Pairwise Closed Sets in Biclosure Spaces

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Abstract

The purpose of this paper is to introduce the concept of pairwise closed sets in biclosure spaces and study their fundamental properties. We introduce the notion of preserve pairwise closed maps by using pairwise closed sets and investigate some of their characterizations.

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1 Introduction

The concept of closure spaces was introduced by E. Čech [2] and then studied by many authors, see e.g. [3, 4, 6, 7]. J.C. Kelly [5] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. In this paper, we introduce and study the concept of pairwise closed sets in biclosure spaces. Using the notion of pairwise closed sets, we introduce preserve pairwise closed maps, which are studies.

2 Preliminaries

A map \( u : P(X) \rightarrow P(X) \) defined on the power set \( P(X) \) of a set \( X \) is called a closure operator on \( X \) and the pair \((X, u)\) is called a closure space if the following axioms are satisfied:

\[(N1) \; u\emptyset = \emptyset,\]
(N2) \( A \subseteq uA \) for every \( A \subseteq X \),

(N3) \( A \subseteq B \Rightarrow uA \subseteq uB \) for all \( A, B \subseteq X \).

A closure operator \( u \) on a set \( X \) is called additive (respectively, idempotent) if \( A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB \) (respectively, \( A \subseteq X \Rightarrow uuA = uA \)). A subset \( A \subseteq X \) is closed in the closure space \( (X, u) \) if \( uA = A \) and it is open if its complement in \( X \) is closed. The empty set and the whole space are both open and closed. A closure space \((Y, v)\) is said to be a subspace of \((X, u)\) if \( Y \subseteq X \) and \( vA = uA \cap Y \) for each subset \( A \subseteq Y \). If \( Y \) is closed in \((X, u)\), then the subspace \((Y, v)\) of \((X, u)\) is said to be closed too. Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \( f : (X, u) \to (Y, v) \) is said to be continuous if \( f(A) \subseteq vf(A) \) for every subset \( A \subseteq X \).

One can see that a map \( f : (X, u) \to (Y, v) \) is continuous if and only if \( uf^{-1}(B) \subseteq f^{-1}(vfB) \) for every subset \( B \subseteq Y \).

Clearly, if \( f : (X, u) \to (Y, v) \) is continuous, then \( f^{-1}(F) \) is a closed subset of \((X, u)\) for every closed subset \( F \) of \((Y, v)\).

Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \( f : (X, u) \to (Y, v) \) is said to be closed (resp. open) if \( f(F) \) is a closed (resp. open) subset of \((Y, v)\) whenever \( F \) is a closed (resp. open) subset of \((X, u)\).

The product of a family \( \{ (X_\alpha, u_\alpha) : \alpha \in I \} \) of closure spaces, denoted by \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \), is the closure space \(( \prod_{\alpha \in I} X_\alpha, u ) \) where \( \prod_{\alpha \in I} X_\alpha \) denotes the cartesian product of sets \( X_\alpha, \alpha \in I \), and \( u \) is the closure operator generated by the projections \( \pi_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha, \alpha \in I \), i.e., is defined by \( uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A) \) for each \( A \subseteq \prod_{\alpha \in I} X_\alpha \).

Clearly, if \( \{ (X_\alpha, u_\alpha) : \alpha \in I \} \) is a family of closure spaces, then the projection map \( \pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta) \) is closed and continuous for every \( \beta \in I \).

**Proposition 2.1.** Let \( \{ (X_\alpha, u_\alpha) : \alpha \in I \} \) be a family of closure spaces and let \( \beta \in I \). Then \( F \) is a closed subset of \((X_\beta, u_\beta)\) if and only if \( F \times \prod_{\alpha \neq \beta} X_\alpha \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

**Proof.** Let \( F \) be a closed subset of \((X_\beta, u_\beta)\). Since \( \pi_\beta \) is continuous, \( \pi_\beta^{-1}(F) \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). But \( \pi_\beta^{-1}(F) = F \times \prod_{\alpha \neq \beta} X_\alpha \), hence \( F \times \prod_{\alpha \in I} X_\alpha \) is a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \).

Conversely, let \( F \times \prod_{\alpha \neq \beta} X_\alpha \) be a closed subset of \( \prod_{\alpha \in I} (X_\alpha, u_\alpha) \). Since \( \pi_\beta \) is closed, \( \pi_\beta \left( F \times \prod_{\alpha \in I} X_\alpha \right) = F \) is a closed subset of \((X_\beta, u_\beta)\). \( \square \)
Proposition 2.2. Let \{(X_\alpha, u_\alpha) : \alpha \in I\} be a family of closure spaces and let \(\beta \in I\). Then \(G\) is an open subset of \((X_\beta, u_\beta)\) if and only if \(G \times \prod_{\alpha \neq \beta} X_\alpha\) is an open subset of \(\prod_{\alpha \in I} (X_\alpha, u_\alpha)\).

Definition 2.3. A biclosure space is a triple \((X, u_1, u_2)\) where \(X\) is a set and \(u_1, u_2\) are two closure operators on \(X\).

Definition 2.4. A subset \(A\) of a biclosure space \((X, u_1, u_2)\) is called closed if \(u_1 u_2 A = A\). The complement of closed set is called open.

Clearly, \(A\) is a closed subset of a biclosure space \((X, u_1, u_2)\) if and only if \(A\) is both a closed subset of \((X, u_1)\) and \((X, u_2)\).

Let \(A\) be a closed subset of a biclosure space \((X, u_1, u_2)\). The following conditions are equivalent

(i) \(u_2 u_1 A = A\),

(ii) \(u_1 A = A\), \(u_2 A = A\).

Definition 2.5. Let \((X, u_1, u_2)\) be a biclosure space. A biclosure space \((Y, v_1, v_2)\) is called a subspace of \((X, u_1, u_2)\) if \(Y \subseteq X\) and \(v_i A = u_i A \cap Y\) for each \(i \in \{1, 2\}\) and each subset \(A \subseteq Y\).

Proposition 2.6. Let \((X, u_1, u_2)\) be a biclosure space and let \((Y, v_1, v_2)\) be a closed subspace of \((X, u_1, u_2)\). If \(F\) is a closed subset of \((Y, v_1, v_2)\), then \(F\) is a closed subset of \((X, u_1, u_2)\).

Proof. Let \(F\) be a closed subset of \((Y, v_1, v_2)\). Then \(v_1 F = F\) and \(v_2 F = F\). Since \(Y\) is both a closed subset of \((X, u_1)\) and \((X, u_2)\), \(u_1 F = F\) and \(u_2 F = F\). Consequently, \(F\) is both a closed subset of \((X, u_1)\) and \((X, u_2)\). Therefore, \(F\) is a closed subset of \((X, u_1, u_2)\).

3 Pairwise Closed Sets

In this section, we introduce the concept of pairwise closed sets in biclosure spaces and study some of their properties. The pairwise closed sets are then used to introduce preserve pairwise closed maps and investigate some of their characterizations.

Definition 3.1. A subset \(A\) of a biclosure space \((X, u_1, u_2)\) is called pairwise closed if \(u_1 u_2 A = A = u_2 u_1 A\). The complement of pairwise closed set is called pairwise open.
Remark 1. Every closed set is pairwise closed. The converse is not true as can be seen from the following example.

Example 3.2. Let \( X = \{1, 2\} \) and define a closure operator \( u_1 \) on \( X \) by \( u_1 \emptyset = \emptyset, u_1 \{1\} = u_1 \{2\} = u_1 X = X \). Define a closure operator \( u_2 \) on \( X \) by \( u_2 \emptyset = \emptyset, u_2 \{1\} = u_2 \{2\} = u_2 X = X \). Then \( \{1\} \) is pairwise closed but it is not closed.

Proposition 3.3. Let \( (X, u_1, u_2) \) be a biclosure space. If \( A \) and \( B \) are pairwise closed subsets of \( (X, u_1, u_2) \), then \( A \cap B \) is pairwise closed.

Proof. Let \( A \) and \( B \) be pairwise closed. Then \( u_1 u_2 A = A = u_2 u_1 A \) and \( u_1 u_2 B = B = u_2 u_1 B \). Therefore, \( u_1 u_2 (A \cap B) = u_1 (u_2 A \cap u_2 B) = u_1 u_2 A \cap u_1 u_2 B = A \cap B \) and \( u_2 u_1 (A \cap B) = u_2 (u_1 A \cap u_1 B) = u_2 u_1 A \cap u_2 u_1 B = A \cap B \). Consequently, \( u_1 u_2 (A \cap B) = A \cap B = u_2 u_1 (A \cap B) \). Hence, \( A \cap B \) is pairwise closed.

The union of two pairwise closed sets need not be a pairwise closed set as can be seen from the following example.

Example 3.4. Let \( X = \{1, 2, 3, 4\} \) and define a closure operator \( u_1 \) on \( X \) by \( u_1 \emptyset = \emptyset, u_1 \{1\} = \{1\}, u_1 \{2\} = \{2\}, u_1 \{3\} = \{3\}, u_1 \{4\} = \{4\}, u_1 \{1, 2\} = \{1, 2, 4\}, u_1 \{1, 3\} = \{1, 3\}, u_1 \{1, 4\} = \{1, 4\}, u_1 \{2, 3\} = \{2, 3\}, u_1 \{2, 4\} = \{2, 4\}, u_1 \{3, 4\} = \{3, 4\}, u_1 \{1, 2, 4\} = \{1, 2, 4\} \) and \( u_1 \{1, 2, 3\} = u_1 \{1, 3, 4\} = u_1 \{2, 3, 4\} = u_1 X = X \). Define a closure operator \( u_2 \) on \( X \) by \( u_2 \emptyset = \emptyset, u_2 \{1\} = \{1\}, u_2 \{2\} = \{2\}, u_2 \{3\} = \{3\}, u_2 \{4\} = \{4\}, u_2 \{1, 2\} = \{1, 2, 4\}, u_2 \{1, 3\} = \{1, 3\}, u_2 \{1, 4\} = \{1, 4\}, u_2 \{2, 3\} = \{2, 3\}, u_2 \{2, 4\} = \{2, 4\}, u_2 \{3, 4\} = \{3, 4\} \) and \( u_2 \{1, 2, 3\} = u_2 \{1, 2, 4\} = u_2 \{1, 3, 4\} = u_2 \{2, 3, 4\} = u_2 X = X \). Then \( \{1\} \) and \( \{2\} \) are pairwise closed but \( \{1\} \cup \{2\} = \{1, 2\} \) is not pairwise closed.

Proposition 3.5. Let \( (X, u_1, u_2) \) be a biclosure space and let \( u_1, u_2 \) be additive. If \( A \) and \( B \) are pairwise closed subsets of \( (X, u_1, u_2) \), then \( A \cup B \) is pairwise closed.

Proof. Let \( A \) and \( B \) be pairwise closed. Then \( u_1 u_2 A = A = u_2 u_1 A \) and \( u_1 u_2 B = B = u_2 u_1 B \). Since \( u_2 \) and \( u_1 \) are additive, \( u_1 u_2 (A \cup B) = u_1 (u_2 A \cup u_2 B) = u_1 u_2 A \cup u_1 u_2 B = A \cup B \) and \( u_2 u_1 (A \cup B) = u_2 (u_1 A \cup u_1 B) = u_2 u_1 A \cup u_2 u_1 B = A \cup B \). Consequently, \( u_1 u_2 (A \cup B) = A \cup B = u_2 u_1 (A \cup B) \). Hence, \( A \cup B \) is pairwise closed.

Proposition 3.6. Let \( (X, u_1, u_2) \) be a biclosure space and let \( (Y, v_1, v_2) \) be a closed subspace of \( (X, u_1, u_2) \). If \( F \) is a pairwise closed subset of \( (Y, v_1, v_2) \), then \( F \) is a pairwise closed subset of \( (X, u_1, u_2) \).
Proof. Let $F$ be a pairwise closed subset of $(Y, v_1, v_2)$. Then $v_1v_2F = F$ and $v_2v_1F = F$. Since $Y$ is both a closed subset of $(X, u_1)$ and $(X, u_2)$, $u_1F = F$ and $u_2F = F$. Therefore, $F = v_1v_2F = v_1(u_2F \cap Y) = v_1(u_2F \cap Y)) = v_1(u_2F) = u_1(u_2F \cap Y) = u_1(u_2(F \cap Y)) = u_1u_2F$ and $F = v_2v_1F = v_2(u_1F \cap Y) = v_2(u_1(F \cap Y)) = v_2(u_1F) = u_2(u_1F) \cap Y = u_2u_1F$. Consequently, $u_1u_2F = F = u_2u_1F$. Hence, $F$ is a pairwise closed subset of $(X, u_1, u_2)$.

The following statement is obvious:

**Proposition 3.7.** Let $(X, u_1, u_2)$ be a biclosure space and let $A \subseteq X$. Then

(i) $A$ is pairwise open if and only if $A = X - u_1u_2(X - A) = X - u_2u_1(X - A)$.

(ii) If $G$ is pairwise open and $G \subseteq A$, then $G \subseteq X - u_1u_2(X - A) = X - u_2u_1(X - A)$.

**Proposition 3.8.** Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then $F$ is a pairwise closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\alpha \in I} X_\alpha$ is a pairwise closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

Proof. Let $\beta \in I$ and let $F$ be a pairwise closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $F$ is both a closed subset of $(X_\beta, u^1_\beta)$ and $(X_\beta, u^2_\beta)$. Since $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha) \rightarrow (X_\beta, u^1_\beta)$ is continuous, $\pi^{-1}_\beta(F) = F \times \prod_{\alpha \in I} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$. Since $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^2_\alpha) \rightarrow (X_\beta, u^2_\beta)$ is continuous, $\pi^{-1}_\beta(F) = F \times \prod_{\alpha \in I} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^2_\alpha)$. Consequently, $F \times \prod_{\alpha \in I} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $F \times \prod_{\alpha \in I} X_\alpha$ is a pairwise closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $F \times \prod_{\alpha \in I} X_\alpha$ be a pairwise closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $F \times \prod_{\alpha \in I} X_\alpha$ is both a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$ and $\prod_{\alpha \in I} (X_\alpha, u^2_\alpha)$. Since $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha) \rightarrow (X_\beta, u^1_\beta)$ is closed, $\pi_\beta(F \times \prod_{\alpha \in I} X_\alpha) = F$ is a closed subset of $(X_\beta, u^1_\beta)$. Since $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^2_\alpha) \rightarrow (X_\beta, u^2_\beta)$ is closed, $\pi_\beta(F \times \prod_{\alpha \in I} X_\alpha) = F$ is a closed subset of $(X_\beta, u^2_\beta)$. Consequently, $F$ is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $F$ is a pairwise closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □
Definition 3.9. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. A map \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) is said to be preserve pairwise closed (resp. preserve pairwise open) if \(f(F)\) is a pairwise closed (resp. pairwise open) subset of \((Y, v_1, v_2)\) whenever \(F\) is a pairwise closed (resp. pairwise open) subset of \((X, u_1, u_2)\).

The following statement is evident:

Proposition 3.10. Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces. If \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) and \(g : (Y, v_1, v_2) \to (Z, w_1, w_2)\) are preserve pairwise closed, then \(g \circ f : (X, u_1, u_2) \to (Z, w_1, w_2)\) is preserve pairwise closed.

Proposition 3.11. Let \(\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}\) be a family of biclosure spaces. Then for each \(\beta \in I\), the projection map \(\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)\) is preserve pairwise closed.

Proof. Let \(F\) be a pairwise closed subset of \(\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)\). Then \(F\) is both a closed subset of \(\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)\) and \(\prod_{\alpha \in I} (X_\alpha, u^2_\alpha)\). Since \(\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)\) is closed, \(\pi_\beta(F)\) is a closed subset of \((X_\beta, u^1_\beta)\). Since \(\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)\) is closed, \(\pi_\beta(F)\) is a closed subset of \((X_\beta, u^2_\beta)\). Consequently, \(\pi_\beta(F)\) is a pairwise closed subset of \(\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)\). Then \(\pi_\beta(F)\) is a pairwise closed subset of \(\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)\). Hence, \(\pi_\beta\) is preserve pairwise closed. \(\square\)

Proposition 3.12. Let \((X, u_1, u_2)\) be a biclosure space, \(\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}\) be a family of biclosure spaces and \(f : X \to \prod_{\alpha \in I} Y_\alpha\) be a map. Then \(f : (X, u_1, u_2) \to \prod_{\alpha \in I} (Y_\alpha, v^1_\alpha, v^2_\alpha)\) is preserve pairwise closed if and only if \(\pi_\alpha \circ f : (X, u_1, u_2) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)\) is preserve pairwise closed for each \(\alpha \in I\).

Proof. Let \(f\) be preserve pairwise closed. Since \(\pi_\alpha\) is preserve pairwise closed for each \(\alpha \in I\), also \(\pi_\alpha \circ f\) is preserve pairwise closed for each \(\alpha \in I\).

Conversely, let \(\pi_\alpha \circ f\) be preserve pairwise closed for each \(\alpha \in I\). Suppose that \(f\) is not preserve pairwise closed. Therefore, there exists a pairwise closed subset \(F\) of \((X, u_1, u_2)\) such that \(\prod_{\alpha \in I} v^1_\alpha \pi_\alpha \left( \prod_{\alpha \in I} v^2_\alpha \pi_\alpha (f(F)) \right) \not\subseteq f(F)\) or \(\prod_{\alpha \in I} v^2_\alpha \pi_\alpha \left( \prod_{\alpha \in I} v^1_\alpha \pi_\alpha (f(F)) \right) \not\subseteq f(F)\). If \(\prod_{\alpha \in I} v^1_\alpha \pi_\alpha \left( \prod_{\alpha \in I} v^2_\alpha \pi_\alpha (f(F)) \right) \not\subseteq f(F)\), then there exists \(\beta \in I\) such that \(v^1_\beta v^2_\beta \pi_\beta (f(F)) \not\subseteq \pi_\beta (f(F))\). But \(\pi_\beta \circ f\) is preserve pairwise closed, \(\pi_\beta (f(F))\) is a pairwise closed subset of \((Y_\beta, v^1_\beta, v^2_\beta)\).

This is a contradiction. If \(\prod_{\alpha \in I} v^2_\alpha \pi_\alpha \left( \prod_{\alpha \in I} v^1_\alpha \pi_\alpha (f(F)) \right) \not\subseteq f(F)\), then there
exists $\beta \in I$ such that $v_3^2v_1^3\pi_\beta(f(F)) \notin \pi_\beta(f(F))$. But $\pi_\beta \circ f$ is preserve pairwise closed, $\pi_\beta(f(F))$ is a pairwise closed subset of $(Y_\beta, v_1^\beta, v_2^\beta)$. This is a contradiction.

**Proposition 3.13.** Let $\{(X_\alpha, u_1^\alpha, u_2^\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v_1^\alpha, v_2^\alpha) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. Then $f : \prod_{\alpha \in I} (X_\alpha, u_1^\alpha, u_2^\alpha) \to \prod_{\alpha \in I} (Y_\alpha, v_1^\alpha, v_2^\alpha)$ is preserve pairwise closed if and only if $f_\alpha : (X_\alpha, u_1^\alpha, u_2^\alpha) \to (Y_\alpha, v_1^\alpha, v_2^\alpha)$ is preserve pairwise closed for each $\alpha \in I$.

**Proof.** Let $\beta \in I$ and let $F$ be a closed subset of $(X_\beta, u_1^\beta, u_2^\beta)$. Then $F \times \prod_{\alpha \in I \setminus \{\beta\}} X_\alpha$ is a pairwise closed subset of $\prod_{\alpha \in I} (X_\alpha, u_1^\alpha, u_2^\alpha)$. Since $f$ is preserve pairwise closed, $f(F \times \prod_{\alpha \in I \setminus \{\beta\}} X_\alpha)$ is a pairwise closed subset of $\prod_{\alpha \in I} (Y_\alpha, v_1^\alpha, v_2^\alpha)$. But $f(F \times \prod_{\alpha \in I \setminus \{\beta\}} X_\alpha) = f_\beta(F) \times \prod_{\alpha \in I \setminus \{\beta\}} Y_\alpha$, hence $f_\beta(F) \times \prod_{\alpha \in I \setminus \{\beta\}} Y_\alpha$ is a pairwise closed subset of $\prod_{\alpha \in I} (Y_\alpha, v_1^\alpha, v_2^\alpha)$. By Proposition 3.8, $f_\beta(F)$ is a pairwise closed subset of $(Y_\beta, v_1^\beta, v_2^\beta)$. Hence, $f_\beta$ is preserve pairwise closed.

Conversely, let $f_\beta$ be preserve pairwise closed for each $\beta \in I$. Suppose that $f$ is not preserve pairwise closed. Therefore, there exists a pairwise closed subset $F$ of $\prod_{\alpha \in I} (X_\alpha, u_1^\alpha, u_2^\alpha)$ such that $\prod_{\alpha \in I} v_1^\alpha\pi_\alpha(\prod_{\alpha \in I} v_2^\alpha\pi_\alpha(f(F))) \not\subseteq f(F)$ or $\prod_{\alpha \in I} v_2^\alpha\pi_\alpha(\prod_{\alpha \in I} v_1^\alpha\pi_\alpha(f(F))) \not\subseteq f(F)$. If $\prod_{\alpha \in I} v_1^\alpha\pi_\alpha(\prod_{\alpha \in I} v_2^\alpha\pi_\alpha(f(F))) \not\subseteq f(F)$. Then there exists $\beta \in I$ such that $v_1^\beta v_2^\beta\pi_\beta(f(F)) \not\subseteq \pi_\beta(f(F))$. But $\pi_\beta(F)$ is a pairwise closed subset of $(X_\beta, u_1^\beta, u_2^\beta)$ and $f_\beta$ is preserve pairwise closed, $f_\beta(\pi_\beta(F))$ is a pairwise closed subset of $(Y_\beta, v_1^\beta, v_2^\beta)$. This is a contradiction. If $\prod_{\alpha \in I} v_2^\alpha\pi_\alpha(\prod_{\alpha \in I} v_1^\alpha\pi_\alpha(f(F))) \not\subseteq f(F)$, then there exists $\beta \in I$ such that $v_1^\beta v_2^\beta\pi_\beta(f(F)) \not\subseteq \pi_\beta(f(F))$. But $\pi_\beta(F)$ is a pairwise closed subset of $(X_\beta, u_1^\beta, u_2^\beta)$ and $f_\beta$ is preserve pairwise closed, $f_\beta(\pi_\beta(F))$ is a pairwise closed subset of $(Y_\beta, v_1^\beta, v_2^\beta)$. This is a contradiction.

**References**


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