 \) and \( \lim_{\epsilon \downarrow \alpha} g(\epsilon)f(\epsilon) = l \neq 0, \infty \), i.e., in establishing that \( f(\epsilon) \sim l/g(\epsilon) \) as \( \epsilon \downarrow \alpha \).

For almost exhaustive references on this area, see Gut and Špătaru(2000; a, b). Consider the linear model \( Y_i = X'_i \beta + e_i, i = 1, ..., n \), where \( \beta \) is a q-dimensional unknown parametric vector, and \( \{e_i\} \) is a sequence of i.i.d. trial errors with \( Ee_1 = 0 \) and \( 0 < \sigma^2 = Ee_i^2 < \infty \). By ordinary least squares and the characteristic of linear models, the estimator of \( \sigma^2 \) always takes the following form:

\[ \sigma^2_n = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 \]

where \( \hat{Y}_i = X'_i \hat{\beta} \).
\begin{equation}
\hat{\sigma}_n^2 = \frac{1}{n - \kappa} \left\{ \sum_{i=1}^{n} e_i^2 - \sum_{j=1}^{\kappa} \left( \sum_{i=1}^{n} a_{nji} e_i \right)^2 \right\},
\end{equation}

where \( \kappa = \kappa_n \) is the rank of the design matrix \( X_n = (X_1, \ldots, X_n) \) satisfying \( \kappa_n \leq q \) and \( \{a_{nli}\} \) is a sequence of real numbers satisfying

\begin{equation}
\sum_{i=1}^{n} a_{nli} a_{nmi} = \begin{cases} 1, & l = m, \\ 0, & l \neq m, \end{cases}
\end{equation}

and \( X_n' (X_n X_n')^{-1} X_n = (a_{nli})' \left( \begin{array}{cc} I_r \\ 0 \end{array} \right) (a_{nli}) \), where \( I_r \) is a \( \kappa \times \kappa \) identity matrix. The limit properties of the error variance estimator have been widely discussed, and we refer the reader to the [1,2,7,9] and references therein.

When \( \{X_n, n \geq 1\} \) is a sequence of i.i.d. random variables, Li et al. (1992b) derived convergence rates of moderate deviations and the precise asymptotics in the law of the iterated logarithm. Chen (1978) and Gut and Spataru (2000) studied the precise asymptotics in the Baum-Katz law of large numbers as \( \epsilon \downarrow 0 \). One of their results is as follow:

**Theorem 1 (A)** Suppose that \( \{X_n; n \geq 1\} \) is a sequence of i.i.d. random variables with \( EX_1 = 0 \) and \( 0 < EX_1^2 = \kappa^2 \). Then, for \( 1 \leq p < r \),

\begin{equation}
\lim_{\epsilon \downarrow 0} \epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} P\left\{ \sum_{k=1}^{n} X_k \geq \epsilon n^{1/p} \right\} = \frac{p}{r-p} E|Z|^{2(r-p)/2-p},
\end{equation}

where \( Z \) follows a normal distribution with 0 mean and variance \( \kappa^2 \).

On the other hand, Chow [1988] discussed complete moment convergence of i.i.d. random variables. He got

**Theorem 2 (B)** Suppose that \( \{X_n; n \geq 1\} \) is a sequence of i.i.d. random variables with \( EX_1 = 0 \). for \( 1 \leq p < 2 \) and \( r > p \), if \( E[|X_1|^r + |X_1| \log(1 + |X_1|)] < \infty \), then for any \( \epsilon > 0 \), we have

\begin{equation}
\sum_{n=1}^{\infty} n^{r/p-2-1/p} E\left\{ \left| \sum_{k=1}^{n} - \epsilon n^{1/p} \right|^r \right\} + < \infty.
\end{equation}

Chow [3] discussed the complete moment convergence, and got the following result.
**Theorem 3 (C)** Let \( \{X_n; n \geq 1\} \) be a sequence of i.i.d. random variables with \( EX_1 = 0 \), and set \( S_n = \sum_{i=1}^{n} X_i \), \( n \geq 1 \). Assume \( p \geq 1 \), \( \alpha > 1/2 \), \( p\alpha > 1 \) and \( E(|X_1|^p + |X_1|\log(1 + |X_1|)) < \infty \). then for any \( \epsilon > 0 \),

\[
\sum_{n=1}^{\infty} n^{\beta\alpha - 2 - \alpha} E\{ \max_{j \leq n} |S_j| - \epsilon n^\alpha \} < \infty.
\]

Recently, Jiang and Zhang (2007) establishing the following precise rates in the law of the logarithm for the moment convergence of i.i.d. random variables by using the strong approximation method.

**Theorem 4** let \( X_n; n \geq 1 \) be a sequence of i.i.d. random variables with \( EX_1 = 0 \), \( EX_1^2 = \sigma^2 < \infty \) and \( E(|X_1|^r/(\log |X_1|)^r) < \infty \). set \( S_n = \sum_{k=1}^{n} X_k \), \( n \geq 1 \). then for \( r < 1 \), we have

\[
\lim_{\epsilon \downarrow \sqrt{r-1} \log(\epsilon^2 - (r - 1))} \frac{1}{n} \sum_{n=1}^{\infty} n^{-2 \alpha - 1} E\{ |S_n| - \sigma \epsilon \sqrt{2n \log n} \}^+ = \frac{\sigma}{(r-1)\sqrt{2\pi}}.
\]

Inspired by Chow [1,88] and Jiang and Zhang (2007) and Chen (1978) and Gut and Spataru (2000), here we study the precise asymptotics in the law of the logarithm for the first moment of the error variance estimator.

Our results are stated as follow:

**Theorem 1.1** Suppose \( Ee_1 = 0 \), \( 0 < \sigma^2 = Ee_1^2 < \infty \) and \( Ee_1^4 < \infty \), and \( \nu = \text{var}(e_1^2) \).

i) If \( 1 < p < 2 \), \( r > 1 + p/2 \), then

\[
\lim_{\epsilon \downarrow 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n=1}^{\infty} n^{r/p-2-1/p} E\{ n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| - \epsilon \nu^{1/2} n^{1/p-1/2} \}^+ = \frac{p(2-p)}{(r-p)(2r - p - 2)} E|N|^{2(r-p)/(2-p)}.
\]

ii) If \( 1 \leq p < 2 \) and \( \delta > -1 \), then

\[
\lim_{\epsilon \downarrow 0} \epsilon^{2p(1+\delta)/(2-p)} \sum_{n \geq 3} \frac{(\log n)^\delta}{n} E\{ n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| - \epsilon \nu^{1/2} (\log n)^{1/p-1/2} \}^+ = \frac{1}{1 + \delta} E|N|^{2p(1+\delta)/(2-p)}.
\]
\[ \lim_{\epsilon \to 0} \epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} E \left\{ n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| - \epsilon n^{1/2} n^{1/p-1/2} \right\} + \]
\[ = \frac{p}{(r-p)} E|N|^{2(r-p)/(2-p)}, \]

where \( N \) is the standard normal random variable.

2 Proofs

First, we review three theorems which will be used in the following proofs.

**Theorem 1** (Lemma 2.4 of Huang and Zhang (2005)). For \( n \geq 1 \), let \( \alpha_n(\epsilon) > 0 \), \( \beta_n(\epsilon) > 0 \) and \( f(\epsilon) > 0 \) satisfy

\[ \alpha_n(\epsilon) \sim \beta_n(\epsilon), \quad \text{as} \quad n \to \infty \quad \text{and} \quad \epsilon \to \epsilon_0, \]

and

\[ f(\epsilon) \beta_n(\epsilon) \to 0, \quad \text{as} \quad \epsilon \to \epsilon_0, \quad \forall n \geq 1. \]

then

\[ \limsup_{\epsilon \to \epsilon_0} (\liminf_{\epsilon \to \epsilon_0}) f(\epsilon) \sum_{n=1}^{\infty} \alpha_n(\epsilon) = \limsup_{\epsilon \to \epsilon_0} (\liminf_{\epsilon \to \epsilon_0}) f(\epsilon) \sum_{n=1}^{\infty} \beta_n(\epsilon). \]

**Theorem 2** (Theorem 1 of Chen (1980)). Suppose that \( Ee_1 = 0 \), \( 0 < \sigma^2 = Ee_1^2 < \infty \) and \( Ee_1^4 < \infty \), and set \( \nu = \text{var}(e_1^2) \). Then we have

\[ n^{1/2} \nu^{-1/2} (\hat{\sigma}_n^2 - \sigma^2) \xrightarrow{d} N, \]

where \( \xrightarrow{d} \) and \( N \) denote convergence in distribution and the standard normal random variable, respectively.

**Theorem 3** (Lemma 3.2.3 of Stout [1995,p.120]). Let \( \{\alpha_{ni}\} \) be a matrix of real numbers and \( \{x_i\} \) a sequence of real numbers. Let \( x_i \to x \) as \( i \to \infty \). then

\[ \sum_{i=1}^{\infty} |a_{ni}| \leq M < \infty \quad \text{for all} \quad n \geq 1, \]

\[ \sum_{i=1}^{\infty} a_{ni} \to 1 \quad \text{as} \quad n \to \infty \]

and

\[ a_{ni} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad i \geq 1 \]

imply that

\[ \sum_{i=1}^{\infty} a_{ni} x_i \to x \quad \text{as} \quad n \to \infty. \]
3 Proofs

Let $\alpha(\epsilon) := \epsilon^{-2p/(2-p)}$ and $0 < \epsilon < 1$, $M > 2$ and let $\nu = 1$, $a_{nl} = a_{ni}$ for the same $l$ in the sequel and let $C$ denote a positive constant whose value possibly varies from place to place. The proof of theorem 1.1 is based on the following three propositions.

**Proposition 3.1** For $1 < p < 2$ and $r > 1 + p/2$, we have

$$\lim_{\epsilon \to 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n=1}^{\infty} n^{r/p-2-1/p+1/2} E\left\{ |N| - \epsilon n^{1/p-1/2} \right\}^+ = \frac{p(2-p)}{(r-p)(2r-p-2)} E|N|^{2(r-p)/(2-p)}.$$  \hspace{1cm} (3)

**proof.** For $r > 1 + p/2$, we have

$$\lim_{\epsilon \to 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n=1}^{\infty} n^{r/p-2-1/p+1/2} E\left\{ |N| - \epsilon n^{1/p-1/2} \right\}^+$$

$$= \lim_{\epsilon \to 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n=1}^{\infty} n^{r/p-2-1/p+1/2} \int_{\epsilon n^{1/p-1/2}}^{\infty} P\{ |N| \geq t \} dt$$

$$= \lim_{\epsilon \to 0} \epsilon^{2(r-p)/(2-p)-1} \int_{\epsilon}^{\infty} x^{r/p-2-1/p+1/2} \int_{\epsilon n^{1/p-1/2}}^{\infty} P\{ |N| \geq t \} dt dx$$

$$= \frac{2p}{2-p} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \int_{\epsilon n^{1/p-1/2}}^{\infty} P\{ |N| \geq t \} dt dy \quad \text{by letting } y = \epsilon x^{1/p-1/2}$$

$$= \frac{2p}{2-p} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} P\{ |N| \geq t \} \int_{\epsilon n^{1/p-1/2}}^{t} y^{2(r-p)/(2-p)-2} dy dt$$

$$= \frac{2p}{(2-p)} \left( \frac{2r-2}{2-p} - 1 \right)^{-1} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} t^{2(r-p)/(2-p)-1} P\{ |N| \geq t \} dt$$

$$= \frac{2p}{2r-2} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} t^{2(r-p)/(2-p)-1} P\{ |N| \geq t \} dt$$

$$= \frac{2-r}{2(r-p)(2r-2-p)} E|N|^{2(r-p)/(2-p)}$$

$$= \frac{p(2-p)}{(r-p)(2r-p-2)} E|N|^{2(r-p)/(2-p)}.$$

Thus the proof of (3.1) is completed.

**Proposition 3.2** For $1 < p < 2$, $r > 1 + p/2$ and all $M > 2$, we have

$$\lim_{\epsilon \to 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n \leq a(\epsilon) M} n^{r/p-2-1/p} E\left\{ \left| |N| - \epsilon n^{1/p-1/2} \right| \right\}^+$$

$$- E\left\{ n^{1/2} \left| \sigma_n^2 - \sigma^2 \right| - \epsilon n^{1/p-1/2} \right\}^+ = 0.$$  \hspace{1cm} (4)
proof. Set \( \Delta_n = \sup_{x \in \mathbb{R}} |P(|N| \geq x) - P(n^{1/2}|\hat{\sigma}_n^2 - \sigma^2| \geq x)| \). Then, from Theorem 2, it follows that \( \Delta_n \to 0 \) as \( n \to \infty \). Notice that

\[
\epsilon^{2(r-p)/(2-p)-1} \sum_{n \leq a(e) M} n^{r/p - 2 - 1/p} P \left( |N| \geq x + \epsilon n^{1/p - 1/2} \right) + E \left\{ \left| \frac{1}{n^{1/2}} (\hat{\sigma}_n^2 - \sigma^2) - \epsilon n^{1/p - 1/2} \right| \right\} \\
= \epsilon^{2(r-p)/(2-p)-1} \sum_{n \leq a(e) M} n^{r/p - 2 - 1/p} \int_0^\infty P \left( |N| \geq x + \epsilon n^{1/p - 1/2} \right) dx - \\
\int_0^\infty P \left( n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| \geq x + \epsilon n^{1/p - 1/2} \right) dx \\
\leq \epsilon^{2(r-p)/(2-p)-1} \sum_{n \leq a(e) M} n^{r/p - 5/2} \int_0^\infty P \left( |N| \geq (x + \epsilon) n^{1/p - 1/2} \right) - \\
P \left( n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| \geq (x + \epsilon) n^{1/p - 1/2} \right) dx \\
\leq \epsilon^{2(r-p)/(2-p)-1} \sum_{n \leq a(e) M} n^{r/p - 5/2} (\Delta_{n1} + \Delta_{n2}),
\]

where

\[
\Delta_{n1} := \int_0^{n^{-(p-\frac{1}{2})}} P \left( |N| \geq (x + \epsilon) n^{1/p - \frac{1}{2}} \right) - P \left( n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| \geq (x + \epsilon) n^{(1/p - \frac{1}{2})} \right) dx \\
\Delta_{n2} := \int_{n^{-(p-\frac{1}{2})}}^{\infty} P \left( |N| \geq (x + \epsilon) n^{1/p - \frac{1}{2}} \right) - P \left( n^{1/2} |\hat{\sigma}_n^2 - \sigma^2| \geq (x + \epsilon) n^{(1/p - \frac{1}{2})} \right) dx
\]

Thus, for \( \Delta_{n1} \), by applying Theorem 3, we have

\[
\epsilon^{2(r-p)-1} \sum_{n \leq a(e) M} n^{\frac{r}{p} - \frac{3}{2}} \Delta_{n1} \leq \epsilon^{2(r-p)-1} \sum_{n \leq a(e) M} n^{\frac{r}{p} - \frac{3}{2}} n^{-(p-\frac{1}{2})} \Delta_{n}^{1/2} \Delta_{n} \\
= \epsilon^{2(r-p)-1} \sum_{n \leq a(e) M} n^{\frac{r}{p} - \frac{3}{2}} n^{\Delta_{n}^{1/2} \Delta_{n}} \to 0, \quad \text{as } \epsilon \downarrow 0.
\]

For \( \Delta_{n2} \), by Markov’s inequality and theorem 3, we have
Precise asymptotics for the first moment

For Proposition 3.3

\[ C \epsilon^{2(r-p)-1} \sum_{n \leq u(\epsilon)M} n^{p-\frac{3}{2}} \Delta_n \leq C \epsilon^{2(r-p)-1} \sum_{n \leq u(\epsilon)M} n^{p-\frac{3}{2}} \int_{n-(\frac{1}{2}+\frac{1}{2})}^{\infty} \frac{1}{(x+\epsilon)^{2n^{2(r-p)-1}}} \, dx \]

\[ = C \epsilon^{2(r-p)-1} \sum_{n \leq u(\epsilon)M} n^{p-\frac{3}{2}} \Delta_n^{1/2} \to 0, \quad \text{as} \quad \epsilon \downarrow 0. \]

Hence the proof of (3.2) is completed.

Proposition 3.3 For \( 1 < p < 2 \) and \( r > 1 + p/2 \), we have

\[ \lim_{M \to \infty} \lim_{\epsilon \downarrow 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n > u(\epsilon)M} n^{r/p-2-1/p} E\{ |N| - \epsilon n^{1/p-1/2} \} \geq 0. \]

\[ \lim_{M \to \infty} \lim_{\epsilon \downarrow 0} \epsilon^{2(r-p)/(2-p)-1} \sum_{n > u(\epsilon)M} n^{r/p-2-1/p} E\{ |N| - \epsilon n^{1/p-1/2} \} = 0. \]

Notice that \( a(\epsilon)M - 1 \geq a(\epsilon)M/2 \) for \( M > 2 \) and \( 0 < \epsilon < 1 \). It follow that

\[ \epsilon^{2(r-p)/(2-p)-1} \sum_{n > u(\epsilon)M} n^{r/p-2-1/p} E\{ |N| - \epsilon n^{1/p-1/2} \} \geq 0. \]

\[ \leq \epsilon^{2(r-p)/(2-p)-1} \int_{a(\epsilon)M-1}^{\infty} y^{r/p-2-1/p} \int_{y^{1/p-1/2}}^{\infty} P\{ |N| \geq x \} \, dx \, dy \]

\[ \leq \epsilon^{2(r-p)/(2-p)-1} \int_{a(\epsilon)M/2}^{\infty} y^{r/p-2-1/p} \int_{y^{1/p-1/2}}^{\infty} P\{ |N| \geq x \} \, dx \, dy \]

\[ = \frac{2}{2-p} \int_{(M/2)^{2-r/p}}^{\infty} \int_{t^{1/p-1/2}}^{\infty} P\{ |N| \geq x \} \, dx \, dt \quad \text{by letting} \quad t = \epsilon y^{1/p-1/2} \]

\[ = \frac{2}{2-p} \int_{(M/2)^{2-r/p}}^{\infty} \int_{t^{1/p-1/2}}^{\infty} P\{ |N| \geq x \} \, dx \, dt \]

\[ = \frac{2}{2-p} \left( \frac{2(r-p)}{2-p} \right)^{-1} \int_{(M/2)^{2-r/p}}^{\infty} x^{2(r-p)/2-p-1} P\{ |N| \geq x \} \, dx \to 0. \]
as $\epsilon \downarrow 0$ and $M \to \infty$ uniformly in $0 < \epsilon < 1$, $1 < p < 2$ and $r > 1 + p/2$.

thus (3.3) is proved.

By the representation of $\hat{\sigma}^2$-(1,1), we have that

$$n(\hat{\sigma}^2 - \sigma^2) = \frac{n}{n - \kappa} \sum_{i=1}^{n} (e_i^2 - \sigma^2) + \frac{n\kappa}{n - \kappa} \sigma^2 - \frac{n}{n - \kappa} \sum_{j=1}^{\kappa} (\sum_{i=1}^{n} a_{nj}e_i)^2.$$ 

For prove (3.4), it suffices to show that

$$\lim_{M \to \infty} \lim_{\epsilon \downarrow 0} \epsilon^{2(r-p)} \sum_{n>a(\epsilon)M} n^{\frac{p}{2} - \frac{1}{p} - \frac{1}{2}} E \left\{ \sum_{i=1}^{n} (e_i^2 - \sigma^2) \right\}_{+} = 0, \quad (7)$$

$$\lim_{M \to \infty} \lim_{\epsilon \downarrow 0} \epsilon^{2(r-p)} \sum_{n>a(\epsilon)M} n^{\frac{p}{2} - \frac{1}{p} - \frac{1}{2}} E \left\{ \kappa \sigma^2 - \epsilon n^{\frac{1}{2}} n^{\frac{1}{p} - \frac{1}{2}} \right\}_{+} = 0, \quad (8)$$

and

$$\lim_{M \to \infty} \lim_{\epsilon \downarrow 0} \epsilon^{2(r-p)} \sum_{n>a(\epsilon)M} n^{\frac{p}{2} - \frac{1}{p} - \frac{1}{2}} E \left\{ \sum_{i=1}^{n} a_{ni}e_i \right\}^{2} - \epsilon n^{\frac{1}{2}} n^{\frac{1}{p} - \frac{1}{2}} = 0, \quad (9)$$

For (7), set $S_n = \sum_{i=1}^{n} (e_i^2 - \sigma^2)$. thus $S_n$ are partial sums of i.i.d. random variables with mean zero and finite variance. then we have that

$$\epsilon^{2(r-p)} \sum_{n>a(\epsilon)M} n^{\frac{p}{2} - \frac{1}{p} - \frac{1}{2}} E \left\{ |S_n| - \epsilon n^{\frac{1}{2}} n^{\frac{1}{p} - \frac{1}{2}} \right\}_{+}$$

$$= \epsilon^{2(r-p)} \sum_{n>a(\epsilon)M} n^{\frac{p}{2} - \frac{1}{p} - \frac{1}{2}} \int_{\epsilon n^{\frac{1}{2}} n^{\frac{1}{p} - \frac{1}{2}}}^{\infty} P(|S_n| \geq x)dx$$

$$\leq C \epsilon^{2(r-p)} \sum_{n>a(\epsilon)M} n^{\frac{p}{2} - \frac{1}{p} - \frac{1}{2}} \int_{\epsilon n^{\frac{1}{2}} n^{\frac{1}{p} - \frac{1}{2}}}^{\infty} x^{-2}dx$$

$$\leq C \epsilon^{2(r-p)} \sum_{n>a(\epsilon)M} n^{\frac{p}{2} - \frac{1}{p} - \frac{1}{2}} \to 0, \quad \text{as} \quad \epsilon \downarrow 0$$

then (7) is proved.

For (8), since $\epsilon n^{1/p} \geq \epsilon^{\frac{1}{2}} n^{\frac{1}{p}} M_n^{\frac{1}{p}} = \epsilon^{\frac{1}{2}} n^{\frac{1}{p}} M_n^{\frac{1}{p}} \to \infty$ as $n > a(\epsilon)M$ and $M \to \infty$, it is easily seen that $E \left\{ \kappa \sigma^2 - \epsilon n^{\frac{1}{2}} n^{\frac{1}{p} - \frac{1}{2}} \right\}_{+} = 0$ as $M \to \infty$, and hence
Precise asymptotics for the first moment follows.

Define $T_n = \sum_{i=1}^{n} a_{ni} e_i$, and then by the Chebychev’s inequality and (2), we have that

$$P(\{|T_n| \geq \sqrt{x}\} \leq C x^{-2} \left( (Ee_1^4) \sum_{i=1}^{n} a_{ni}^4 + (Ee_1^2) \sum_{i \neq j} a_{ni}^2 a_{nj}^2 \right).$$

therefore,

$$\epsilon^{2(r-p)}/(2-p)^{-1} \sum_{n>a(\epsilon)M} n^{2-2/p-1/2} E\left\{ |T_n|^2 - \epsilon n^{1/p} n^{1/2} \right\} +$$

$$= \epsilon^{2(r-p)}/(2-p)^{-1} \sum_{n>a(\epsilon)M} n^{2-2/p-1/2} \int_{en^{1/p}}^{\infty} P\{|T_n| \geq \sqrt{x}\} \ dx$$

$$\leq C \epsilon^{2(r-p)}/(2-p)^{-1} \sum_{n>a(\epsilon)M} n^{2-2/p-1/2} \int_{en^{1/p}}^{\infty} \frac{1}{x^2} \ dx$$

$$\leq C \epsilon^{2(r-p)-2} \sum_{n>a(\epsilon)M} n^{-2-2/p-1/2} \to 0, \ \text{as} \ \epsilon \downarrow 0$$

The proof of parts (ii) and (iii) are similar to (i), so we omit them.

References


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