Helly-Type Theorems for Countable Intersections of Planar Sets Starshaped via Polygonal $k$-Paths

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Abstract. Let $k$ be a fixed integer, $k \geq 1$, and let $\mathcal{K}$ be a family of simply connected sets in the plane.

1) If every countable intersection of members of $\mathcal{K}$ is starshaped via (at most) $k$-paths and the corresponding $k$th-order kernel has nonempty interior, then $\cap\{K : K \in \mathcal{K}\}$ has these properties as well.

2) When members of $\mathcal{K}$ are closed, if every countable intersection of members of $\mathcal{K}$ is starshaped via (at most) $k$-paths, then $\cap\{K : K \in \mathcal{K}\}$ is starshaped via $k$-paths, also.

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1. Introduction

Let $S$ be a nonempty set in the plane, and let $k$ be an integer, $k \geq 1$. We define a $k$-path in $S$ to be a simple polygonal path in $S$ consisting of at most $k$ segments. If a $k$-path consists of exactly $k$ segments, we call $k$ the length of the path. For distinct points $x, y$ in set $S$ we say $x$ sees $y$ ($x$ is visible from $y$) via $k$-paths if and only if $S$ contains a $k$-path from $x$ to $y$. If $x$ sees $y$ via $k$-paths and $k$ is minimal, we say that $k$ is the distance from $x$ to $y$, denoted $\text{dist}_S(x, y)$ or $\text{dist}(x, y)$. For convenience of notation, we say $x$ sees itself via a 0-path, and $\text{dist}(x, x) = 0$. Set $S$ is called an $L_k$ set if and only if for all $x, y$ in $S$, $x$ sees $y$ via $k$-paths. (Alternately, we may think of an $L_k$ set as convex via $k$-paths.) Similarly, set $S$ is starshaped via $k$-paths if and only if for some point $p$ in $S$, $p$ sees each point of $S$ via $k$-paths. The associated set of all such points $p$ is the $k$th order kernel of $S$ (the $k$-kernel of $S$), denoted $k - \text{Ker} S$ or, if no confusion arises, simply $\text{Ker} S$. 
A familiar theorem by Victor Klee \cite{5} establishes the following Helly-type theorem for countable intersections of convex sets: Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^d$. If every countable subfamily of $\mathcal{C}$ has nonempty intersection, then $\bigcap\{C : C \in \mathcal{C}\}$ is nonempty as well. Results in \cite{1} provide an analogue of Klee’s theorem for sets in $\mathbb{R}^d$ that are starshaped via segments, while \cite[Theorem 1]{2} supplies a staircase version for families of simply connected sets in the plane. Here we consider the problem for families of simply connected planar sets whose countable intersections are starshaped via $k$-paths.

Throughout the paper, we will use the following terminology and notation. We say that a planar set $S$ is \textit{simply connected} if and only if, for every simple closed curve $\delta \subseteq S$, the bounded region determined by $\delta$ lies in $S$. If $\lambda$ is a simple path containing points $x$ and $y$, then $\lambda(x, y)$ will denote the subpath of $\lambda$ from $x$ to $y$, ordered from $x$ to $y$. When $x$ and $y$ are distinct, $L(x, y)$ will represent the corresponding line, while $R(x, y)$ will be the ray emanating from $x$ through $y$. For any set $S$, $\text{int}S$ will denote its interior. Readers may refer to Valentine \cite{8}, to Lay \cite{6}, to Danzer, Grünbaum, Klee \cite{3}, and to Eckhoff \cite{4} for discussions concerning Helly-type theorems, visibility via straight line segments, and starshaped sets.

2. The results

We begin with two easy propositions. The first appears in \cite[Proposition 1]{2} while the second is a variation of \cite[Proposition 2]{2}.

Proposition 1. Let $\mathcal{K}$ be any family of sets in $\mathbb{R}^d$. If every countable intersection of members of $\mathcal{K}$ has a nonempty interior, then $\bigcap\{K : K \in \mathcal{K}\}$ has a nonempty interior as well.

\textit{Proof}. The proof, which holds in any second countable topological space, is given in \cite{2}.

Proposition 2. Let $k$ be a fixed integer, $k \geq 0$. Assume that $G$ is a simply connected set in the plane, with $\text{int}G = \emptyset$. If $G$ is starshaped via $k$-paths, then for each pair of points $x, y$ in $G$ there is a unique simple path (a unique geodesic) $\lambda$ in $G$. Moreover, $\lambda$ is an $n$-path for some $n \leq 2k$.

\textit{Proof}. The proof is a variation of the argument in \cite[Proposition 2]{2}.

Our first lemma and theorem concern sets having empty interior.

Lemma 1. Let $k$ be a fixed integer, $k \geq 0$, and let $G$ be a simply connected set in the plane with $\text{int}G = \emptyset$. Assume that $G$ is starshaped via $k$-paths, where $k$ is as small as possible. Then $k - \text{Ker}G$ is a convex subset of a line. Moreover, if $k - \text{Ker}G$ contains distinct points $x$ and $y$, then $k - \text{Ker}G$ is exactly the component $C$ of $L(x, y) \cap G$ containing $x$ and $y$.

\textit{Proof}. Observe that set $G$ satisfies the hypotheses of Proposition 2. If $k - \text{Ker}G$ is a singleton set, the result is established. Hence we assume that $k - \text{Ker}G \ni \text{Ker}G$ contains at least two distinct points $x$ and $y$. Let $\lambda(x, y)$ denote the unique $x - y$ geodesic in $G$. We assert that $\lambda \subseteq \text{Ker}G$: Choose
for every $\lambda$ have another contradiction. Our supposition is false, and $\lambda$ has at least $t$ edges, impossible since $\lambda(t,y)$.

We have proved that for all distinct pairs $x,y$ in $\operatorname{Ker} G$, the unique $x-y$ geodesic $\lambda(x,y)$ in $G$ is a segment and lies in $\operatorname{Ker} G$. Since $G$ is simply connected and has no interior points, this implies that all points of $\operatorname{Ker} G$ lie on the same line $L$. We will show that for $C$ the component of $L \cap G$ containing $x$ and $y$, $C = \operatorname{Ker} G$. To see that $C \subseteq \operatorname{Ker} G$, let $z$ belong to $C$. For $w$ in
Let \( \delta_x, \delta_y \) denote the unique \( w - x, w - y \) geodesics in \( G \). As in an earlier argument, \( \delta_x \) and \( \delta_y \) share all points off \( C \) and share a first point \( t \) of \( C \). Since \( x \neq y, \delta_x(w, t) = \delta_y(w, t) \) is at most a \( (k - 1) \)-path, and this \( (k - 1) \)-path may be joined to \([t, z]\) to produce a \( k \)-path from \( w \) to \( z \). That is, the distance from \( w \) to \( z \) is at most \( k \). Since this is true for all \( w \) in \( G, z \in \text{Ker} G, \) and \( C \subseteq \text{Ker} G \). Moreover, since every distinct pair in \( \text{Ker} G \) is joined by a segment, it follows that \( \text{Ker} G \subseteq C \), and the sets are equal, finishing the proof of Lemma 1.

**Theorem 1.** Let \( j \) be a fixed integer, \( j \geq 0 \), and let \( \mathcal{K} \) be a family of simply connected sets in the plane. Assume that every countable intersection of members of \( \mathcal{K} \) is starshaped via (at most) \( j \)-paths. Also assume that some countable intersection of members of \( \mathcal{K} \) has empty interior. Then \( \cap\{K : K \in \mathcal{K}\} \) is nonempty and starshaped via \( k_0 \)-paths for some smallest \( k_0 \leq j \).

**Proof.** Assume that for some countable subfamily \( \mathcal{G} = \{G_n : n \geq 1\} \) of \( \mathcal{K}, \cap\{G_n : n \geq 1\} \equiv G \) has empty interior. Moreover, assume that from all such countable subfamilies of \( \mathcal{K}, \mathcal{G} \) has been selected so that \( G \) is starshaped via \( k \)-paths for \( k \) is small as possible.

Of course, \( 0 \leq k \leq j \). If \( k = 0 \), then \( \cap\{G_n : G_n \in \mathcal{G}\} \) is a singleton set, as is \( \cap\{K : K \in \mathcal{K}\} \), and there is nothing left to prove. Hence we assume that \( k \geq 1 \). If possible, choose \( \mathcal{G} \) so that \( \text{Ker} G \) contains at least two distinct points. Let \( \mathcal{J} \) denote the family of all countable intersections of members of \( \cap\{G \cap K : K \in \mathcal{K}\} \). Then members of \( \mathcal{J} \) are nondegenerate, have empty interior, are starshaped via (at most) \( j \)-paths, and are not starshaped via \( (k - 1) \)-paths. Observe that members of \( \mathcal{J} \) satisfy Proposition 2 and Lemma 1 as well.

We will prove that for \( J \) in \( \mathcal{J} \), \( J \) is starshaped via \( k \)-paths, and \( k - \text{Ker} J \subseteq k - \text{Ker} G \). The following definitions will be helpful: We define \( A_0 = \text{Ker} G \equiv k - \text{Ker} G \). For \( 1 \leq n \leq k \), define \( A_n = \{x : x \in G, \text{dist}(x, \text{Ker} G) = n\} \).

(That is, the distance from \( x \) to a closest point of \( \text{Ker} G \) is exactly \( n \). For \( 1 \leq n \leq k \), notice that each component of \( A_n \) is either a (nonclosed) segment or a (nonclosed) ray having one endpoint (boundary point) in \( A_{n-1} \). Moreover, for \( x, y \) in distinct components of \( A_n \), any \( x - y \) path in \( G \) uses points from \( A_{n-1} \).

We will need the following observations concerning paths in \( G \) : For \( p \) in \( \text{Ker} G \) and \( x, y \) in \( G, x \neq y \), let \( \lambda_x(p, x), \lambda_y(p, y) \) denote the unique \( p - x, p - y \) geodesics in \( G \), respectively. The paths \( \lambda_x \) and \( \lambda_y \) share a last point \( t \), and \( \lambda_x(x, t) \cup \lambda_y(t, y) \) is the unique \( x - y \) geodesic in \( G \). Certainly \( \lambda_x(p, x) \) moves through \( A_n \) sets with increasing subscripts, from \( A_0 \) to \( A_{n(x)} \), where \( x \in A_{n(x)} \).

Moreover, \( \lambda_x \) uses points from exactly one component of each corresponding \( A_n \) set. A parallel statement holds for \( \lambda_y(p, y) \). Thus \( \lambda_y(x, t) \cup \lambda_y(t, y) \) passes through \( A_n \) sets with decreasing, then increasing subscripts (or with decreasing subscripts, or with increasing subscripts, or with just one subscript, depending on the location of \( t \)).
We will show that for every $J$ in $\mathcal{J}$, $J$ is starshaped via $k$-paths and $k - \text{Ker } J \equiv \text{Ker } J \subseteq \text{Ker } G$. In fact, we will establish the following stronger result:

Proposition 3. If $\mathcal{J}$ may be chosen so that $\text{Ker } G$ contains distinct points (and hence, by Lemma 1, a segment), then for all $J$ in $\mathcal{J}$, $J$ is starshaped via $k$-paths, $\text{Ker } J$ contains a segment, and $\text{Ker } J \subseteq \text{Ker } G$. If no such $\mathcal{J}$ exists, then $\text{Ker } G$ is a singleton set $\{p\}$, and $\{p\} = \text{Ker } J$ for all $J$ in $\mathcal{J}$.

Proof of Proposition 3. Let $J$ belong to $\mathcal{J}$. Choose the smallest $n_0$ such that $J$ contains points of $A_{n_0}$. Select $q \in A_{n_0} \cap J$ to show that $q \in \text{Ker } J$. Certainly $q$ sees each point of $J$ via a unique geodesic in $J$ and in $G$. If $q \in A_0 \equiv \text{Ker } G$, then $q$ sees each point of $J$ via a unique geodesic (in $J$ and in $G$) whose length is at most $k$. By our choice of $k$, this means that $J$ is starshaped via $k$-paths with $q \in \text{Ker } J$. This remark will be useful in the proof.

We will show that this situation must occur. If $q \notin A_0$, select $x$ in $J$ whose distance to $q$ in $J$ is as large as possible. By our observations above, the unique $q - x$ geodesic $\lambda(q, x)$ in $J$ (which is also the unique $q - x$ geodesic in $G$) passes through $A_n$ sets having decreasing, then increasing subscripts. Since $q \in A_{n_0}$ and $n_0$ is minimal in $J$, the associated subscripts must increase from $n_0$ to $n(x)$, where $x \in A_{n(x)}$. The path $\lambda(q, x)$ uses points from exactly one component (segment or ray) of each corresponding $A_n$ set. If $n_0 \geq 2$, then $\lambda(q, x)$ would have length at most $k - 1$. By our choice of $x$, this would force $J$ to be starshaped via $(k - 1)$-paths, impossible. Thus $n_0 = 1$ and $q \in A_1$.

Moreover, since the distance from $x$ to $q$ is maximal, this distance must be at least $k$ (and hence exactly $k$ by the description of $\lambda$ above). It follows that $J$ is starshaped via $k$-paths.

Let $C_1$ denote the component of $A_1$ in $G$ for which $q \in C_1$, and let $p$ denote the endpoint of $C_1$ in $A_0 \equiv \text{Ker } G$. To complete the proof of Proposition 3, we consider two cases.

Case 1. Assume that $\text{Ker } G$ contains a segment. Select $p'$ in $(\text{Ker } G)\{p\}$. Then $[p', p] \cup [q, p] \cup \lambda(q, x)$ is a $p' - x$ geodesic in $G$, and $\text{dist}_G(p', x) \geq k + 1$, impossible since $G$ is starshaped via $k$-paths with $p' \in \text{Ker } G$. We have a contradiction, and this situation cannot occur. That is, if $\text{Ker } G$ contains a segment, then $q \in A_0$ and (by an earlier remark) $q \in \text{Ker } J$. Thus $J$ contains points of $A_0$ and $A_0 \cap J \subseteq \text{Ker } J$.

We assert that $\text{Ker } J$ contains a segment as well. Otherwise, for $q$ selected above in $A_0 \cap J$, $\{q\} = \text{Ker } J$. Again consider $x$ in $J$ whose distance to $q$ in $J$ is maximal (and hence $k$). Examine the $q - x$ geodesic $\lambda(q, x)$ in $J$. Observe that $\lambda(q, x)$ meets $\text{Ker } G$ only in $\{q\}$, since $\lambda(q, x) \cap \text{Ker } G \subseteq J \cap A_0 \subseteq \text{Ker } J = \{q\}$. This implies that the first segment $[q, x_1]$ of $\lambda(q, x)$ cannot be collinear with $\text{Ker } G$. (Otherwise, $[q, x_1] \subseteq (\text{Ker } G) \cap J = A_0 \cap J \subseteq \text{Ker } J$, impossible.) But then for $p' \in (\text{Ker } G)\{q\}$, $p'$ is not on line $L(q, x_1)$, and the $p' - x$ geodesic in $G$ is exactly $[p', q] \cup \lambda(q, x)$, a path of length $k + 1$. However, since $p' \in \text{Ker } G$, the $p - x$ geodesic must have length at most $k$. We have a contradiction and
$Ker\ J$ cannot be a singleton set. That is, $Ker\ J$ contains a segment, and the assertion is established. Finally, by Lemma 1, for $L$ the associated line, $Ker\ J$ will be a component of $L \cap J$, so $Ker\ J$ will be exactly the nondegenerate convex set $A_0 \cap J = (Ker\ G) \cap J$. This finishes Case 1.

Case 2. Assume that $Ker\ G$ is a singleton set. Using our earlier notation, this means that $Ker\ G = \{p\}$, where $p$ is the endpoint of $C_1$ in $A_0$ and where $C_1$ is the component of $A_1$ in $G$ for which $q \in C_1$. Recall that $q \notin A_1 \cap J$ and $A_0 \cap J = \emptyset$. Again consider point $x$ in $J$ whose distance to $q$ is maximal, hence exactly $k$. As in Case 1, let $[q, x_1]$ denote the first segment of the $q - x$ geodesic $\lambda(q, x)$ in $J \subseteq G$. Since $p$ sees $x$ via a $k$-path in $G$ and $[p, q] \subseteq C_1 \subseteq G$, $p$ must be collinear with $[q, x_1]$. For any point in $J$ whose distance to $q$ is $k$, a parallel statement holds.

Let $C$ be the (nondegenerate) component of $L(p, q) \cap J = L(q, x_1) \cap J$ that contains $q$. We will show that $C \subseteq Ker\ J$. Let $c \in C$, let $w \in J$, and let $\delta(q, w)$ be a $q - w$ geodesic in $J \subseteq G$. If $\delta$ has length at most $k - 1$, then $[c, q] \cup \delta$ contains a $c - w$ $k$-path in $J$. If $\delta$ has length fully $k$, then, by the argument above, the first edge of $\delta$ lies in $L(p, q) \cap J$. That is, $c$ is collinear with the first edge of $\delta(q, w)$, and again $[c, q] \cup \delta$ contains a $c - w$ $k$-path in $J$. But then $C \subseteq Ker\ J$, and $Ker\ J$ contains a segment, contradicting our choice of $g$. This situation cannot occur either. That is, if $Ker\ G$ is a singleton set, then $q \in A_0$. By an earlier remark, $q \in Ker\ J$. Moreover, since $A_0 = \{p\}, q = p$. By our choice of $g$, $Ker\ J$ is a singleton set, so $Ker\ J = \{p\}$, finishing Case 2 and completing the proof of Proposition 3.

Now we may finish the argument for Theorem 1. If $Ker\ G$ contains a segment $s$, let $L$ be the corresponding line. Then $Ker\ G = C$, where $C$ is the component of $L \cap G$ containing segment $s$. Moreover, for $J$ in $\mathcal{J}$, $Ker\ J \subseteq Ker\ G = C$, and $Ker\ J$ is a nondegenerate convex subset $C_J$ of $C$. Clearly every countable intersection of $C_J$ sets is associated with another member of $\mathcal{J}$ and hence has a full one-dimensional intersection in $C$. Thus by an easy argument in $\mathbb{R}^1$ or by [1, Theorem 1, Corollary 2], $\cap\{C_J : J \in \mathcal{J}\}$ is fully one-dimensional as well. That is, $\cap\{J : J \in \mathcal{J}\} = \cap\{K : K \in \mathcal{K}\}$ contains a nondegenerate segment.

Moreover, it is easy to show that $\cap\{K : K \in \mathcal{K}\}$ is starshaped via $k$-paths, with $\cap\{C_J : J \in \mathcal{J}\}$ in the associated $k^{th}$ order kernel: For $p \in \cap\{C_J : J \in \mathcal{J}\}$ and $z \in \cap\{K : K \in \mathcal{K}\}$, $p$ sees $z$ via a $k$-path in $J$ for all $J$ in $\mathcal{J}$. This path $\lambda(p, z)$ is unique in $G$ and hence unique in $J$ for all $J$. Thus $\lambda \subseteq \cap\{J : J \in \mathcal{J}\}$, and $p$ sees $z$ via a $k$-path in $\cap\{J : J \in \mathcal{J}\} = \cap\{K : K \in \mathcal{K}\}$. That is, $\cap\{K : K \in \mathcal{K}\}$ is starshaped via $k$-paths, and its $k^{th}$-order kernel contains $\cap\{C_J : J \in \mathcal{J}\}$. In case $k$ is minimal, then it is easy to see that $\cap\{C_J : J \in \mathcal{J}\}$ will be exactly the $k^{th}$ order kernel of $\cap\{K : K \in \mathcal{K}\}$.

Finally, if $Ker\ G$ is a singleton set $\{p\}$, then $Ker\ J = \{p\}$ for all $J$ in $\mathcal{J}$, and $p$ belongs to $\cap\{J : J \in \mathcal{J}\} = \cap\{K : K \in \mathcal{K}\}$. An argument like the one above shows that $p$ belongs to the $k^{th}$ order kernel of $\cap\{K : K \in \mathcal{K}\}$, so
∩\{K : K \text{ in } \mathcal{K}\} is starshaped via \(k\)-paths. If \(k\) is minimal, the associated kernel will be either \(\{p\}\) or a one-dimensional convex set containing \(p\). This finishes the proof of Theorem 1.

We have the following corollaries.

Corollary 1.1. Let \(\mathcal{K}, \mathcal{S}, G\) and \(k\) be as defined in the proof of Theorem 1. If \(\text{Ker} \, G\) contains a segment, then \(\cap\{K : K \text{ in } \mathcal{K}\}\) is starshaped via \(k\)-paths, and the associated \(k\)-th order kernel contains a segment. If \(\text{Ker} \, G\) is a singleton set \(\{p\}\), then \(\cap\{K : K \text{ in } \mathcal{K}\}\) is starshaped via \(k\)-paths, and the associated \(k\)-kernel contains \(p\). (Of course, \(k\) need not be minimal.)

\textbf{Proof.} This follows immediately from the proof of Theorem 1.

Corollary 1.2. Let \(j\) be a fixed integer, \(j \geq 1\), and let \(\mathcal{K}\) be a family of simply connected sets in the plane. Assume that every countable intersection of members of \(\mathcal{K}\) is starshaped via (at most) \(j\)-paths. Then \(S \equiv \cap\{K : K \text{ in } \mathcal{K}\}\) is nonempty.

\textbf{Proof.} If every countable intersection of members of \(\mathcal{K}\) has nonempty interior, then \(S\) has nonempty interior by Proposition 1. If some countable intersection of members of \(\mathcal{K}\) has empty interior, then \(S\) is nonempty by Theorem 1.

Using the notation in Theorem 1, it is interesting to observe that if \(\cap\{K : K \text{ in } \mathcal{K}\}\) is starshaped via \(k_0\)-paths for \(k_0 < k\), the associated \(k_0\)-kernel may be disjoint from \(\text{Ker} \, G\). Moreover, the dimension of its \(k_0\)-kernel need not be the dimension of \(k - \text{Ker} \, G\). Consider the following examples.

Example 1. In \(\mathbb{R}^2\), let \(T\) be the segment from \((0,0)\) to \((4,0)\), and let \(S_0\) denote the polygonal 3-path whose consecutive vertices are \((0,0), (0,1), (1,2)\), and \((1,3)\). (See Figure 2.) For each \(q\) irrational, \(0 < q < 4\), define \(S_q = (q,0) + S_0\). Finally, for \(r\) irrational, \(0 < r < 4\), define \(K_r = \cup\{T \cup S_0 \cup S_q : q\) irrational, \(0 < q < 4, q \neq r\}\).

Clearly each countable intersection of \(K_r\) sets will be starshaped via \(4\)-paths, with \(T\) the associated kernel. However, \(\cap\{K_r : 0 < r < 4\} = T \cup S_0\) is starshaped via \(2\)-paths, and its one-point kernel is \(\{(0,1)\}\).

The example may be adapted for sets \(K_r\) starshaped via \(2k\)-paths, with \(\cap\{K_r : 0 < r < 4\}\) starshaped via \(k\)-paths, \(k \geq 2\).
Example 2. In $\mathbb{R}^2$, let $\theta$ denote the origin and let $W = \{(x, y) : x^2 + y^2 = 1, x \text{ irrational}, x > 0, y > 0\} \cup \{(1, 0)\}$. For each $w$ in $W$, let $S_w$ denote the 3-path whose consecutive vertices are $\theta, w, w + (0, 1), w + (1, 1)$. (See Figure 3.) For $u$ in $W, u \neq (1, 0)$, define $K_u = \cup\{S_w : w \in W, w \neq u\}$. Clearly each countable intersection of $K_u$ sets will be starshaped via 3-paths, with $\{\theta\}$ the associated kernel. However, $\cap\{K_u : u \in W, u \neq (1, 0)\} = S_{(1,0)}$ is starshaped via 2-paths, and its kernel consists of all points on the segment from $(1, 0)$ to $(1, 1)$.

![Figure 3](image-url)

The example may be adapted for sets $K_u$ starshaped via $(2k + 1)$-paths, with $\cap\{K_u : u \in W, u \neq (1, 0)\} = S_{(1,0)}$ starshaped via $(k + 1)$-paths, $k \geq 1$.

The next results concern the more general case in which sets may have nonempty interior. Theorem 2 is a polygonal $n$-path analogue of a staircase result in [2, Lemma 1].

**Theorem 2.** Let $n$ be a fixed integer, $n \geq 1$. Let $\mathcal{K}$ be a family of simply connected sets in the plane, and let $x, s \in \cap\{K : K \in \mathcal{K}\}$. If every countable intersection of members of $\mathcal{K}$ contains a polygonal $n$-path from $x$ to $s$, then $\cap\{K : K \in \mathcal{K}\}$ contains such a path.

**Proof.** We use induction on $n$. If $n = 1$, the result is trivial. Assume that the result is true for all $j, 1 \leq j \leq k$, to prove for $k + 1$. For convenience, let $\mathcal{J}$ denote the family of all countable intersections of members of $\mathcal{K}$. Also, assume that for at least one $J_0$ in $\mathcal{J}, J_0$ contains no $x - s$ $k$-path, for otherwise the result follows immediately from our induction hypothesis. For every $J_0$ in $\mathcal{J}$, define the associated set $E_{\alpha,1}$ of first endpoints (after $x$) of $(k + 1)$-paths from $x$ to $s$ in $J_0 \cap J_\alpha$. Continue, denoting the set of $i$th endpoints by $E_{\alpha,i}, 1 \leq i \leq k$. If for some $i$ fixed, every set $E_{\alpha,i}$ has interior points, then certainly countable intersections of these sets have interior points, and by Proposition 1 all $E_{\alpha,i}$ sets share an interior point $p$. We may use our induction hypothesis to find an $i$-path from $x$ to $p$ in $\cap\{J : J \in \mathcal{J}\}$ and a $(k + 1 - i)$-path from $p$ to $s$ in $\cap\{J : J \in \mathcal{J}\}$. The union of these two paths contains a $(k + 1)$-path from $x$ to $s$ in $\cap\{J : J \in \mathcal{J}\}$, the desired result. Hence we may assume that for every $i, 1 \leq i \leq k$, there is a $J_\alpha_i$ in $\mathcal{J}$ whose associated $E_{\alpha,i}$ has empty interior. Define $\cap\{J_0 \cap J_\alpha_i : 1 \leq i \leq k\} \equiv J_B \in \mathcal{J}$. In $J_B$, there is no $k$-path from $x$ to $s$. Moreover, $E_{B,i}$ has no interior points for every $i, 1 \leq i \leq k$. 


In $J_B$, there is no $k$-path from $x$ to $s$. Moreover, $E_{B,i}$ has no interior points for every $i, 1 \leq i \leq k$.

In case $J_B$ contains only one $x-s(k+1)$-path, this path belongs to $\bigcap \{J: J \in \mathcal{J}\}$, finishing the argument. Hence we assume that $J_B$ contains at least two such paths, say $\lambda_y = [x, y_1] \cup \ldots \cup [y_k, s]$ and $\lambda_z = [x, z_1] \cup \ldots \cup [z_k, s]$ are two such simple paths in $J_B$.

We will make some observations about any pair $\lambda_y$ and $\lambda_z$. First we assert that $\lambda_y(x, y_2)$ meets $\lambda_z(x, z_2)$ at a point other than $x$ and $s$. (Here if $k = 1$, then $y_2 = z_2 = s$.) If $\lambda_y(x, y_2)$ is disjoint from $\lambda_z$ (except at $x, s$) and $\lambda_z(x, z_2)$ is disjoint from $\lambda_y$ (except at $x, s$), then $E_{B,1}$ has interior points near $z_1$ and near $y_1$, impossible. Thus, for an appropriate labeling of $\lambda_y$ and $\lambda_z$, $\lambda_z(x, z_2)$ meets $\lambda_y$ (at a point not $x$, not $s$). Observe that if $k \geq 2$ the third segment of $\lambda_y$ cannot meet $[x, z_1]$, for such an intersection would yield an $x-sk$-path in $J_B$, impossible. Similarly, no segment of $\lambda_y$ beyond the third can meet $[x, z_1]$. By a parallel argument, the fourth segment of $\lambda_y$ cannot meet $[z_1, z_2]$, and no segment of $\lambda_y$ beyond the fourth can meet $[z_1, z_2]$. If the third segment and no previous segment of $\lambda_y$ meets $[z_1, z_2]$, then we have interior points of $E_{B,1}$ near $y_1$, again impossible. (See Figure 4.)

Thus there must be an intersection of $\lambda_y(x, y_2)$ and $\lambda_z(x, z_2)$ at a point not $x$, not $s$. This establishes the first assertion.

Clearly for any pair $\lambda_y$ and $\lambda_z$ in $J_B$, either $\lambda_y(x, y_2)$ meets $\lambda_z(x, z_2)$ in isolated points or their intersection contains a segment. First consider the case in which, for some pair $\lambda_y$ and $\lambda_z$, $\lambda_y(x, y_2) \cap \lambda_z(x, z_2)$ consists only of isolated points.

We will show that there must be an intersection point (not $x$, not $s$) in the first segment of one path and in the second segment of the other path. (For convenience, we call this our second assertion.) Suppose on the contrary that all such points are in $(z_1, z_2) \cap (y_1, y_2)$. Let $x_1$ denote such a point of intersection. Clearly $x_1 \neq z_2$ and $x_1 \neq y_2$, for otherwise $E_{B,1}$ would have interior points near $z_1$ or near $y_1$, impossible. (See Figure 5a.) If $\lambda_y$ meets $[z_1, z_2]$ at a different point, the point must be in $[y_2, y_3]$. Similarly for $\lambda_z$ meeting $[y_1, y_2]$. If the first situation occurs, this produces interior points of $E_{B,1}$ near $y_1$, impossible. (See Figure 5b.) A similar argument holds for the second situation and point $z_1$. 

![Figure 4](image-url)
Thus \( \lambda_y \) cannot meet \([z_1, z_2]\) except at \( x_1 \). Using earlier observations, \( \lambda_y \setminus \{x, s\} \) meets \( \lambda_z(x, z_2) \) only at \( x_1 \), \( \lambda_z \setminus \{x, s\} \) meets \( \lambda_y(x, y_2) \) only at \( x_1 \). But then there must be interior points of \( E_{B,1} \) near \( y_1 \) and other interior points of \( E_{B,1} \) near \( z_1 \), impossible.

We conclude that if the intersection of \( \lambda_y \) and \( \lambda_z \) consists only of isolated points, such a point (not \( x \), not \( s \)) must lie in the first segment of one path and in the second segment of the other path. This establishes the second assertion.

Suppose that \( J_B \) contains paths \( \lambda_y, \lambda_z \) described above. That is, these are \((k + 1)\)-paths from \( x \) to \( s \), and \( \lambda_y(x, y_2) \cap \lambda_z(x, z_2) \) consists of isolated points. By our second assertion above, \( \lambda_y(x, y_2) \) meets \( \lambda_z(x, z_2) \) at some point \( x_1 \) in the first segment of one path and in the second segment of the other path. We assert that for \( x_1 \neq z_1 \), the paths also meet at a (different) point is \([y_1, y_2] \cap [z_1, z_2]\). (We call this assertion 3.) For an appropriate labeling of \( \lambda_y \) and \( \lambda_z \), \( x_1 \in (x, z_1] \cap (y_1, y_2] \). Furthermore, assume that \( x_1 \neq z_1 \). If \( \lambda_y \) doesn’t meet \( \lambda_z(x, z_2) \) again and \( \lambda_z \) doesn’t meet \( \lambda_y(x, y_2) \) again (except at \( x, s \)), then \( E_{B,1} \) has interior points near \( y_1 \), impossible. If \( \lambda_y \) meets \( \lambda_z(x, z_2) \) again, by earlier observations, an intersection must occur in the second or third edge of \( \lambda_y \) and in \([z_1, z_2] \). (See Figure 6.) If such an intersection occurs in the third edge of \( \lambda_y \) and not in the second, then there are interior points of \( E_{B,1} \) near \( y_1 \), again impossible. Thus such an intersection occurs in the second edge of \( \lambda_y \) and in \([z_1, z_2] \). A parallel argument holds if \( \lambda_z \) meets \( \lambda_y(x, y_2) \) again. Either way, \([z_1, z_2] \) meets \([y_1, y_2] \) at a point other than \( x_1 \). This establishes the third assertion.
Fix \((k + 1)\)-path \(\lambda_y\) in \(J_B\) from \(x\) to \(s\). Using our earlier notation, there are six possibilities, listed below:

1) For each \(J\) in \(J\), there exists a \((k + 1)\)-path \(\lambda_z\) from \(x\) to \(s\) in \(J \cap J_B\) such that \(\lambda_z(x, z_2)\) meets \([y_1, y_2]\) twice, once in \((x, z_1)\) and once in \((z_1, z_2)\).

2) For each \(J\) in \(J\), there is a \((k + 1)\)-path \(\lambda_z\) from \(x\) to \(s\) in \(J \cap J_B\) such that \(\lambda_y(x, y_2)\) meets \([z_1, z_2]\) twice, once in \((x, y_1)\) and once in \((y_1, y_2)\).

3) Each \(J\) set contains a \((k + 1)\)-path from \(x\) to \(s\) that starts in a segment along ray \(R(x, y_1)\).

4) Each \(J\) set contains a \((k + 1)\)-path from \(x\) to \(s\) whose second segment meets \(R(y_1, y_2)\) in a segment.

5) Each \(J\) set contains a \((k + 1)\)-path \(\lambda_z\) from \(x\) to \(s\) whose first endpoint \(z_1\) lies on \([y_1, y_2]\).

6) Each \(J\) set contains a \((k + 1)\)-path \(\lambda_z\) from \(x\) to \(s\) whose second segment \([z_1, z_2]\) contains \(y_1\).

Of course, any statement true for each \(J\) in \(J\) is true for countable intersections of members of \(J\) as well. Similarly, if some \(J_i\) in \(J\) failed to satisfy property \(i\) above for each \(1 \leq i \leq 6\), then \(\cap\{J_i \cap J_B : 1 \leq i \leq 6\}\) in \(J\) would not contain any appropriate \(x - s\) path by assertions 1, 2, and 3 above. Hence at least one of these properties must hold.

We will prove that each of the six situations above yields a ray \(R\) at \(x\) such that every \(J\) contains a suitable path that begins along \(R\). Consider each situation in turn.

1) Suppose that for each \(J\) in \(J\), there is a corresponding \((k + 1)\)-path \(\lambda_z\) from \(x\) to \(s\) in \(J \cap J_B\) such that \(\lambda_z(x, z_2)\) meets \([y_1, y_2]\) twice, once in \((x, z_1)\) and once in \((z_1, z_2)\). Without loss of generality, we may assume that the associated order of the two points along \([y_1, y_2]\) is the same for every \(J \cap J_B\). For each \(J \cap J_B\), consider all suitable \(x - s\) paths \(\lambda_z\) in \(J \cap J_B\). Each \(\lambda_z\) determines a point of \([y_1, y_2]\) \(\cap (x, z_1)\), and we let \(A_J\) denote the union of all such points.

In case every \(J \cap J_B\) contains two suitable paths \(\lambda_z\) and \(\lambda_z'\) whose first edges are non collinear, then (using our first assertion) \(A_J\) will have relative interior points in \([y_1, y_2]\). Hence every countable intersection of \(A_J\) sets will have relative interior points in \([y_1, y_2]\) and, by Proposition 1, \(\cap\{A_J : J \in J\}\) has nonempty relative interior in \([y_1, y_2]\). For \(w\) in this intersection, every \(J\) set contains an appropriate path that begins along ray \(R(x, w)\). In case, for some \(J \cap J_B\), every suitable path \(\lambda_z\) begins along the same ray \(R\) at \(x\), then every \(J\) set contains a suitable path that begins along ray \(R\). Either way, we have the desired result.

2) Suppose that for each \(J\) in \(J\) there is a \((k + 1)\)-path \(\lambda_z\) from \(x\) to \(s\) in \(J \cap J_B\) such that \(\lambda_y(x, y_2)\) meets \([z_1, z_2]\) twice, once in \((x, y_1)\) and once in \((y_1, y_2)\). As in situation 1 above, without loss generality we may assume that the associated order of the two points along \([z_1, z_2]\) is the same for all \(J \cap J_B\) sets. To each \(J \cap J_B\) we associate the family of rays determined by first edges
of suitable paths $\lambda_2$. Let $C_J$ denote the union of these rays. If every $J \cap J_B$ contains two suitable paths whose first edges are non collinear, then every $C_J$ will have interior points. For $w \neq x, w$ interior to $\cap \{C_J : J \in \mathcal{J}\}$, the ray $R(x, w)$ yields the desired result. If for some $J \cap J_B$, every suitable $\lambda_2$ has its first edge along the same ray $R$, then ray $R$ satisfies the result.

3) If every $J$ set contains a suitable path whose first edge lies on ray $R(x, y_1)$, then this ray satisfies our result.

4) If every $J$ set contains a suitable path $\lambda_2$ whose second segment meets $R(y_1, y_2)$ in a segment, again without loss of generality assume that the associated order on $R(y_1, y_2)$ is the same for all $J$ in $\mathcal{J}$. For each $J$, consider each suitable $\lambda_2$ and each corresponding first endpoint $z_1$ on $L(y_1, y_2)$. Again using our first assertion, the collection $D_J$ of first endpoints is a convex set. Every countable family of $D_J$ sets has a nonempty intersection, and by [5, Lemma 3.1], $\cap \{D_J : J \in \mathcal{J}\} \neq \emptyset$. For $w$ in this intersection, $R(x, w)$ satisfies the result.

5) Suppose that each $J$ set contains a suitable $(k + 1)$-path $\lambda_2$ from $x$ to $s$ whose first endpoint $z_1$ lies on $[y_1, y_2]$. In case for some $J$ there is only one corresponding first edge $[x, z_1]$ meeting our requirement, then ray $R(x, z_1)$ satisfies the result. Otherwise, each set $J \cap J_B$ contains at least two such paths having distinct first segments. For $J_\gamma$ fixed, let $\lambda_2$ and $\lambda_2'$ represent appropriate paths in $J_\gamma \cap J_B$, with $\lambda_2(x, z_2) = [x, z_1] \cup [z_1, z_2]$ and $\lambda_2'(x, z_2') = [x, z_1'] \cup [z_1', z_2']$ associated subpaths consisting of their first and second edges, $z_1 \neq z_1'$. Using assertion 2, for an appropriate labeling, $[x, z_1]$ meets $[z_1', z_2']$. Hence $x, z_2'$ are on the same side of line $L(y_1, y_2)$. (See Figure 7.) Thus there is an interval at $z_1'$ on $[z_1', z_2']$, all of whose points belong to $E_{\gamma, 1}$. Consider the family of rays from $x$ to points in $E_{\gamma, 1}$. The union $C_{J_\gamma}$ of these rays has interior points, as do countable intersections of the $C_J$ sets. By Proposition 1, $\cap \{C_J : J \in \mathcal{J}\}$ has interior points as well. Select $w \neq x, w$ interior to $\cap \{C_J : J \in \mathcal{J}\}$. Ray $R(x, w)$ satisfies our result.

![Figure 7](image-url)

6) Finally, suppose that each $J$ set contains a $(k + 1)$-path $\lambda_2$ from $x$ to $s$ whose second segment $[z_1, z_2]$ contains $y_1$. Without loss of generality, we may assume that each $J$ set contains an appropriate $\lambda_2$ with $z_1$ in the open halfplane $L_1$ determined by line $L = L(x, y_1)$. (See Figure 8.) An argument like the one in situation 5 above produces an appropriate ray.
We have proved that there exists a ray $R$ at $x$ such that every $J$ set contains a suitable $x - s$ $(k + 1)$-path whose first segment lies on ray $R$. Without loss of generality, assume that second endpoints of the paths lie in the same open halfplane determined by $R$. We assert that, for some $w_1$ on $R \setminus \{x\}$, every $J$ set contains an appropriate $x - s$ $(k + 1)$-path whose first endpoint is $w_1$.

In case for some $J$ set there is only one suitable first endpoint on $R \setminus \{x\}$, this point will be $w_1$. Hence we consider the case in which every $J$ set has at least two suitable first endpoints on $R \setminus \{x\}$. Fix $J_\gamma$ in $J$ and let $z_1, z'_1$ denote distinct first endpoints on $R \setminus \{x\}$ for suitable paths in $J_\gamma \cap J_B, x < z_1 < z'_1$. As usual, let paths and associated vertices be $\lambda_z = [x, z_1] \cup \ldots \cup [z_k, s], \lambda_{z'} = [x, z'_1] \cup \ldots \cup [z'_k, s]$. For $k = 1$, $[z_1, z_2]$ meets $[z'_1, z'_2]$ at $z_2 = z'_2 = s$, producing a nondegenerate segment $[z_1, z'_1]$ on ray $R$ in $E_{\gamma,1}$. For $k \geq 2$, using arguments like those in assertions 1, 2, and 3, $[z'_1, z'_2]$ meets $[z_1, z_2]$ and, for an appropriate labeling, $[z'_2, z'_3]$ meets $[z_1, z_2]$ as well. (Otherwise, we produce interior points in $E_{\gamma,2}$.) However, then again $E_{\gamma,1}$ contains a nondegenerate segment on ray $R$ at $z'_1$. (See Figure 9.)

Hence each set $E_{\gamma,1}$ contains an open interval on $R$, as do countable intersections of these sets. By Proposition 1, the intersection of all sets $E_{\gamma,1}$ contains an interval. For $w_1$ in this intersection, $w_1$ satisfies our final assertion. That is, every $J$ set contains an $x - s$ $(k + 1)$-path whose first segment is $[x, w_1]$, and countable intersections of members of $K$ contain such a path. Therefore,
countable intersections of members of $\mathcal{K}$ contain a $w_1 - s k$-path. By our induction hypothesis, $\cap\{K : K \in \mathcal{K}\}$ contains a $k$-path $\lambda(w_1, s)$ from $w_1$ to $s$, and $[x, w_1] \cup \lambda(w_1, s)$ contains a $(k + 1)$-path from $w_1$ to $s$ in $\cap\{K : K \in \mathcal{K}\}$. The result holds for $k + 1$ and, by induction, holds for every natural number $n$, finishing the proof of Theorem 2.

Corollary 2.1. Let $k$ be a fixed integer, $k \geq 1$. Let $\mathcal{K}$ be a family of simply connected sets in the plane, and let $\mathcal{J}$ denote the family of all countable intersections of members of $\mathcal{K}$. Assume that each member of $\mathcal{J}$ is starshaped via (at most) $k$-paths, and let $S \equiv \cap\{J : J \in \mathcal{J}\} = \cap\{K : K \in \mathcal{K}\}$. For each $J_\alpha$ in $\mathcal{J}$, define $M_\alpha = \{x : x \in J_\alpha, x$ sees each point of $S$ via $k$-paths in $J_\alpha\}$, and let $\mathcal{M}$ denote the collection of all the $M_\alpha$ sets. Then $S$ is nonempty, each $M_\alpha$ is nonempty, and $\cap\{M : M \in \mathcal{M}\}$ is exactly the $k^{th}$-order kernel of $S$.

Proof. By Corollary 1.2, $S \neq \emptyset$. Moreover, for each $J_\alpha$ and associated $M_\alpha$, $S \subseteq J_\alpha$, so $k - \text{Ker} J_\alpha \subseteq M_\alpha \neq \emptyset$. We assert that $\cap\{M : M \in \mathcal{M}\} \subseteq k - \text{Ker} S$: Fix $x$ in $\cap\{M : M \in \mathcal{M}\} \subseteq S$ and $s$ in $S$. For every $J_\alpha$ in $\mathcal{J}$, $x$ sees $s$ via $k$-paths in $J_\alpha$. That is, every countable intersection of members of $\mathcal{K}$ contains a $k$-path from $x$ to $s$. By Theorem 2, $\cap\{K : K \in \mathcal{K}\} = S$ contains such a path as well. Thus $x$ sees $s$ via a $k$-path in $S$. Since this is true for every $s$ in $S$, $x \in k - \text{Ker} S$, and $\cap\{M : M \in \mathcal{M}\} \subseteq k - \text{Ker} S$, establishing the assertion. The reverse inclusion is obvious, and the sets are equal.

Theorem 3. Let $k$ be a fixed integer, $k \geq 1$, and let $\mathcal{K}$ be a family of simply connected sets in the plane. Assume that every countable intersection of members of $\mathcal{K}$ is starshaped via (at most) $k$-paths and that the corresponding $k^{th}$ order kernel has nonempty interior. Then $\cap\{K : K \in \mathcal{K}\}$ has these properties as well.

Proof. As in Corollary 2.1, let $\mathcal{J}$ denote the family of countable intersections of members of $\mathcal{K}$, and let $S = \cap\{J : J \in \mathcal{J}\} = \cap\{K : K \in \mathcal{K}\} \neq \emptyset$. For each $J_\alpha$ in $\mathcal{J}$, define $M_\alpha = \{x : x \in J_\alpha, x$ sees each point of $S$ via $k$-paths in $J_\alpha\}$, and let $\mathcal{M}$ denote the collection of all the $M_\alpha$ sets. Notice that, for each $J_\alpha$ in $\mathcal{J}$ and associated $M_\alpha$ in $\mathcal{M}$, $k - \text{Ker} J_\alpha \subseteq M_\alpha$, so $M_\alpha$ has nonempty interior. Furthermore, for any countable family $\{M_n : n \geq 1\}$ in $\mathcal{M}$ and corresponding family $\{J_n : n \geq 1\}$ in $\mathcal{J}$, $\cap\{J_n : n \geq 1\} \equiv J_0$ belongs to $\mathcal{J}$. For the associated $M_0$ in $\mathcal{M}$, $M_0 \subseteq \cap\{M_n : n \geq 1\}$, so $\cap\{M_n : n \geq 1\}$ has nonempty interior as well. Hence by Proposition 1, $\cap\{M : M \in \mathcal{M}\}$ has nonempty interior, and by Corollary 2.1 this intersection is exactly the $k^{th}$ order kernel of $S$, finishing the proof.

It is interesting to observe that, for $k \geq 2$, Theorems 2 and 3 fail without the requirement that members of $\mathcal{K}$ be simply connected. Consider the following example.

Example 3. Let $D$ denote the closed unit disk centered at the origin. For every point $p$ of $D$ on the $x$-axis, let $K_p = D \setminus \{p\}$, and let $\mathcal{K}$ be the collection of all the $K_p$ sets. Clearly any countable intersection $J$ of members of $\mathcal{K}$ will
be an \(L_2\) set. Hence \(J\) will be starshaped via 2-paths, and its 2-kernel will have nonempty interior. However, \(\cap\{K : K \in \mathcal{K}\}\) is not even connected.

**Theorem 4.** Let \(k\) be a fixed integer, \(k \geq 1\), and let \(\mathcal{K}\) be a family of closed, simply connected sets in the plane. If every countable intersection \(J\) of members of \(\mathcal{K}\) is starshaped via (at most) \(k\)-paths, then \(S = \cap\{K : K \in \mathcal{K}\}\) is starshaped via \(k\)-paths, also.

**Proof.** By Corollary 1.2, \(S \neq \emptyset\). As in Corollary 2.1, let \(\mathcal{J}\) denote the family of all countable intersections of members of \(\mathcal{K}\). For each \(J_\alpha\) in \(\mathcal{J}\), define \(M_\alpha = \{x : x \in J_\alpha, x \text{ sees each point of } S \text{ via } k\text{-paths in } J_\alpha\}\), and let \(\mathcal{M}\) denote the collection of all the \(M_\alpha\) sets. Since members of \(\mathcal{K}\) are closed, members of \(\mathcal{J}\) are closed, and \(S\) is closed. Moreover, a standard convergence argument shows that members of \(\mathcal{M}\) are closed as well.

Further, it is easy to see that countable intersections of members of \(\mathcal{M}\) are nonempty. Let \(\{M_n : n \geq 1\}\) be a countable subfamily of \(\mathcal{M}\), with \(\{J_n : n \geq 1\}\) the associated subfamily of \(\mathcal{J}\). For convenience, let \(J_0 = \cap\{J_n : n \geq 1\}\), with \(M_0\) the corresponding set in \(\mathcal{M}\). By hypothesis, \(k - \text{Ker} J_0 \neq \emptyset\), and clearly \(k - \text{Ker} J_0 \subseteq M_0 \subseteq \cap\{M_n : n \geq 1\}\). Thus \(\cap\{M_n : n \geq 1\} \neq \emptyset\). Since \(\mathcal{M}\) is a family of closed sets whose countable intersections are nonempty, a standard argument involving the Lindelöf property shows that \(\cap\{M : M \in \mathcal{M}\} \neq \emptyset\). By Corollary 2.1, this intersection is exactly the \(k\)-kernel of \(S\), so \(S\) is starshaped via \(k\)-paths, finishing the proof.

We conclude with an observation concerning \(k - \text{Ker} S\) in Theorem 4. Results by Sparks [7, Theorems 3.1 and 3.2] for compact, simply connected sets (in \(\mathbb{R}^2\)) may be modified easily for closed, simply connected sets and hence for set \(S\) in Theorem 4. Using these theorems, for \(x, y\) in \(k - \text{Ker} S\), there is an associated \(k\)-path \(\lambda(x, y)\) in \(k - \text{Ker} S\), and thus \(k - \text{Ker} S\) is an \(L_k\) set (is convex via \(k\)-paths).

**References**


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