Abstract. We have given necessary and sufficient conditions that a finite direct products of rings have maximal subrings. We proved that if $R$ is any ring and $\alpha$ is a ring automorphism of $R$, then the skew polynomial ring $R[x; \alpha]$, always has a maximal subring. It is shown that for any ring $R$ and for any natural number $n > 1$, the ring $M_n(R)$ has maximal subring, and we determine some of them. Finally, we show that if $R$ is a non-quasi duo ring, then $R$ has a maximal subring. Consequently, every ring has either a maximal subring or its every maximal one-sided ideal is two-sided. In particular, if $R$ is a non-division ring which is one-sided primitive, then $R$ has a maximal subring.

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INTRODUCTION

All rings in this note are associative with $1 \neq 0$; all homomorphisms, and modules are unital. The characteristic of a ring $R$ is denoted by $c(R)$. A proper subring $S$ of a ring $R$ is said to be maximal if there is no subring of $R$ properly between $S$ and $R$. Unlike maximal ideals, maximal subrings need not always exist. If $S$ is a maximal subring of a ring $R$, then $R$ is called a minimal ring extension of $S$. Minimal rings extension have recently received some attention and been studied by several authors, see for example, [3], [4], [5], [6], [7], [10], [11], [12], [13] and [33] for commutative rings. But it seems, maximal subrings for commutative rings, were first studied systematically in [16], [15] and [32]. For the history of minimal rings extension, see also [33].

For non-commutative rings, J. Lewin in [23], proved that if a ring $R$ has a finite maximal subring, then $R$ has only finitely many ideals. In [27], [26], [8] (for commutative rings) and [9] (for $PI$-rings) it is proved that, if $R$ is a ring
with finite maximal subrings, then $R$ is a finite ring. Also, in [22], it is proved that, if $k$ is a field, then a $k$-algebra which has a finite-dimensional maximal subalgebra must be, itself, finite-dimensional. Recently, in [14], the theory of minimal rings extension are generalized to non-commutative rings.

In [4], some useful criteria for the existence of maximal subrings are given. It is observed in [5], that fields which are uncountable or of characteristic zero, have maximal subrings. Moreover, we have proved that the only fields $F$ which have no maximal subrings are those of the form $F = \bigcup_{n \in T} F_{p^n}$, where $p$ is a prime number, $F_{p^n}$ is the unique subfield, with $p^n$ elements, of $F_p$, the algebraic closure of $F_p = \frac{Z}{(p)}$, and $T$ is a subset of $\mathbb{N}$, the positive integers, such that $T = \{n \in \mathbb{N} : $ every prime divisor of $n$ is in a fixed set of prime numbers, say $P\} \cup \{1\}$, see ([5], Theorem 2.24). We have also characterized Artinian rings which have maximal subrings, see [6]. In particular, in [6], it is shown that if a commutative ring $R$ has a maximal subring $S$, then $S$ is Artinian if and only if $R$ is Artinian and integral over $S$. This generalizes the main theorem of [22]. Recently, in ([13], Theorem 2.4) it has been shown that every ring can be embedded in a larger ring as a maximal subring. Let us cite a simple proof of this interesting result which is presented in the introduction of [7]. Let $R$ be a ring and put $R_M^* = \begin{pmatrix} R & 0 \\ M & R \end{pmatrix} = \{ \begin{pmatrix} r \\ m \\ r \end{pmatrix} : r \in R, m \in M \}$, where $M$ is an $R$-module. It is clear that $R_M^*$ is a ring and we may consider $R$ as the subring \{\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} : r \in R\} of $R_M^*$. It is also trivial to see that every subring of $R_M^*$, containing $R$, is of the form $R_N^*$, where $N$ is an $R$-submodule of $M$. Consequently, $R$ is a maximal subring of $R_M^*$ if and only if $M$ is a simple $R$-module (cf. [21], Proposition 3.6, where $R_M^*$ is used for another purpose). We also note that if $N$ is a maximal submodule of $M$, then $R_N^*$ is a maximal subring of $R_M^*$ containing $R$.

In this paper we study the existence of maximal subrings in general non-commutative rings. It seems the existence of maximal subrings in non-commutative rings, in general, are easier to come by (we observe that every ring $R$ is either quasi duo ring, i.e., no maximal one-sided ideal exist, or it contains maximal subrings, see the final section).

Now, let us sketch a brief outline of this paper. In Section 1, we give some preliminary results which are proved in [4], [5], [6] and [7]. Section 2, is devoted to the study of the existence of maximal subrings in polynomial rings and finite direct product of rings. In Section 3, we show that for any ring $R$ and natural number $n > 1$, the ring $M_n(R)$ has maximal subrings and we present some of them. Finally, in Section 4, we show that every non-quasi duo ring have maximal subrings. In particular, if $R$ is a non-division ring which is one sided primitive (or simple), then $R$ has a maximal subring too.
Non quasi duo rings have maximal subrings

1. Review of Our Results in Commutative Rings and Preliminaries

In this section we cite some results from [4], [5], [6] and [7] about the existence of maximal subrings in commutative rings.

First, we recall that in [4], we show that maximal subrings of a field $F$ are just maximal $G$-domains which are contained in $F$, as well as, maximal valuation rings which are contained in $F$ (note, in any case, the quotient fields of these $G$-domains or valuation domains are not necessarily $F$).

In [5], we characterize fields without maximal subrings. In fact, we show that if $F$ is a field with at least one of the following conditions:

1. $F$ is uncountable.
2. $F$ has zero characteristic.
3. $F$ is not algebraic over its prime subfield.

Then $F$ has maximal subring.

Moreover, we have proved that the only fields $F$ which have no maximal subrings are those of the form $F = \bigcup_{n \in T} F_{p^n}$, where $p$ is a prime number, $F_{p^n}$ is the unique subfield, with $p^n$ elements, of $\bar{F}_p$, the algebraic closure of $F_p = \mathbb{Z}^{(p)}$, and $T$ is a subset of $\mathbb{N}$, the positive integers, such that $T = \{n \in \mathbb{N} : \text{every prime divisor of } n \text{ is in a fixed set of prime numbers, say } P\} \cup \{1\}$, see ([5], Theorem 2.24). Consequently, in [5], we see that the set of fields without maximal subrings (up to isomorphism) has exactly $2^{\aleph_0}$ elements; and therefore if $R$ is a commutative ring with $|\text{Max}(R)| > 2^{\aleph_0}$ or $|R/J(R)| > 2^{\aleph_0}$, then $R$ has a maximal subring. Also, we prove that if $D$ is a non-field domain, then the quotient field of $D$ has maximal subrings. In particular, if $P$ is non-maximal prime ideal of a ring $R$, then the ring $R_P$ has maximal subring.

Next, we cite the following results from [6], which are generalizations of the results in [5].

**Proposition 1.1.** Let $R$ be an Artinian ring which is either uncountable or of characteristic zero, then $R$ has a maximal ideal $M$ such that $R$ and $R/M$ have maximal subrings.

**Lemma 1.2.** Let $R_1, \ldots, R_n$, $n \geq 2$ be rings. Then $R = \prod_{i=1}^n R_i$ has a maximal subring if and only if one of the following conditions holds:

1. For some $i$, $R_i$ has a maximal subring.
2. There exist $i \neq j$ and maximal ideals $M_i$ of $R_i$ and $M_j$ of $R_j$ such that $R_i/M_i \cong R_j/M_j$, as ring isomorphism (thus there exist two distinct maximal ideals $M, N$ in $R$ with $R/M \cong R/N$, as ring isomorphism).
Theorem 1.3. Let \((R, M)\) be a local Artinian ring with \(c(R) = p^n\), where \(p\) is a prime number. If \(R\) is not integral over \(\mathbb{Z}_{p^n}\), then \(R\) and \(R/M\) have maximal subrings.

Corollary 1.4. Let \(R\) be an Artinian ring without maximal subrings. Then \(R\) is integral over a (every) finite subring.

Theorem 1.5. Let \(R\) be an Artinian ring. Then \(R\) has a maximal subring if and only if \(R\) satisfies one of the following conditions:

1. There exists a maximal ideal \(M\) such that \(R/M\) has a maximal subring.
2. There exist a maximal ideal \(M\) and an ideal \(I\) such that \(M^2 \subseteq I \subseteq M\) and \(R/I \cong K[\frac{1}{x}]\), where \(K = R/M\) (in fact, in this case \(R/I\) has a maximal subring isomorphic to \(K\)).
3. There exist distinct maximal ideals \(M\) and \(N\) in \(R\) with \(R/M \cong R/N\).

Finally, we review some results from [7].

Proposition 1.6. Let \(R\) be a ring with \(|R| > \text{Max}(2^{2^\aleph_0}, |U(R)|)\), then \(R\) has a maximal subring.

Theorem 1.7. Let \(R\) be a Noetherian ring with \(|R| > 2^{\aleph_0}\). Then \(R\) has a maximal subring containing \(N\), the nilradical of \(R\).

Theorem 1.8. Let \(R\) be a ring. Then \(R\) has a maximal subring if and only if there exists an ideal \(I\) of \(R\) which satisfies one of the following conditions.

1. \(I = N(R)\), the nilradical of \(R\) and \(R/I\) has a maximal subring.
2. There exists a field \(K\) such that \(R/I \cong K[\frac{1}{x}]\). In other words there exists a maximal ideal \(M\) of \(R\) such that \(M^2 \subseteq I \subseteq M\) and \(R/I\) has a maximal subring.

Proposition 1.9. Let \(S\) be a maximal subring of a ring \(R\) such that \(P = (S : R)\) is a maximal ideal in \(S\). Then either \(P\) is a prime ideal in \(R\), in which case \(P\) is maximal in \(R\), or \(R\) as an \(S\)-module can be generated by at most two elements.

Proposition 1.10. Let \(S\) be a maximal subring of a zero-dimensional ring \(R\) which contains no maximal ideal of \(R\). Then \(S\) is Noetherian if and only if both \(R\) and \(S\) are Artinian. Moreover, in this case, \(R\) is integral over \(S\).

Theorem 1.11. Let \(S\) be a maximal subring of a ring \(R\). Then \(S\) is Artinian if and only if \(R\) is Artinian and integral over \(S\).

Proposition 1.12. Let \(R\) be a zero-dimensional ring of characteristic zero. Then \(R\) has a maximal subring.

Corollary 1.13. Let \(R\) be a ring of characteristic zero. If \(R\) contains a zero-dimensional subring, then \(R\) contains maximal subring.
Corollary 1.14. Let \( R \) be a ring with \( c(R) \neq 0 \) and \( S \) be a zero-dimensional subring of \( R \). If \( S \) is not integral over its prime subring, then \( R \) has a maximal subring.

Theorem 1.15. Let \( \{R_i\}_{i \in I} \) be an infinite family of rings. Then \( R = \prod_{i \in I} R_i \) has maximal subrings.

Corollary 1.16. Let \( \{R_i\}_{i \in I} \) be a family of rings. Then \( R = \prod_{i \in I} R_i \) has maximal subrings if and only if at least one of the following conditions holds:

1. There exists \( i \in I \), such that \( R_i \) has maximal subrings.
2. There exist \( i \neq j \) in \( I \), and maximal ideals \( M_i \) of \( R_i \) and \( M_j \) of \( R_j \), such that \( \frac{R_i}{M_i} \cong \frac{R_j}{M_j} \) (Thus in this case \( R \) has distinct maximal ideals \( M \) and \( N \) such that \( \frac{R}{M} \cong \frac{R}{N} \)).
3. There exists a maximal ideal \( M \) of \( R \), such that \( \frac{R}{M} \) has maximal subrings.

Now, let us present some preliminaries that are needed in the next sections. Let \( R \) be any ring (not necessarily commutative) and let \( A \) be any left ideal of it, then the idealizer of \( A \) is denoted by \( I(A) \) and is the subset \( \{r \in R \mid Ar \subseteq A\} \). In fact the idealizer of \( A \) is the largest subring of \( R \), in which \( A \) is an ideal, see [31] and [34].

The ring \( R \) is called left quasi duo if every left maximal ideal of \( R \) is two sided. Right quasi duo rings are defined similarly. A ring \( R \) is called quasi duo if \( R \) is a left and right quasi duo. Recently in [19], [24], [25] and [30], quasi duo rings are fully investigated. In [30], some useful equivalent conditions that a ring \( R \) must be right quasi duo, are given. We summarize some of these results in the following proposition.

Proposition 1.17. (1) If \( R \) is a right quasi duo ring, then \( R \) is a Dedkind finite ring.
(2) If \( R \) is a right quasi duo ring, then \( R \) is a right primitive ring if and only if \( R \) is a division ring.
(3) If \( R \) is a right quasi duo ring, then \( R \) is simple ring if and only if \( R \) is a division ring.
(4) The ring \( R \) is right quasi duo ring if and only if \( \frac{R}{J(R)} \) is right quasi duo ring.
(5) If the ring \( R \) is a right quasi duo ring, then \( \frac{R}{J(R)} \) is subdirect product of division rings.
(6) If the ring \( R \) is right quasi duo, then \( \frac{R}{J(R)} \) is a reduced ring and thus \( N(R) \subseteq J(R) \), where \( N(R) \) is the set of all nilpotent element of the ring \( R \).
(7) If \( R \) is a right Artinian and right quasi duo ring, then \( \frac{R}{J(R)} \) is a finite direct product of division rings. In particular \( N(R) = J(R) \). Conversely,
if $R$ is a right Artinian ring with $\text{Nil}(R) = J(R)$, then $R$ is quasi duo ring.

(8) If $R$ is a semi local ring, then $R$ is right quasi duo ring if and only if $\frac{R}{J(R)}$ is a finite direct product of division rings.

(9) If $R$ is a Von Neumann regular ring, then $R$ is a right quasi duo ring if and only if $R$ is a strongly regular ring. In particular $R$ must be a quasi duo ring.

(10) If $R$ is right quasi duo ring and $x, y \in R$, $x + y \in Rxy$, then $Rx = Ry$.

2. Polynomial Rings and Finite Direct Product of Rings

At first we have the following essential theorem, whose proof is similar to Theorem 2.5 of [4]. In what follows, if $S$ is a subring of a ring $R$ and $A \subseteq R$ is a subset, then $S[A]$ is the subring of $R$ generated by $S$ and $A$.

**Theorem 2.1.** Let $R$ be any ring. Then $R$ has a maximal subring if and only if there exist a subring $S$ of $R$ and an element $x \in R \setminus S$ such that $S[x] = R$.

Now, we have the following immediate corollaries.

**Corollary 2.2.** The ring $R$ has maximal subrings if and only if there exist a proper subring $S$ of $R$ and a nonempty subset $A$ of $R$ such that $S[A] = R$ and $A$ is minimal with this property. In other words, for all proper subset $B$ of $A$, $S[B] \neq R$.

**Corollary 2.3.** Let $R$ be any ring. Then

1. For any index set $I$, with $|I| \geq 2$, the ring $\prod_{i \in I} R$ has maximal subring. In particular $R \times R$ has always maximal subring.

2. For any infinite index set $I$, the ring $\bigoplus_{i \in I} \frac{R}{i}$ has maximal subrings.

3. For any ring $R$ and any set of independent commutative variable $X$ over $R$, the ring $R[X]$ has maximal subrings.

4. For any ring $R$, and $n > 1$, the ring $M_n(R)$ has maximal subrings (similarly up and down triangular matrix rings have maximal subrings).

**Proof.** It suffices to prove item (1) for $|I| = 2$ and item (2). For (1), note that $S = \{(r, r) \mid r \in R\}$ is a proper subring of $R$ and it is clear that $S[(1, 0)] = R \times R$, thus by the above theorem $R \times R$ has a maximal subring; and therefore for the set $I$, with $|I| \geq 2$, the ring $\prod_{i \in I} R$ has maximal subrings. For (2), note that since $I$ is an infinite set thus we have a partition $I = I_1 \cup I_2$ for $I$, with $I_k$, $k = 1, 2$, have the same cardinality as $I$. Now, we have the natural ring isomorphisms

$$T = \prod_{i \in I} R \cong \prod_{i \in I_1} R \times \prod_{i \in I_2} R \cong T \times T$$

therefore $T$ has a maximal subring. □

**Corollary 2.4.** Let $R$ be any ring and $I$ and $J$ be proper ideals of $R$. Then

Proof. It is clear that for any ideal \( I + J = R \) and \( \frac{R}{I} \cong \frac{R}{J} \), then \( R \) has a maximal subring containing \( I \cap J \).

(2) If \( I + J \neq R \), then \( \frac{R \times R}{I \times J} \cong \frac{R}{I} \times \frac{R}{J} \) has maximal subrings.

Example 2.5. Let \( R \) be the ring \( \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k \) and \( p > 2 \), be a prime number. It is well known that \( pR \) is a maximal ideal of \( R \) and in fact \( \frac{R}{pR} \cong M_2(\mathbb{Z}_p) \). Thus, \( R \) has a maximal subring containing \( pR \), see ([18], Page 49).

Example 2.6. Let \( R \) be a non-reduced Artinian semisimple ring, then \( R \) has a maximal subring. More generally, if \( R \) is a right Artinian ring, with \( J(R) \subseteq N(R) \), where \( N(R) \) is the set of nilpotent element of \( R \), then \( R/J(R) \) has maximal subrings. For proof, note that by Artin-Wedderburn Theorem, \( R \) must has a component which has the form \( M_n(D) \), where \( n > 1 \).

In the following theorems we give some natural subrings of \( R \times R \) and skew polynomial rings corresponding to maximal ideals of the base rings.

Theorem 2.7. Let \( R \) be any ring. Then every subring of \( R \times R \), which contains \( S = \{(r, r) \mid r \in R\} \), has the form \( S + I \times I \), where \( I \) is an ideal of \( R \). In particular, for any ideal \( I \), this subring is a maximal subring of \( R \times R \) if and only if \( I \) is a maximal ideal of \( R \).

Proof. It is clear that for any ideal \( I \) of \( R \), the ring \( S + I \times I \) is subring of \( R \times R \), which contains \( S \). For the converse, assume that \( T \) is a subring of \( R \times R \) containing \( S \), and define \( I = \{a \mid (a, 0) \in T\} \) and \( J = \{b \mid (0, b) \in T\} \). Since \( S \subseteq T \), we infer that \( I \) and \( J \) are ideals of \( R \) and in fact \( I = J \). It is clear that \( S + I \times I \subseteq T \), note that \( T \) is a subring. Now, let \((x, y) \in T\), then we have \((0, y - x) = (x, y) - (x, x) \in T\), and therefore \( y - x \in I \), so we infer that \((x, y) = (x, x) + (0, y - x) \in S + I \times I \). Thus \( T = S + I \times I \) and we are done. The final part is now evident. \( \square \)

Corollary 2.8. The ring \( R \) is simple if and only if \( S = \{(r, r) \mid r \in R\} \) is a maximal subring of \( R \times R \).

In the following theorem we exactly determine when finite direct product of rings have maximal subrings.

Theorem 2.9. Let \( R_1 \) and \( R_2 \) be two rings. Then \( R = R_1 \times R_2 \) has a maximal subring if and only if one of the following conditions holds.

(1) \( R_1 \) or \( R_2 \) has a maximal subring.

(2) There exists ideals \( I_j \subseteq R_j \), \( j = 1, 2 \), such that \( \frac{R_1}{I_1} \cong \frac{R_2}{I_2} \), as ring isomorphism.

Proof. If (1) holds, then it is evident that \( R = R_1 \times R_2 \) has a maximal subring. Next, we first recall that for any ring \( R \) the ring \( R \times R \) has a maximal subring.
Hence if (2) holds, then \( \frac{R}{I_1 \times I_2} \cong \frac{R_1}{I_1} \times \frac{R_2}{I_2} \) and therefore \( \frac{R}{I_1 \times I_2} \cong R \times R \), a fortiori, \( R \) have maximal subrings (note, if \( T = \frac{R}{I_1} \), then \( \frac{R}{I_1 \times I_2} \cong T \times T \)). Conversely, let \( S \) be a maximal subring of \( R \) and put \( I = R_1 \times \{0\}, J = \{0\} \times R_2 \). If \( I \subseteq S \) \((J \subseteq S)\), then \( \frac{S}{T} (\frac{S}{T}) \) is a maximal subring of \( \frac{R}{T} \cong R_2 (\frac{R}{T} \cong R_1) \) and therefore (1) holds. Now let \( I \) and \( J \) be not in \( S \) and then by the maximality of \( S \) we have \( R = S + I = S + J \). Consequently, \( R_2 \cong \frac{R}{I} \cong \frac{S}{S \cap I} \) and \( R_1 \cong \frac{R}{J} \cong \frac{S}{S \cap J} \).

Let \( I' = I \cap S = I_1 \times \{0\}, I_1 \subseteq R_1 \) and \( J' = J \cap S = \{0\} \times J_1, J_1 \subseteq R_2; \) and note that \( I' + J' = I_1 \times J_1 \) is an ideal of \( S \). We now consider two cases:

Case(I): \((1,0)\) and thus \((0,1)\) do not belong to \( S \). In this case \( I' + J' = I_1 \times J_1 \) is a proper ideal of \( S \). Since \( \frac{S}{T} \cong R_2 (\frac{S}{T} \cong R_1) \) we infer that \( \frac{S}{I'} + \frac{S}{J'} \cong \frac{R_1}{A} \cong \frac{R_2}{B} \), where \( A, B \) are ideals in \( R_1, R_2 \), respectively and (2) holds.

Case(II): \((1,0)\) and thus \((0,1)\) belong to \( S \). We show that this case leads us to a contradiction and hence the proof is complete. Clearly, in this case we have \( S = I_1 \times J_1 \) and since \( I_1 \neq R_1, J_1 \neq R_2 \) we also have \( S \subseteq R_1 \times J_1 \), which is a contradiction (note, in this case \( I_1 \), \( J_1 \) are subrings in \( R_1, R_2 \), respectively). \( \square \)

The following is now evident.

**Corollary 2.10.** Let \( R_1 \) and \( R_2 \) be two rings. Then \( R = R_1 \times R_2 \) has a maximal subring if and only if one of the following conditions holds:

1. \( R_1 \) or \( R_2 \) has a maximal subring.
2. There exists maximal ideals ideal \( M_j \subset R_j, j = 1, 2, \) with \( \frac{R_1}{M_1} \cong \frac{R_2}{M_2} \) as ring isomorphism.

**Corollary 2.11.** Let \( R_1, \ldots, R_n, n \geq 2 \) be rings. Then \( R = R_1 \times R_2 \times \cdots \times R_n \) has a maximal subring if and only if one of the following conditions holds:

1. For some \( i, R_i \) has a maximal subring.
2. There exist \( i \neq j \) and maximal ideals \( M_i \) of \( R_i \) and \( M_j \) of \( R_j \) such that \( \frac{R_i}{M_i} \cong \frac{R_j}{M_j} \) as ring isomorphism. In other words, there exist distinct maximal ideals \( M, N \) in \( R \) with \( \frac{R}{M} \cong \frac{R}{N} \) as ring isomorphism.

**Proof.** If either (1) or (2) holds, then by Theorem 2.9, \( R \) has a maximal subring. Conversely, we proceed by induction on \( n \). For \( n = 2 \), we are done, by the previous corollary. Now suppose that the theorem is true for \( n - 1 \) and let \( R = R_1 \times R_2 \times \cdots \times R_n \). If \( R_1 \) or \( T = R_2 \times \cdots \times R_n \) has a maximal subring, then by the induction hypotheses we are done. Hence we may assume that \( R_1 \) and \( T = R_2 \times \cdots \times R_n \) have no maximal subrings. Now in view of Corollary 2.10, we infer that there exist maximal ideals \( M_1 \) in \( R_1 \) and \( M \) in \( T \) such that \( \frac{R_1}{M_1} \cong \frac{T}{M} \). But \( M = R_2 \times R_3 \times \cdots \times R_{i-1} \times M_i \times R_{i+1} \times \cdots \times R_n \), where \( M_i \) is a maximal ideal of \( R_i \) and \( i \geq 2 \), and \( \frac{T}{M} \cong \frac{R_1}{M_1} \) and we are through. \( \square \)

By the above result, we note that if \( S \) is a maximal subring of \( \prod_{i=1}^n R_i \), where \( R_i \) are rings and \( n \geq 2 \) is a natural number, if and only if \( S \) has
the form $T \times \prod_{k \neq i,j} R_k$, where $T$ is a maximal subring of $R_i \times R_j$, for some $1 \leq i \neq j \leq n$. Thus we have the following corollary.

**Corollary 2.12.** Let $R_i$, $1 \leq i \leq n$, $n \geq 2$, be simple rings and $S$ is a maximal subring of $\prod_{i=1}^n R_i$, then there exist $j$, $1 \leq j \leq n$, such that either $S \cong \prod_{i \neq j} R_i$, or $S \cong S_j \times \prod_{i \neq j} R_i$, where $S_j$ is a maximal subring of $R_j$.

In the next theorem we get some interesting properties of a rings with a simple maximal subring.

**Theorem 2.13.** Let $S$ be simple ring which is a maximal subring of a ring $R$, then every nonzero ideal of $R$ is a maximal ideal and $|\text{Max}(R)| \leq 2$ and one of the following holds:

1. $R$ is a simple ring and is a minimal ring extension of $S$.
2. $\text{Max}(R) = \{M,N\}$ and $R/M \cong R/N \cong S$ and $M \cap N = 0$ and $R \cong S \times S$.
3. $\text{Max}(R) = \{M\}$ and $R/M \cong S$ and $M^2 = M$ (in this case either $R$ is (two sided) primitive ring or $J(R) = M$).
4. $\text{Max}(R) = \{M\}$ and $R/M \cong S$ and $M^2 = 0$, in this case $J(R) = M$.

**Proof.** If $I$ is a nonzero ideal of $R$ then $S \cap I = 0$ and thus $S + I = R$, so we have $R/I \cong S$, hence $I$ is maximal ideal of $R$. Thus if $M$ and $N$ are two different maximal ideals of $R$ then we must have $M \cap N = 0$ and therefore we have

$$R \cong R/M \times R/N \cong S \times S$$

so $|\text{Max}(R)| = 2$, i.e., $|\text{Max}(R)| \leq 2$ in any case. If $\text{Max}(R) = \{M\}$, where $M \neq 0$, then since $M$ is the only nonzero ideal of $R$, thus either $M^2 = M$ or $M^2 = 0$ and in the latter case it is trivial to see that $J(R) = M$. \hfill $\Box$.

The above theorem can be generalized, to a prime maximal subring, in the following proposition.

**Proposition 2.14.** Let $S$ be prime maximal subring of a ring $R$, then either $R$ is a prime ring, or there exist a prime ideal $P$ of $R$ such that $P \cap S = 0$ and $S \cong R/P$.

**Proof.** Let $X = S \setminus \{0\}$, since $S$ is a prime ring, we infer that $X$ is a $m$-system in $S$ and thus in $R$ with $X \cap (0) = \emptyset$. Thus there exist a prime ideal $P$ of $R$ which is maximal with respect to the property that $P \cap X = \emptyset$. Now if $P = 0$, then $R$ is a prime ring, otherwise $P \cap S = 0$ and by the maximality of $S$, we infer that $S + P = R$ and thus $R/P \cong S$. \hfill $\Box$.

Finally, in this section we determine some forms of maximal subrings of skew polynomial rings. We recall that if $\alpha$ is a ring automorphism of a ring $R$, then we have the equation $xr = \alpha(r)x$.

**Theorem 2.15.** Let $R$ be any ring, and $\alpha$ be any ring automorphism of it. Then
For any maximal subring $S$ of $R$, the ring $S + R[x; \alpha]x$ is a maximal subring of $R[x; \alpha]$.

(2) For any maximal ideal $M$ of $R$, the ring $R + Mx + R[x; \alpha]x^2$ is a maximal subring of $R[x; \alpha]$.

Proof. For (1), it suffices to prove that if $T = S + R[x; \alpha]x$, then for any $f(x) \in R[x; \alpha] \setminus T$, we have $T[f(x)] = R[x; \alpha]$ (note that $T$ is a subring of $R[x; \alpha]$). Since $f(x) = a_0 + g(x)$, where $a_0 \in R \setminus S$ and $g(x) \in R[x; \alpha]x$, we have $a_0 \in T[f(x)]$. Now, since $S$ is a maximal subring of $R$, we infer that $R = S[a_0] \subseteq T[f(x)]$. Thus $R + R[x; \alpha]x \subseteq T[f(x)]$ and therefore $T[f(x)] = R[x; \alpha]$.

For (2), one can see that $T = R + Mx + R[x; \alpha]x^2$ is a subring of $R[x; \alpha]$. Now, we show that for any $f(x) \in R[x; \alpha] \setminus T$, we have $T[f(x)] = R[x; \alpha]$. Since $f(x) \in R[x; \alpha] \setminus T$, we infer that $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where $a_i \in R$ and $a_1 \in R \setminus M$. Since $M$ is a maximal ideal of $R$, there exists $m \in M$ and $n \in N$ and $r_i, s_i \in R$, such that $m + \sum_{i=1}^n r_ia_1s_i = 1$ (note $M + (a_1) = R$). Now, we have

$$x = 1x = mx + \sum_{i=1}^n r_i(a_1x)\alpha^{-1}(s_i) \in T[f(x)]$$

and therefore $T[f(x)] = R[x; \alpha]$. \hfill \Box

Corollary 2.16. Let $R$ be any ring and $\alpha$ be any ring automorphism of it. Then $R$ is a simple ring if and only if $R + R[x; \alpha]x^2$ is a maximal subring of $R[x; \alpha]$.

3. Matrix rings and Group Rings

We see that for any ring $R$ and any integer $n > 1$, the ring $\mathbb{M}_n(R)$ has maximal subrings. In what follows, we characterize some of them.

Theorem 3.1. Let $R$ be a ring and $n > 1$ be an integer number, then we have the following statements.

(1) If $M$ is a maximal ideal of $R$ then the ring

$$\begin{pmatrix} R & \cdots & R & R \\ \vdots & \ddots & \vdots & \vdots \\ R & \cdots & R & R \\ M & \cdots & M & R \end{pmatrix}$$

is a maximal subring of $\mathbb{M}_n(R)$ (it contains the maximal ideal $\mathbb{M}_n(M)$).

(2) If $S$ is maximal subring of $R$, then $\mathbb{M}_n(S)$ is maximal subring of $\mathbb{M}_n(R)$. Moreover, if $A$ is maximal subring of $\mathbb{M}_n(R)$ such that for all $i, j$, $E_{ij} \in A$, then there exists a maximal subring $S$ of $R$ such that $A = \mathbb{M}_n(S)$.
Proof. (1) Let \( T = M_n(R) \) and \( S \) be the subset of \( T \) mentioned in part (1) of the theorem. We first show that \( S \) is a subring of \( T \). Let \( A \) and \( B \) be two elements in \( S \). It is trivial that \( A - B \in S \). Now for any \( 1 \leq j \leq n - 1 \), we have

\[
(AB)_{nj} = \sum_{i=1}^{n} A_{ni}B_{ij} = \left( \sum_{i=1}^{n-1} A_{ni}B_{ij} \right) + A_{nn}B_{nj}
\]

Since \( A \in S \) we have \( A_{ni} \in M \) for any \( 1 \leq i \leq n - 1 \). But \( B \in S \) implies that \( B_{nj} \in M \), thus we have \( (AB)_{nj} \in M \) for all \( 1 \leq j \leq n - 1 \), therefore \( AB \in S \). Thus \( S \) is a subring of \( T \).

It remains to be shown that \( S \) is a maximal subring of \( T \). It is trivial that \( S \) is a proper subring of \( T \). Thus it suffices to prove the maximality of \( S \). For this, let \( A \in T - S \), we show that \( S[A] = T \). By the form of \( S \) we may assume that

\[
A = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
x_1 & \cdots & x_{n-1}
\end{pmatrix}
\]

such that at least one of the \( x_j \), \( 1 \leq j \leq n - 1 \) is not in \( M \). By the maximality of \( M \) we have \( M + Rx_jR = R \), so there exist \( m \in M \) and \( r_i, s_i \) in \( R \), \( (i = 1, \ldots, k) \), such that \( m + \sum_{i=1}^{k} r_i x_j s_i = 1 \). Now, take \( E_{rs} = [e_{ij}] \) where \( e_{ij} = 0 \) if \( (i,j) \neq (r,s) \) and \( e_{rs} = 1 \), then we have

\[
mE_{nj} + \sum_{i=1}^{k} (r_i)A(E_{jj}s_i) = E_{nj} \in S[A].
\]

So for all \( i, j \) where \( 1 \leq i \leq n - 1, 1 \leq j \leq n - 1 \), we have \( E_{ni} = E_{nj}E_{ji} \in S[A] \), and thus \( S[A] = T \).

(2) In this case we first recall that by ([29], Theorem 17.5), if \( A \) is a subring of \( M_n(R) \), containing all elementary matrix \( E_{ij} \), then \( A = M_n(S) \), for some subring \( S \) of \( R \). Thus we infer that \( M_n(S) \) is a maximal subring of \( M_n(R) \) if and only if \( S \) is a maximal subring of \( R \).

\[\Box\]

Remark 3.2. Let \( T \) be maximal subring of \( M_n(R) \), then for any invertable matrix \( P \) of \( M_n(R) \), the subset \( PTP^{-1} \) is a maximal subring of \( M_n(R) \).

Corollary 3.3. Let \( R \) be any ring and \( n > 1 \) be a natural number. If \( \phi : M_n(R) \rightarrow T \), is a ring homomorphism, then \( T \) has maximal subrings.

Proof. We recall that by ([29], Corollary 17.7) \( T \) has the form \( M_n(R') \) for some ring \( R' \). \[\Box\]
Theorem 3.4. Let $R$ and $S$ be two rings and $M$ be $(R, S)$-bimodule and $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, then we have:

1. If $R \times S$ has maximal subrings, then $T$ has maximal subrings too.
2. If $R M S$ has maximal subbimodules, then $T$ has maximal subrings.

Proof. (1) We know that $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is an ideal of $T$ and we have:

$$
T = \begin{pmatrix} 0 & 0 \\ M & 0 \\ 0 & 0 \end{pmatrix} \cong R \times S
$$

which shows that $T$ has maximal subrings.

(2) Let $RNS$ be a maximal subbimodule of $M$, we show that $W = \begin{pmatrix} R & N \\ 0 & S \end{pmatrix}$ is a maximal subring of $T$. Too see this, we show that for any $X \in T - W$ we have $W[X] = T$. We may assume that $X = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ where $x \notin N$. But, by the maximality of $N$, we have $N + RxS = M$. Hence for any $m \in M$ there exist $n \in N$ and $r_i \in R$ and $s_i \in S$ for $1 \leq i \leq k$ such that $m = n + \sum_{i=1}^{k} r_i x s_i$ therefore

$$
\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^{k} \begin{pmatrix} r_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & s_i \end{pmatrix}
$$

is in $W[X]$. Thus $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \subseteq W[X]$ and therefore $W[X] = T$, so $W$ is a maximal subring of $T$.

$\square$

Corollary 3.5. Let $R$ and $S$ be two rings and $RMS$ be $(R, S)$-bimodule which is Noetherian as $R$-module ($S$-module), then $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ has a maximal subring.

Corollary 3.6. Let $R$ be ring and $RM_R$ be $(R, R)$-bimodule, then $T = \begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ has a maximal subring.

In the next theorem we show that every ring, up to isomorphism, is a maximal subring of some ring. This is the main theorem of [13] for commutative rings.
Theorem 3.7. Let $R$ be a ring and $M$ be a maximal ideal of $R$, then $S = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \mid r \in R \right\}$ is a maximal subring of $T = \left\{ \begin{pmatrix} r & R/M \\ 0 & r \end{pmatrix} \mid r \in R \right\}$ moreover, if $R$ is commutative, then $T$ is commutative too.

Proof. It is easy to see that $T$ is a ring and $S = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \mid r \in R \right\}$ is a proper subring of $T$ and $R \cong S$ and if $R$ is commutative then so is $T$. Thus we show the maximality of $S$ in $T$. To see this, we show for any $A \in T \setminus S$ we have $S[A] = T$. Let $A$ be any such element, then we can suppose that $A$ has the form $\begin{pmatrix} 0 & a + M \\ 0 & 0 \end{pmatrix}$ where $a \notin M$, so by the maximality of $M$ we have $M + RaR = R$. Therefore there exist an element $m \in M$ and $r_i, s_i, 1 \leq i \leq n$, such that $m + \sum_{i=1}^{n} r_i a s_i = 1$. This implies that 

$$\sum_{i=1}^{n} \begin{pmatrix} r_i & 0 \\ 0 & r_i \end{pmatrix} \begin{pmatrix} 0 & a + M \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix} = \begin{pmatrix} 0 & 1 + M \\ 0 & 0 \end{pmatrix}$$

is in $S[A]$ and so $\begin{pmatrix} 0 & R/M \\ 0 & 0 \end{pmatrix} \subseteq S[A]$ which show that $S[A] = T$. \hfill \Box

Theorem 3.8. Let $R$ be a ring and $G$ be a group, then in the group ring $RG$ we have:

1. If $S$ is a maximal subring of $R$ then $SG$ is contained in a maximal subring of $RG$.
2. If $H$ is maximal subgroup of $G$, then $RH$ is contained in a maximal subring of $RG$.

Proof. For (1), let $S$ be a maximal subring of $R$, then for any $x \in R \setminus S$, we have $S[x] = R$. Thus $SG[x] = S[x]G = RG$. This show that $RG$ has a maximal subring containing $SG$. For (2), let $H$ be a maximal subgroup of $G$, and $g \in G \setminus H$, then we have $< H, g > = G$. Thus we infer that $RH[g, g^{-1}] = RG$ and therefore $RG$ has a maximal subring which contains $RH$. \hfill \Box

4. Non Quasi Duo rings

In the final section, we begin with the following interesting fact, which shows that either every non-commutative ring $R$ has a maximal subring or every maximal one-sided ideal is two-sided.

Theorem 4.1. Let $R$ be a ring and $A$ be a left maximal ideal of $R$. If $A$ is not a two sided ideal then the idealizer of $A$ is a maximal subring of $R$. 
Proof. Let \( S = I(A) \), we show that for any \( x \in R \setminus S \), we have \( S[x] = R \). Since \( x \notin S \), we have \( Ax \notin A \). Thus, since \( A \) is a left maximal ideal of \( R \) and \( A + Ax \) is a left ideal which properly contains \( A \), we infer that \( A + Ax = R \). Therefore we have \( R = A + Ax \subseteq S[x] \), so \( S[x] = R \) and thus \( S \) is a maximal subring of \( R \). \( \square \)

**Proposition 4.2.** Let \( S \) be a maximal subring of a ring \( R \), which contains a non-two sided maximal one sided ideal \( A \) of \( R \), then \( S = I(A) \).

**Proof.** Assume that \( A \) is a maximal left ideal of \( R \). We show that \( S \subseteq I(A) \), which proves the proposition. If not, then let \( x \in S \setminus I(A) \). Thus we have \( Ax \notin A \) and therefore \( R = A + Ax \subseteq S \), which is a contradiction. \( \square \)

**Corollary 4.3.** Let \( R \) be a non quasi duo ring, then \( R \) has a maximal subring.

Since if \( R \) is quasi duo left (right) primitive ring, then \( R \) is division ring, thus we have the following immediate Theorem.

**Theorem 4.4.** Let \( R \) be a ring which is not a division ring. If \( R \), at least, has one of the following conditions, then \( R \) has maximal subring.

1. \( R \) is a simple ring.
2. \( R \) is a left (right) primitive ring.

**Corollary 4.5.** Let \( R \) be a one-sided primitive ring which is not two sided primitive, then \( R \) has maximal subring.

**Corollary 4.6.** Let \( R \) be a ring without maximal subrings, then \( R \) is a quasi duo ring and the following sets of ideals of \( R \) coincide.

Maximal ideals, left primitive ideals, right primitive ideals, left maximal ideals, right maximal ideals.

In [1] it is shown that if \( D \) is a division ring with center \( F \), and \( M \) be maximal subgroup of \( D^* \), then either \( D = F(M) \) or \( M \cup \{0\} \) is a division ring. Using the latter property we conclude this article with the next result.

**Theorem 4.7.** Let \( D \) be a division ring which contains a maximal subgroup \( M \) such that \( M \cup \{0\} \) is a division ring, then \( D \) has a maximal subring.

**Proof.** Let \( S = M \cup \{0\} \). If \( S \) is a maximal subring of \( D \), then we are done. Thus assume \( S \) is not a maximal subring of \( D \). Let \( R \) be a subring of \( D \) such that \( S \subseteq R \subseteq D \). Note that since \( M \) is a maximal subgroup of \( D^* \), we infer that \( R \) is not a division ring. Thus \( M \subseteq U(R) \subseteq R \setminus \{0\} \). Now, let \( x \in R \setminus \{0\} \) and \( x \notin U(R) \). So by the maximality of \( M \), we have \( <M,x> = D^* \). Thus we have \( D^* \subseteq U(R[x^{-1}]) \) and therefore \( R[x^{-1}] = D \). So \( D \) has a maximal subring. \( \square \)

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