Chromatic Transversal Domatic Number
of Graphs

L. Benedict Michael Raj\textsuperscript{1}, S. K. Ayyaswamy\textsuperscript{2}
and I. Sahul Hamid\textsuperscript{3}

\textsuperscript{1} Department of Mathematics, St. Joseph’s College
Tiruchirappalli - 620 002, India
benedict.mraj@gmail.com

\textsuperscript{2} Department of Mathematics, SASTRA University
Thanjavur–612001, India
sjcayya@yahoo.co.in

\textsuperscript{3} Department of Mathematics, The Madura College
Madurai–625011, India
sahulmat@yahoo.co.in

Abstract

The chromatic number $\chi(G)$ of a graph $G$ is the minimum number of colours required to colour the vertices of $G$ in such a way that no two adjacent vertices of $G$ receive the same colour. A partition of $V$ into $\chi(G)$ independent sets (called colour classes) is said to be a $\chi$-partition of $G$. A set $S \subseteq V$ is called a dominating set of $G$ if every vertex in $V - S$ is adjacent to a vertex in $S$. A dominating set $S$ of $G$ is called a chromatic transversal dominating set (ctd-set) if $S$ has non-empty intersection with every colour class of every $\chi$-partition of $G$. The minimum order of a ctd-set of $G$ is the chromatic transversal domination number of $G$ and is denoted by $\gamma_{ct}(G)$. The chromatic transversal domatic number of a graph $G$ is the maximum order of a partition of $V$ into ctd-sets of $G$ and is denoted by $d_{ct}(G)$ . In this paper we obtain some bounds for $d_{ct}(G)$ and characterize graphs attaining the bounds. Also we characterize uniquely colourable graphs with $d_{ct}(G) = 1$. Finally we obtain Nordhaus–Gaddum inequalities for $d_{ct}(G)$ and characterize graphs for which $d_{ct}(G) + d_{ct}(\overline{G}) = p$ and $p - 1$. 
Mathematics Subject Classification: 05C35

Keywords: Domination number, Domatic number, Chromatic Transversal Domination number, Chromatic Transversal Domatic number

1 Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For graph theoretic terminology we refer to Harary [3].

Graph colouring theory and domination in graphs are two areas within graph theory which have been extensively studied. Graph colouring deals with the fundamental problem of partitioning a set of objects into classes according to certain rules. Time tabling, sequencing and scheduling problems in their many terms are basically of this nature. The fundamental parameter in the theory of graph colouring is the chromatic number $\chi(G)$ of a graph $G$ which is defined to be the minimum number of colours required to colour the vertices of $G$ in such a way that no two adjacent vertices of $G$ receive the same colour. A partition of $V$ into $\chi(G)$ independent sets is called a $\chi$-partition of $G$.

Another fastest growing area within graph theory is the study of domination and related subset problems such as independence, covering and matching. A set $S \subseteq V$ is said to be a dominating set of $G$ if every vertex in $V - S$ is adjacent to a vertex in $S$. The minimum cardinality of a dominating set is called the domination number of $G$ and is denoted by $\gamma(G)$. A comprehensive treatment of the fundamentals of domination is given in the book by Haynes et al. [5]. A survey of several advanced topics in domination can be seen in Haynes et al. [4]. Benedict et al. [1] introduced the concept of chromatic transversal domination using the concept of graph colouring and domination. A dominating set $S$ of a graph $G$ is called a chromatic transversal dominating set (ctd-set) if $S$ is a transversal of every $\chi$-partition of $G$. That is, $S$ has non-empty intersection with every colour class of every $\chi$-partition of $G$. The minimum cardinality of a ctd-set of $G$ is called the chromatic transversal domination number of $G$ and is denoted by $\gamma_{ct}(G)$. Obviously $\chi(G) \leq \gamma_{ct}(G)$.

Cockayne and Hedetniemi [2] introduced the concept of domatic number of a graph. A partition $\{V_1, V_2, \ldots, V_n\}$ of $V$ is a domatic partition of $G$ if each $V_i$ is a dominating set. The maximum order of a domatic partition of $G$ is called the domatic number of $G$ and is denoted by $d(G)$. For further details one can refer to Zelinka [6]. In this paper we extend the concept of domatic partition of a graph to a chromatic transversal domatic partition.
Chromatic transversal domatic number of graphs

We need the following definition and theorems.

**Definition 1.1.** A vertex \( v \in V(G) \) is called a \( \chi \)-critical vertex if \( \chi(G - v) < \chi(G) \). If every vertex in a graph is a \( \chi \)-critical vertex, then \( G \) is called a vertex \( \chi \)-critical graph.

**Theorem 1.2 ([2]).** For any graph \( G \)

(i) \( 1 \leq d(G) \leq \delta + 1 \).

(ii) For any tree \( T \) with \( p \geq 2 \), \( d(T) = 2 \).

**Theorem 1.3 ([2]).** Let \( G \) be a graph with \( p \) vertices. Then \( d(G) + \overline{d}(G) \leq p + 1 \). The equality is attained if and only if \( G \simeq K_p \) or \( \overline{K}_p \).

**Theorem 1.4 ([1]).** For any graph \( G \), \( \gamma_{ct}(G) = p \) if and only if \( G \) is either a vertex \( \chi \)-critical graph or \( K_p \).

**Theorem 1.5 ([3]).** In the \( n \)-colouring of a uniquely \( n \)-colourable graph, the subgraph induced by the union of any two colour classes is connected.

## 2 The chromatic transversal domatic number

**Definition 2.1.** A partition \( \{V_1, V_2, \ldots, V_n\} \) of \( V \) is a chromatic transversal domatic partition (ct-domatic partition) of \( G \) if each \( V_i \) is a ctd-set. The maximum order of a ct-domatic partition of a graph \( G \) is called the chromatic transversal domatic number (ct-domatic number) of \( G \) and is denoted by \( d_{ct}(G) \).

**Example 2.2.** (i) If \( G \) is \( K_p \) or \( \overline{K}_p \), then \( d_{ct}(G) = 1 \).

(ii) If \( (X,Y) \) is the bipartition of the complete bipartite graph \( K_{m,n} \), \( 1 \leq m \leq n \), where \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \), then \( \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m, y_{m+1}, \ldots, y_n\}\} \) is a ct-domatic partition of \( K_{m,n} \) of maximum order so that \( d_{ct}(K_{m,n}) = m \).

(iii) If a graph \( G \) has a \( \chi \)-critical vertex, then the vertex must be found in every ctd-set and hence \( d_{ct}(G) = 1 \).

**Remark 2.3.** Since every ct-domatic partition of a graph \( G \) is a domatic partition of \( G \), we have \( d_{ct}(G) \leq d(G) \). The difference between these two parameters can be made arbitrarily large as \( d_{ct}(K_p) = 1 \) and \( d(K_p) = p \).
Remark 2.4. Since every member of any ct-domatic partition of a graph $G$ on $p$ vertices has at least $\gamma_{ct}(G)$ vertices, it follows that $d_{ct}(G) \leq \frac{p}{\gamma_{ct}(G)}$. This inequality can be strict. For the complete bipartite graph $K_{m,m}$ we have $d_{ct}(K_{m,m}) = m$ and $\gamma_{ct}(K_{m,m}) = 2$.

Proposition 2.5. For the path $P_p$ and cycle $C_p$ on $p$ vertices,

(i) $d_{ct}(P_p) = \begin{cases} 1, & \text{when } p < 4, \\ 2, & \text{otherwise}. \end{cases}$

(ii) $d_{ct}(C_p) = \begin{cases} 2, & \text{when } p \text{ is even and } p \equiv 1 \text{ or } 2 \pmod{3}, \\ 3, & \text{when } p \text{ is even and } p \equiv 0 \pmod{3}. \end{cases}$

Proof. (i) Let $G$ be $P_p$, where $P_p$ is denoted by $v_1, v_2, \ldots, v_p$. Obviously $d_{ct}(G) = 1$, when $p < 4$. Let $p \geq 4$. If $p \equiv 1$ or $2 \pmod{3}$, then $p = 3k + r$, where $r = 1$ or $2$. In this case, $\gamma_{ct}(G) = \lceil \frac{p}{3} \rceil = k + 1$. By Remark 2.4, $d_{ct}(G) \leq 2$. Obviously the sets $S = \{v_2, v_3, \ldots, v_{p-2}, v_p\}$ and $V - S$ are ctd-sets of $P_p$. Hence $d_{ct}(G) = 2$. If $p \equiv 0 \pmod{3}$, then $p = 3k$. In this case $\gamma_{ct}(G) = \lceil \frac{p}{3} \rceil = k$. By Remark 2.4, $d_{ct}(G) \leq 3$. $d_{ct}(G) \neq 3$, since $S = \{v_2, v_5, \ldots, v_{p-1}\}$ in the only $\gamma_{ct}$-set with $k$ vertices. Hence $d_{ct}(G) \leq 2$. As $S$ and $V - S$ are ctd-sets, $d_{ct}(G) = 2$.

(ii) Let $G$ be $C_p$. When $p$ is odd, $C_p$ is a vertex $\chi$-critical graph. So $d_{ct}(G) = 1$. Assume that $p$ be even. If $p \equiv 1$ or $2 \pmod{3}$, then as in (i), we can prove that $d_{ct}(G) = 2$. If $p \equiv 0 \pmod{3}$, then $p = 3k$. In this case $\gamma_{ct}(G) = \lceil \frac{p}{3} \rceil$, so that $d_{ct}(G) \leq 3$ by Remark 2.4. In this case we can always obtain three disjoint $\gamma_{ct}$-sets in $C_p$. Hence $d_{ct}(G) = 3$. $\blacksquare$

3 Bounds on $d_{ct}$

In the following theorem, we establish a bound for $d_{ct}(G)$ in terms of the order of the graph and characterize the graphs attaining the bound.

Theorem 3.1. If $G$ is a graph with $p \geq 2$ vertices, then $d_{ct}(G) \leq \frac{p}{2}$. Further equality holds if and only if $G \simeq K_{\frac{p}{2},\frac{p}{2}} - E'$, where $E'$ is a matching, except the graph $K_2 \cup K_2$.

Proof. Let $p \geq 2$. If $G = \overline{K}_p$, $d_{ct}(G) = 1 \leq \frac{p}{2}$. For all other graphs $\gamma_{ct}(G) \geq 2$ and so by Remark 2.4, we have $d_{ct}(G) \leq \frac{p}{2}$. Again by Remark 2.4, $d_{ct}(G) \leq \frac{p}{\gamma_{ct}(G)} \leq \frac{p}{\chi(G)}$. So $\chi(G) \leq 2$. If $\chi(G) = 1$, then $d_{ct}(G) = 1$ and $p = 2$. In this case $G \simeq \overline{K}_2 = K_{1,1} - e$, where $e$ is the unique edge in it.
If $\chi(G) = 2$, then $\gamma_{ct}(G) = 2$ and so $G$ becomes a connected bipartite graph with bipartition $(X, Y)$, say. Since $d_{ct}(G) = \frac{p}{2}$, we have $|X| = |Y| = \frac{p}{2}$. Further, since $\gamma_{ct}(G) = 2$, $\deg(v)$ is either $\frac{p}{2}$ or $\frac{p}{2} - 1$ for every vertex $v$ in $G$. If $\deg(v) = \frac{p}{2}$ for all $v$ in $G$, then $G = K_{\frac{p}{2}, \frac{p}{2}}$; otherwise $G = K_{\frac{p}{2}, \frac{p}{2}} - E'$, where $E'$ is a maximum matching in $K_{\frac{p}{2}, \frac{p}{2}}$.

Since $G$ is a connected bipartite graph $K_2 \cup K_2 = K_{2,2} - E'$, where $E'$ is a maximum matching is excluded. The converse is obvious. \hfill \blacksquare

**Corollary 3.2.** If $d_{ct}(G) = \frac{p}{2}$, then $d_{ct}(\overline{G}) = 1$ or $2$.

**Proof.** If $d_{ct}(G) = \frac{p}{2}$, then by the above theorem, $G$ becomes a connected bipartite graph with bipartition $(X, Y)$, where $|X| = |Y|$. Since $|X| = \frac{p}{2}$, $\chi(\overline{G}) \geq \frac{p}{2}$ so that $\gamma_{ct}(\overline{G}) \geq \frac{p}{2}$ and hence it follows from Remark 2.4 that $d_{ct}(\overline{G}) = 1$ or $2$. \hfill \blacksquare

**Theorem 3.3.** For any graph $G$ on $p$ vertices we have $\gamma_{ct}(G) + d_{ct}(G) \leq p + 1$. Further equality holds if and only if $G$ is either vertex $\chi$-critical or $\overline{K}_p$.

**Proof.** If $d_{ct}(G) = 1$, then obviously $\gamma_{ct}(G) + d_{ct}(G) \leq p + 1$ and if $d_{ct}(G) \geq 2$, then by Remark 2.4, we have $d_{ct}(G) \leq \frac{p}{2}$ and so it follows from the bound given in Theorem 3.1 that $\gamma_{ct}(G) + d_{ct}(G) \leq p$. Hence, if $\gamma_{ct}(G) + d_{ct}(G) = p + 1$, then $d_{ct}(G) = 1$ so that $\gamma_{ct}(G) = p$. Thus it follows from Theorem 1.4 that $G$ is either vertex $\chi$-critical or $K_p$. Also the converse is immediate. \hfill \blacksquare

## 4 Graphs with $d_{ct}(G) = 1$

As we already mentioned in Example 2.2 (iii), if a graph $G$ has a critical vertex, then $d_{ct}(G) = 1$. Also obviously $d_{ct}(G) = 1$ for graphs having an isolated vertex as well. Thus, for graphs having either an isolated vertex or a critical vertex, we have $d_{ct}(G) = 1$, whereas the converse is not true. For example, for graphs $G$ given in Figure 1, $d_{ct}(G) = 1$, but it has neither a critical vertex nor an isolated vertex. However the converse is true for all uniquely colourable graphs which we now prove in the following theorem.

**Theorem 4.1.** Let $G$ be a uniquely colourable graph. Then $d_{ct}(G) = 1$ if and only if $G$ has a critical vertex.

**Proof.** Suppose $d_{ct}(G) = 1$. Let $\{V_1, V_2, \ldots, V_\chi\}$ be the $\chi$-partition of $G$. Now, we have to prove that $G$ has a critical vertex.
Figure 1:

Suppose not. Then $|V_i| \geq 2$, for every $i = 1, 2, \ldots, \chi$. Since $G$ is uniquely colourable each $V_i$ is a dominating set. Also, by Theorem 1.5, $\langle V_i \cup V_j \rangle$ is connected for all $i \neq j$ and so in particular $\langle V_1 \cup V_\chi \rangle$ is connected. Hence there exists a vertex $v_1 \in V_1$ such that $\deg v_1 \geq 2$ in $\langle V_1 \cup V_\chi \rangle$. Now, since $V_2$ is a dominating set and $|V_1| \geq 2$, there exists a vertex $v_2 \in V_2$ such that $v_2$ is adjacent to a vertex in $V_1$ other than $v_1$. Similarly, since $V_3$ is a dominating set and $|V_2| \geq 2$, there exists a vertex $v_3 \in V_3$ such that $v_3$ is adjacent to a vertex in $V_2$ other than $v_2$. Thus there exist vertices $v_2, v_3, \ldots, v_i, \ldots, v_\chi$ such that $v_i$ is adjacent to a vertex in $V_{i-1}$ other than $v_{i-1}$ for $i = 2, 3, \ldots, \chi$.

Now let $S = \{v_1, v_2, \ldots, v_\chi\}$. If $V - N[S] = \emptyset$, then $S$ and $V - S$ are ctd-sets so that $d_{ct}(G) \geq 2$, which is a contradiction. Hence $V - N[S] \neq \emptyset$. Let $D$ be a minimal dominating set of $\langle V - N[S] \rangle$. Then $D_1 = S \cup D$ is a ctd-set of $G$.

**Claim:** $V - D_1$ is a ctd-set of $G$.

We now prove that $\{v_i\} \cup D \neq V_i$, for $i = 1, 2, \ldots, \chi$, so that $V - D_1$ will be a transversal of the given $\chi$-partition. Suppose $\{v_i\} \cap D = V_i$ for some $i$, then there exists some $d \in D$ such that $d \in N[v_{i+1}] \subset N[S]$, a contradiction (in the case of $i = \chi$, $v_{i+1} = v_1$). Next we prove that $V - D_1$ is a dominating set of $G$. All the vertices of $S$ are dominated by $N(S) \subset V - D_1$. If $D$ has an isolated vertex, then it must be adjacent to a vertex in $N(S) \subset V - D_1$. All other vertices in $D$ are dominated by $V - D_1$. Hence $V - D_1$ is a dominating set, so that $V - D_1$ is a ctd-set of $G$.

Since, as in the claim, $V - D_1$ is a ctd-set, it follows that $d_{ct}(G) \geq 2$, which is a contradiction to $d_{ct}(G) = 1$. Thus $G$ has a critical vertex. The converse is obvious.

**Corollary 4.2.** Let $G$ be a connected bipartite graph. Then $d_{ct}(G) = 1$ if and only if $G$ is a star.

**Proof.** Follows from Theorem 4.1.
Corollary 4.3. For any tree $T$, $d_{ct}(T) = \begin{cases} 1, & \text{if } T \text{ is a star,} \\ 2, & \text{otherwise.} \end{cases}$

Proof. Since $d_{ct}(G) \leq d(G)$ for any graph $G$, it follows from Theorem 1.2 that $d_{ct}(T) \leq 2$. Hence the result follows from Corollary 4.2.

Corollary 4.4. For a connected unicyclic graph $G$ with $p$ vertices,

$$d_{ct}(G) = \begin{cases} 1, & \text{if } G \text{ has odd cycle,} \\ 3, & \text{if } G \simeq C_{3k}, \text{ where } k \text{ is even,} \\ 2, & \text{otherwise.} \end{cases}$$

Proof. If $G$ has an odd cycle, say $C$, then every vertex $v$ on $C$ is a $\chi$-critical vertex of $G$. Hence $d_{ct}(G) = 1$.

Let $G$ have even cycle. If $G$ has at least one pendant vertex then, $d_{ct}(G) \leq 2$. As no vertex of $G$ is a critical vertex, by Theorem 4.1, $d_{ct}(G) = 2$.

If $G$ has no pendant vertex, then $G \simeq C_p$, where $p$ is even.

So by (ii) of Proposition 2.5, we get the required result.

5 Nordhaus–Gaddum inequalities

For any graph parameter $f$ and any graph $G$, sharp upper and lower bounds for both $f(G) + f(\overline{G})$ and $f(G) \cdot f(\overline{G})$ are referred to as Nordhaus–Gaddum inequalities. In this section we study these inequalities for $d_{ct}(G)$.

Theorem 5.1. Let $G$ be any graph with $p$ vertices. Then

(i) $2 \leq d_{ct}(G) + d_{ct}(\overline{G}) \leq p$

(ii) $1 \leq d_{ct}(G) \cdot d_{ct}(\overline{G}) \leq \frac{p^2}{4}$.

Further the following are equivalent:

(a) $d_{ct}(G) + d_{ct}(\overline{G}) = p$. (b) $d_{ct}(G) \cdot d_{ct}(\overline{G}) = \frac{p^2}{4}$. (c) $G$ is one of the graphs $K_2, \overline{K_2}$ and $P_4$.

Proof. Since $d_{ct}(G) \leq \frac{p}{2}$ for any graph $G$, the inequalities in (i) and (ii) of the theorem follow. Now if either (a) or (b) is true then both $d_{ct}(G)$ and $d_{ct}(\overline{G})$ are equal to $\frac{p}{2}$. By Corollary 3.2, $p = 2$ or 4. Hence it follows from Theorem 3.1 that each of $G$ and $\overline{G}$ is of the form $K_{\frac{p}{2}, \frac{p}{2}} - E'$, where $E'$ is a matching, except the graph $K_2 \cup K_2$. Hence it is not difficult to see that $G$ is one of the graphs $K_2, \overline{K_2}$ and $P_4$.

Theorem 5.2. Let $G$ be a graph with $p > 1$ vertices. Then $d_{ct}(G) + d_{ct}(\overline{G}) =$
This implies that \( d_{ct}(G) \) is one of the graphs \( C_4, C_6, K_3, \overline{C}_4, \overline{C}_6, K_3 \) and \( P_3 \).

**Proof.** Assume that \( d_{ct}(G) + d_{ct}(\overline{G}) = p - 1 \). Suppose \( p \) is even. Then either \( d_{ct}(G) = \frac{p}{2} \) and \( d_{ct}(\overline{G}) = \frac{p}{2} - 1 \) or \( d_{ct}(G) = \frac{p}{2} - 1 \) and \( d_{ct}(\overline{G}) = \frac{p}{2} \).

Consider the first case. As \( d_{ct}(G) = \frac{p}{2} \), by Corollary 3.2, \( d_{ct}(\overline{G}) = 1 \) or 2, so that \( p = 4 \) or 6. So by Theorem 3.1, \( G \) becomes \( C_4 \) when \( p = 4 \) and \( C_6 \) when \( p = 6 \). The other choice gives \( \overline{C}_4 \) or \( \overline{C}_6 \). Suppose \( p \) is odd. In this case, \( d_{ct}(G) = d_{ct}(\overline{G}) = \frac{p-1}{2} \). We claim that \( p \leq 3 \). If not, then, we have \( 0 < \delta(G) < p - 1 \) so that \( \chi(G) \geq 2 \) and hence it follows from Remark 2.4 that \( \chi(G) = \gamma_{ct}(G) = 2 \). Hence \( G \) is a connected bipartite graph. Let \((X,Y)\) be its bipartition. Since \( \gamma_{ct}(G) = 2 \) and \( d_{ct}(G) = \frac{p-1}{2} \), we have \(|X|, |Y| \geq \frac{p-1}{2} \), whereas \(|X| \neq |Y|\). If \(|X| > |Y| \geq \frac{p-1}{2} \), then \( \frac{p}{2} < \frac{p+1}{2} \leq \chi(G) \leq \gamma_{ct}(G) \). This implies that \( d_{ct}(G) = 1 \neq \frac{p-1}{2} \), a contradiction. Similarly we can get a contradiction if \(|X| < |Y|\). Thus \( p = 3 \) and consequently \( G \) is either \( K_3, \overline{K}_3 \) or \( P_3 \).

The converse can be easily verified.

## 6 Conclusion and scope

Theory of domination and graph colouring theory are two important as well as fastest growing areas in combinatorics. A number of variations of domination have been introduced by several authors. In this sequence, Benedict et al. [1] introduced a new variation in domination, namely, chromatic transversal domination which involves both domination and colouring. Also partitioning the vertex set \( V \) of a graph \( G \) into subsets of \( V \) having some property is one direction of research in graph theory. For instance, one such partition is domatic partition which is a partition of \( V \) into dominating sets. Analogously, we here demand each set in the partition of \( V \) to have the property of chromatic transversal domination instead of just domination and call this partition a chromatic transversal domatic partition. Further, the maximum order of such partition is called the chromatic transversal domatic number which is denoted by \( d_{ct}(G) \). In this paper we have just initiated a study of this parameter. However, there is a wide scope for this parameter and we here list some of them:

(A) The following are some interesting open problems.

1. Characterize graphs \( G \) for which
   (i) \( d_{ct}(G) = d(G) \), (ii) \( d_{ct}(G) + \gamma_{ct}(G) = p \), (iii) \( d_{ct}(G) \cdot \gamma_{ct}(G) = p \).
We have characterized all connected bipartite graphs for which $d_{ct} = 1$. Hence it is natural to ask the following.

2. Characterize connected bipartite graphs for which $d_{ct}(G) = 2$.

3. Characterize graphs for which $d_{ct}(G) = 1$.

4. Given two positive integers $a$ and $b$, where $a \leq b$, is it possible to find a graph $G$ for which $d_{ct}(G) = a$ and $d(G) = b$?

(B) For any graph theoretic parameter the effect of removal of a vertex or an edge on the parameter is of practical importance. As far as our parameter $d_{ct}(G)$ is concerned, removal of either a vertex or an edge may increase or decrease the value of $d_{ct}(G)$ or may remain unchanged. For example, for the star $K_{1,p-1}$, we have $d_{ct}(K_{1,p-1}) = 1$, whereas $d_{ct}(K_{1,p-1} - v) = p - 1$, where $v$ is the centre vertex of the star. Also, if $(X,Y)$ is the bipartition of $K_{m,2m}$ ($m > 1$) with $|X| = m$ and $|Y| = 2m$, then $d_{ct}(K_{m,2m}) = m$ whereas $d_{ct}(K_{m,2m} - v) = 1 < d_{ct}(K_{m,2m})$ for all $v \in X$ and $d_{ct}(K_{m,2m} - v) = m = d_{ct}(K_{m,m})$ for all $v \in Y$. Hence it is possible to partition $V$ into the sets $V_0$, $V_+$ and $V_-$, where

$$V_0 = \{v \in V : d_{ct}(G - v) = d_{ct}(G)\},$$

$$V_+ = \{v \in V : d_{ct}(G - v) > d_{ct}(G)\},$$

$$V_- = \{v \in V : d_{ct}(G - v) < d_{ct}(G)\}.$$

Now, one may start investigating the properties of these sets.

(C) Similarly, one can observe that removal of an edge may increase or decrease the value of $d_{ct}(G)$ or may remain unchanged and consequently we can analogously define the sets $E_0$, $E_+$ and $E_-$. Now investigate these sets.

References


Received: September, 2009