On the Gonality of Gorenstein Reducible Curves
(Extremal Cases)

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Abstract. Let $X$ be a reduced, connected and Gorenstein projective curve with $\omega_X$ ample. Here we study when $X$ has $\omega_X$-line bundles or rank 1 “nice” sheaves $F$ with $h^0(X, F) \geq 2$ and $\deg(F) \leq 1$, under the assumption $\text{length}(T \cap X \setminus T) \leq 3$ for every irreducible component $T$ of $X$.

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1. Introduction

Here we extend some of the ideas and of the proofs contained in [1] to the following more general set-up. Let $X$ be a reduced, projective, connected and Gorenstein curve such that $\omega_X$ is ample. Set $g := p_a(X)$. Since $\omega_X$ is ample, it is possible to speak about $\omega_X$-stability and $\omega_X$-semistability for depth 1 sheaves on $X$ with pure rank 1 ([5]). For any reduced curve $Y$ let $B(Y)$ denote the set of its irreducible components. Let $\text{Sing}(X''')$ be the set of all singular points of $X$ lying on at least two irreducible components of $X$. From now on “semistability” means $\omega_X$-semistability. Here we only consider depth 1 sheaves $F$ on $X$ with pure rank 1. The degree $\deg(F)$ of $X$ may be defined by the Riemann-Roch formula $\chi(F) = \deg(F) + \chi(\mathcal{O}_X)$. Let $S(X)$ denote the set of all coherent sheaves with depth 1 and pure rank 1. Set $S(X, d) := \{F \in S(X) : \deg(F) = d\}$. For any $F \in S(X)$ set $\text{Sing}(F) = \{P \in X : F$ is not locally free at $P\}$. Set $S(X'',) := \text{Sing}(F) \cap \text{Sing}(X)'$. Let $S(X')$ denote the set of all $F \in S(X)$ such that $F$ is locally free at each point of $\text{Sing}(X)'$, i.e. such that $\text{Sing}(F)' = \emptyset$. These are the element of $S(X)$ for which the restriction to proper subcurves is well-behaved. For any $d \in \mathbb{Z}$ let $B(X, d)$ denote the set of all semistable elements of $S(X, d)$. Set $A(X, d) := B(X, d) \cap \text{Pic}(X)$

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and $B(X, d)' := B(X, d) \cap S(X)'$. Set $S(X)[2] := \{F \in S(X) : h^0(X, F) \geq 2\}$, $B(X, d)[2] := B(X, d) \cap S(X)[2]$, $B(X, d)'[2] := B(X, d)' \cap S(X)[2]$ and $A(X, d)[2] := A(X, d) \cap S(X)[2]$. For any subcurve $A \subseteq X$ set $\eta_A := \deg(\omega_X|A)$ and, if $A \neq X$, $\ell_A := \text{length}(A \cap X \setminus \overline{A})$. Thus $\ell_A = \ell_{\overline{A} \setminus A}$ and $\eta_X = 2g - 2$.

**Definition 1.** Assume $X$ reducible. Let $V_X$ denote the set of all $T \in B(X)$ such that $T \cong \mathbb{P}^1$, $X \setminus T$ is not connected and, calling $C_1, \ldots, C_z$, $z \geq 2$, the closures in $X$ of the connected components of $X \setminus T$, we have $z = \ell_T$ or, equivalently, $\text{length}(C_i \cap T) = 1$ for all $i$. Since $T \cong \mathbb{P}^1$ and $\omega_X$ is ample, $z \geq 3$. An element of $V_X$ is called a *rational totally disconnecting component of $X$*. Fix $T \in V_X$. Let $C_1, \ldots, C_z$, $z = \ell_T$, be the connected components of $X \setminus T$, with the convention $p_a(C_1) \geq \cdots \geq p_a(C_z)$. Set $g_i := p_a(C_i)$. Hence $g = g_1 + \cdots + g_z$. We will say that $T$ is *allowable* (and write $T \in V_X'$) if $2g_1 \leq g$, i.e. if $g_1 \leq \sum_{i=2}^z g_i$.

**Theorem 1.** Assume $2g - 2 > 2\eta_T$ and $\ell_T \leq 3$ for all $T \in B(X)$. Then $B(X, 1)'[2] = A(X, 1)[2]$ and there is a bijection between $A(X, 1)[2]$ and $V_X'$. Every $L \in A(X, 1)[2]$ is spanned and $h^0(X, L) = 2$.

In the set-up of Theorem 1 every $D \in V_X$ has $z = 3$, because the assumption “$\ell_D \leq 3$” gives $z \leq 3$, while the ampleness of $\omega_X$ gives $z \geq 3$.

Let $Y$ be any connected projective curve. The point $P \in \text{Sing}(Y)$ is called a *disconnecting point of $Y$* if $Y \setminus \{P\}$ is not connected. If $P$ is an ordinary node of $Y$ and a disconnecting point of $Y$, then $Y \setminus \{P\}$ has exactly 2 connected components. In this case we often say that $P$ is a disconnecting node of $P$. Let $v : C \to Y$ be the partial normalization of $Y$ in which we only normalize the disconnecting node $P$. Since $P$ is a disconnecting node of $Y$, $C$ has two connected components. Hence $h^0(C, \mathcal{O}_C) = 2$. Set $F_{[P]} := v_*(\mathcal{O}_C)$. We have $h^0(X, F_{[P]}) = h^0(C, \mathcal{O}_C) = 2$. It is easy to check and well-known that the coherent sheaf $F_{[P]}$ has depth 1, pure rank 1, $\deg(F_{[P]}) = 1$, $\text{Sing}(F_{[P]}) = \{P\}$ and that $F_{[P]}$ is spanned (see e.g. [1]).

**Definition 2.** Let $P$ be a disconnecting node of $X$. Let $C_1$ and $C_2$ be the closures in $X$ of the 2 connected components of $X \setminus \{P\}$. The disconnecting node $P$ is said to be allowable if $g$ is even and $p_a(C_1) = p_a(C_2) = g/2$.

**Theorem 2.** Let $\Phi$ denote the set of all allowable disconnecting nodes of $X$. Let $\Psi$ denote the set of all isomorphism classes of depth 1 semistable sheaves $F$ on $X$ with pure rank 1, degree 1, $h^0(X, F) \geq 2$, $\text{Sing}(F)' \neq \emptyset$ and such that at least one point of $\text{Sing}(F)'$ is an ordinary node of $X$. The map $P \mapsto F_{[P]}$ induces a bijection $\beta : \Phi \to \Psi$. Every $F \in \Psi$ is spanned, with a unique non-locally free point and satisfies $h^0(X, F) = 2$.

2. **Preliminaries and statements for degrees $d \leq 0$**

We will often silently use the following observation. Fix $F \in S(X)'$. Then for every subcurve $C$ of $X$ the sheaf $F|C$ has no torsion. Hence $F|C \in S(C)'$. We have $\deg(F) = \sum_{T \in B(X)} \deg(F|T)$ and $\deg(F|C) = \sum_{T \in B(C)} \deg(F|T)$.
Remark 1. Fix $F \in \mathcal{S}(X, d)'$ and a proper subcurve $A$ of $X$. As in [2] we say that $(F, A)$ satisfies the Basic Inequality if

$$|\deg(F|A) - d\eta_A/\eta_X| \leq \ell_A/2$$

The proofs in [4], pp. 464–466, gives that $F \in B(X, d)'$ if and only if (1) holds for every proper subcurve $C$ of $X$.

Remark 2. Fix any $S \subset \text{Sing}(X)^n$ such that each $P \in S$ is an ordinary node of $X$. Let $u_S : C_S \to X$ be the partial normalization of $X$ in which we normalize only the points of $S$. There is a unique reduced connected curve $X_S$ and morphism $v_S : X_S \to X$ with the following properties: there is an inclusion $j : C_S \to X_S$, $u_S = v_S \circ j$, $X_S/j(C_S)$ is the disjoint union of curves $\{E_P\}_{P \in S}$ such that $E_P \cong \mathbb{P}^1$, $v_S(E_P) = \{P\}$, $\sharp(E_P \cap j(C_S)) = 2$, and each point of $j(C_S) \backslash E_P$ is an ordinary node of $X_S$ for every $P \in S$. The curve $X_S$ is often called the blowing-up of $X$ at $S$. Since $X$ is Gorenstein, $C_S$ and $X_S$ are Gorenstein. If $\omega_X$ is ample, then $\omega_{X_S}$ is semistable and the exceptional components $E_P$, $P \in S$, are the only irreducible components, $T$, of $X_S$ such that $\deg(\omega_{X_S}|T) = 0$. $C_S$ may be disconnected even under our standing assumption that $X$ is connected. However, the connectedness of $X$ implies the connectedness of $X_S$.

Remark 3. Fix $F \in \mathcal{S}(X, d)$. Assume that $X$ has an ordinary node at each point of $S := \text{Sing}(F)^n$. Take the set-up of Remark 2, but set $C := C_S$, $u := u_S$, $D := X_S$ and $v := v_S$. Hence there is an inclusion $j : C \to D$ and $D/j(C)$ is the disjoint union of $\text{Sing}(F)^n$ curves $E_P$, $P \in \text{Sing}(F)^n$; $D$ is nodal in a neighborhood of $D/j(C)$; $E_P \cap j(C) = j(u^{-1}(P))$ for every $P \in \text{Sing}(F)^n$. Notice that $\chi(O_C) = \chi(O_X) + \sharp(\text{Sing}(F)^n)$. Set $M := u^*(F)/\text{Tors}(u^*(F))$. Obviously, $M$ has depth 1 and pure rank 1. The classification of depth 1 modules over a nodal singularity ([5], 164–166) gives $\deg(M) = \deg(F) - \sharp(\text{Sing}(F)^n)$ and $u_*(M) = F$. Hence $h^i(X, F) = h^i(C, M)$, $i = 0, 1$. Using $\text{Sing}(F)^n$ Mayer-Vietor is exact sequences we get $\chi(O_D) = \chi(O_X)$. There is a unique $L \in \mathcal{S}(D, d)$ which is locally free in a neighborhood of each $E_P$, $j^*(L) \cong M$ and $\deg(L|E_P) = 1$. Using $\text{Sing}(F)^n$ Mayer-Vietor is exact sequences we get $h^0(D, L) = h^0(X, F)$ if $F$ is spanned at each point of $\text{Sing}(F)^n$. As in [4], pp. 464–466, we get that $F$ is semistable if and only if (1) holds for every proper subcurve $A$ of $D$ (here $\ell_A := \text{length}(D \cap D/A)$). This inequality is always satisfied by the subcurves $E_P$, because $\ell_{E_P} = 2$, $\deg(L|E_P) = 1$ and $\eta_{E_P} = 0$. Now assume that $F$ is spanned. Since the tensor product is a right exact functor, $M$ is spanned. Using $\text{Sing}(F)^n$ Mayer-Vietor is exact sequences we get that $L$ is spanned.

Proposition 1. Assume $\ell_T \leq 3$ for all $T \in \mathcal{B}(X)$. Fix an integer $d < 0$ and any $F \in B(X, d)'$. Then $h^0(X, F) = 0$.

Proof. Assume $h^0(X, F) > 0$. Since $\eta_T > 0$ and $\ell_T \leq 3$ for all $T \in \mathcal{B}(X)$, $F$ is semistable and $d \leq 0$, from (1) we get $\deg(F|T) \leq 1$ for all $T \in \mathcal{B}(X)$. Set
Since $A$, we have
\[ \deg(h^0(X, F)) = 0. \]
So $\sigma$ and $\eta$ are contained in $A$. Then we have
\[ \deg(h^0(X, F)) = 0. \]
Fix $\sigma$ in $H^0(X, F)$ and set $Z := \{ P \in X : \sigma(P) = 0 \}$. Since $d < 0$, $Z \neq \emptyset$. If $T \in S_-$, then $T \subseteq Z$. If $T \in S_0$, then either $T \subseteq Z$ or $T \cap Z = \emptyset$. If $T \in S_1$, then either $T \subseteq Z$ or $\ell(T) \leq 1$. Since $\emptyset \neq Z \neq X$, we get $S_1 \neq \emptyset$. Set $W := \{ T \in \mathcal{B}(X) : T \subseteq Z \}$ and $Y := \cup_{T \in W} T$. Since $Z \neq X$, $Y \neq \emptyset$. Since $S_- \subseteq W$ and $S_\neq \emptyset$, $Y \neq \emptyset$. Let $C$ be a connected component of $X \backslash Y$. Since $X$ is connected, $m := \ell_C > 0$. Set $A := \{ T \in \mathcal{B}(X) : T \subseteq Y \}$. We just saw that $A \subseteq S_1$ and that $\ell(T \cap A) = 1$ for all $T \in A$. Thus $\deg(F|C) \geq m$. Since $2p_a(C) - 2 + m = \eta_C \geq 0$, either $m \geq 2$ or $p_a(C) = 1$. Since $d \cdot \eta_C \leq 0$, the inequality (1) gives $|m| \leq m/2$, contradiction. \hfill \Box

**Proposition 2.** Assume $\ell_T \leq 3$ for all $T \in \mathcal{B}(X)$. Fix any $F \in \mathcal{B}(X, 0)'$. Then either $h^0(X, F) = 0$ or $F = \mathcal{O}_X$.

**Proof.** Fix $F \in \mathcal{B}(X, 0)'$ such that $h^0(X, F) > 0$ and assume $F \neq \mathcal{O}_X$. Since $\ell_T \leq 3$ for all $T \in \mathcal{B}(X)$, setting $d = 0$ in (1) we get $\deg(F|T) \in \{-1, 0, 1\}$ for all $T \in \mathcal{B}(X)$. Set $S_1 := \{ T \in \mathcal{B}(X) : \deg(F|T) = \iota \}$. Since $\deg(F) = 0$, $\iota(S_1) = \iota(S_{-1})$. Fix $\sigma$ in $H^0(X, F) \setminus \{0\}$ and set $Z := \{ P \in X : \sigma(P) = 0 \}$. Since $F \neq \mathcal{O}_X$, $Z \neq \emptyset$. If $T \in S_{-1}$, then $T \subseteq Z$. If $T \in S_0$, then either $T \subseteq Z$ or $T \cap Z = \emptyset$. If $T \in S_1$, then either $T \subseteq Z$ or $\ell(T) \leq 1$. Since $\emptyset \neq Z \neq X$, we get $S_1 \neq \emptyset$. Set $W := \{ T \in \mathcal{B}(X) : T \subseteq Z \}$ and $Y := \cup_{T \in W} T$. Since $Z \neq X$, $Y \neq \emptyset$. Since $S_{-1} \neq \emptyset$ and $S_{-1} \subseteq W$, $Y \neq \emptyset$. Since $X$ is connected, $m := \ell(Y \cap X \setminus Y) > 0$. Set $A := \{ T \in \mathcal{B}(X \setminus Y) : T \cap Y \neq \emptyset \}$. We just saw that $A \subseteq S_1$ and that $\ell(T \cap \cup_{i=1}^n T_i) \leq 3$. Since $\omega_Y$ is semiample, $\eta_A \geq 0$ for every subcurve $A$ of $Y$. Since $L$ satisfies the inequality (1), $\deg(L|T) \leq 3/2$. Since $d \leq 0$, we get $\deg(L|T) \leq 1$. Set $S_i := \{ T \in \mathcal{B}(X_S) : \deg(L|T) = \iota \}$, $Y_i := \cup_{T \in S_i} T$ and $Y_\cdot := \cup_{i=0}^\infty Y_i$. By the definition of $Y$ the curve $Y_1$ contains every irreducible component of $Y$ contracted by $\iota$. We have $Y_i \subseteq Z$ for all $i > 0$ and if $T \in S_0$, then either $T \subseteq Z$ or $T \cap Z = \emptyset$. If $D$ is an irreducible component of $Y_1$, then either $D \subseteq Z$ or $\ell(D \cap Z) \leq 1$. Since $\deg(L|E_\iota) = 1$ for every exceptional component $E_\iota$ of $u$, $s > 0$, and $\sum_{T \in \mathcal{B}(Y)} \deg(L|T) = d \leq 0$, we obtain $Y_\cdot \neq \emptyset$. Hence $B \neq \emptyset$. Since $\sigma \neq 0$, $B \neq Y$. Let $A$ be any connected component of
We saw that $\deg(L|A) \geq m$. Since $\omega_Y$ is semiample, $\eta_A \geq 0$. Since $d \leq 0$, the inequality (1) gives $\deg(L|A) \leq m/2$, contradiction. \hfill \Box

3. Proofs of Theorems 1 and 2.

Proof of Theorem 1. Fix $L \in B(X,1)^2$. Since $\eta_T/\eta_X < 1/2$ and $\ell_T/2 \leq 1/2$, (1) gives $\deg(L|T) \in \{-1,0,1\}$ for all $T \in B(X)$. Set $S_i := \{T \in B(X) : \deg(L|T) = i\}$ and $Y_i := \bigcup_{T \in S_i} T$. Let $B'$ be the base locus of $L$ and $B$ the union of the irreducible components of $X$ contained in $B'$. We have $Y_{i-1} \subseteq B$. If $T \in S_0$ then either $T \subseteq B$ or $T \cap B' = \emptyset$. If $T \in S_1$, then either $T \subseteq B$ or $\text{length}(T \cap B') \leq 1$.

(a) First assume $B \neq \emptyset$. Since $h^0(X,L) > 0$, $B \neq X$. Since $X$ is connected, $B \cap X \setminus B \neq \emptyset$. Let $A_i$, $1 \leq i \leq w$, be the connected components of $X \setminus B$. Since $A_i$ is a connected component of $X \setminus B$, $A_i \cap X \setminus A_i = A_i \cap B$. Since $X$ is connected, we get $m_i := \text{length}(A_i \cap B) > 0$. We just saw that $\deg(L|T) = 1$ and $\text{length}(T \cap B) = 1$ for every $T \subseteq A_i$ such that $T \cap B \neq \emptyset$. Hence $a_i := \deg(L|A_i) \geq m_i$. By applying (1) to $A_i$ we get $p_a(A_i) > 0$ and

\begin{equation}
(2) \quad a_i(2g - 2) - m_i(g - 1) \leq 2p_a(A_i) - 2 + m_i.
\end{equation}

Since $A_i \neq X$ and $\omega_X$ is ample, $p_a(A_i) \leq g - 2$. Hence

\begin{equation}
(3) \quad (2a_i - m_i)(g - 1) \leq 2g - 6 + m_i
\end{equation}

with $a_i \geq m_i > 0$. The inequality (3) is not satisfied for all $(a_i, m_i, g)$ such that $a_i \geq m_i \geq 1$, $g \geq 3$ and $a_i \geq 2$. Thus if $B \neq \emptyset$ we get a contradiction, unless $a_i = m_i = 1$ for all connected components $A_i$ of $X \setminus B$. Assume $a_i = m_i = 1$ for all $i$. Since $X$ is connected and each connected component of $X \setminus B$ intersects $B$ at a unique point, $B$ must be connected. Since $\deg(L|A_i) = 1$, we also see that the unique point $A_i \cap B$ is the only base point of $L$ contained in $A_i$, that $L|A_i \cong \mathcal{O}_{A_i}(A_i \cap B)$ and that the restriction map $p_i : h^0(X,L) \to h^0(A_i,L|A_i)$ has one-dimensional image. Since $h^0(X,L) \geq 2$, we get $w \geq 2$. From (2) with $a_i = m_i = 1$ we get $p_a(A_i) \geq g/2$ for all $i$. Since $B$ is connected, $p_a(B) \geq 0$. Since $w \geq 2$ and $g \geq p_a(B) + \sum_{i=1}^z p_a(A_i)$, we obtain $w = 2$, $g$ even, $p_a(A_1) = p_a(A_2) = g/2$ and $p_a(B) = 0$. Since $\ell_B = w = 2$ and $p_a(B) = 0$, $\omega_X$ is not ample contradiction. The contradiction gives $B = \emptyset$.

(b) Since $B = \emptyset$, $S_{-1} = \emptyset$. Hence there is $D \in B(X)$ such that $\deg(L|D) = 1$ and $\deg(L|T) = 0$ for all $T \in B(X) \setminus D$. Hence $B' \subseteq D \cap X_{\text{reg}}$. Since $B' \cap X \setminus D = \emptyset$ and $\deg(L|T) = 0$ for all $T \in X \setminus D$, $L|X \setminus D \cong \mathcal{O}_{X \setminus D}$ and there is $\sigma \in h^0(X,L)$ with no zero in a neighborhood of $X \setminus D$. Consider the Mayer-Vietoris exact sequence

\begin{equation}
0 \to L \to L|D \oplus L|X \setminus D \to L|D \cap X \setminus D \to 0
\end{equation}

Since $L|X \setminus D$ is trivial, $h^0(X \setminus D, L|X \setminus D)$ is the number, $z$, of the connected components of $X \setminus D$. Since $\ell_D \leq 3$, we have $z \leq 3$. Call $M_i$, $1 \leq i \leq z$, the
connected components of $X \setminus D$. Since $L|X \setminus D \cong O_{X \setminus D}$ and every connected component of $X \setminus D$ intersects $D$, the restriction map $H^0(X \setminus D, L|X \setminus D) \to H^0(D \cap X \setminus D, L|D \cap X \setminus D)$ is injective. Hence (4) gives that the restriction map $\rho : H^0(X, L) \to H^0(D, L|D)$ is injective. Since $\deg(L|D) = 1$, we get $D \cong \mathbb{P}^1$. Since $\ell_D \leq 3$ by assumption, while $\deg(\omega_X|D) > 0$, we get $\ell_D = 3$. Thus $z = 3$. Since $\deg(L|D) = 1$, $h^0(D, L|D) = 2$. Since $h^0(X, L) \geq 2$, we get that $\rho$ is an isomorphism. Hence $h^0(X, L) = 2$ and $B' \cap D = \emptyset$. Since $B' \subset D \cap X_{\text{reg}}$, we get $B' = \emptyset$, i.e. $L$ is spanned. Let $\phi_L : X \to \mathbb{P}^1$ be the morphism induced by $|L|$. Since $L|M_i \cong O_{M_i}$ and $M_i$ is connected, $\phi_L(M_i)$ is a point $Q_i \in \mathbb{P}^1$. Since $\rho$ is bijective and $\deg(L|D) = 1$, $\phi_L|D$ is an isomorphism. Since $\phi_L|M_i$ is constant, we get $\sharp(M_i \cap D) \leq 1$. Since each $M_i$ is a connected component of $X \setminus D$, we have $M_i \cap D \neq \emptyset$. Thus $\ell(M_i \cap D) = 1$. Hence $D \in V_X$ with $z = 3$. Conversely, take $D \in V_X$ and call $C_i$, $1 \leq i \leq z$, the connected components of $X \setminus D$. The assumption $\ell_D \leq 3$ and $\omega_X|D$ ample gives $z = 3$. Fix an isomorphism $u : D \to \mathbb{P}^1$. Let $f : X \to \mathbb{P}^1$ be unique morphism such that $f|D = u$ and $f(C_i) = u(C_i \cap D)$ for all $i$. Set $L := f^*(O_{\mathbb{P}^1}(1))$. Hence $L$ is a degree 1 spanned line bundle. Obviously, $\deg(L|C) = 1$ for any subcurve $C$ of $X$ containing $D$, while $\deg(L|C') = 0$ for any subcurve $C'$ of $X$ not containing $D$. It is clear that the second construction is the inverse of the first one. Hence to conclude the proof of Theorem 1 it is sufficient to check that $L$ is semistable if and only if $D \in V_X$. Let $C_1, C_2, C_3$ be the connected components of $X \setminus D$. Set $g_i := p_a(C_i)$ and assume $g_1 \geq g_2 \geq g_3$. To check the inequality (1) it is sufficient to check it for all proper connected subcurves of $X$. Let $U$ be a connected proper subcurve of $X$. Set $\tau := \ell_U$ and $q := p_a(U)$. First assume that $U$ does not contain $D$. Hence $\deg(L|U) = 0$. The inequality (1) is satisfied by $U$ if and only if $|2q - 2 + \tau| \leq \tau(g - 1)$. The latter inequality is always satisfied if $\tau \geq 2$, because $0 \leq q \leq g - 1$ and $g \geq 3$. Now assume $\tau = 1$. Since $U$ is connected and does not contain $D$, there is $i \in \{1, 2, 3\}$ such that $U \subseteq C_i$. Since $C_i$ is connected, $q \leq g_i$. Since $g_i \leq g_1$, the pair $(L, U)$ satisfies the Basic Inequality if $2g_1 \geq g$, i.e. if $L \in V_X$. By taking $U := C_1$ we see that if $2g_1 \geq g$, then $L$ is not semistable. Now assume $D \subseteq U$. Hence $\deg(L|U) = 1$. The inequality (1) is satisfied if and only if

$$|2g - 2q - \tau| \leq \tau(g - 1).$$

Since $0 \leq q \leq g - 2$ (here we use that $\omega_X$ is ample), the inequality (5) is satisfied if and only if either $\tau \geq 2$ or $\tau = 1$ and $2g \geq g$. Assume $\tau = 1$. Hence $D \neq U$. Since $U$ is connected, each $C_i$ is connected and $D \subseteq U$, we get that $U$ contains at least two of the curves $C_1, C_2, C_3$, say $C_i$ and $C_j$. Since $U$ is connected, we get $q \geq g_i + g_j$. Thus if $g_i \leq g_2 + g_3$ (as remarked we have $\ell_D = 3$ and hence $z = 3$), then $L$ is semistable. \hfill \Box

Proof of Theorem 2. Take $F \in \Psi$. Let $S$ be the set of all elements of $\text{Sing}(F)''$ which are ordinary nodes of $X$. The definition of $\Psi$ gives $S \neq \emptyset$. Set $s := \sharp(S)$. Let $u_S : X_S \to X$ be the curve obtained by blowing-up $S$.
Let $L$ be the degree 1 sheaf on $X_S$ associated to $F$ (Remark 3). The sheaf $L$ satisfies the inequalities (1) (Remark 1). We have $h^0(X_S, L) \geq h^0(X, F)$ (Remark 3) and hence $h^0(X_S, L) \geq 2$. We write $\ell_U = \text{length}(U \cap X_S\setminus U)$ and $\eta_U := \text{deg}(\omega_{X_S}|U)$ for any proper subcurve $U$ of $X_S$. Notice that $\ell_{E_p} = 2$ for all exceptional component of $X_S$, while for any other component $T$ we have $T \cong u_S(T)$, $\text{deg}(\omega_X|T) = \eta_{u_S(T)} > 0$ and $\ell_T = \ell_{u_S(T)}$. Since $2 \leq \ell_T \leq 3$ and $\eta_T/(2g - 2) < 1/2$ for every $T \in B(X_S)$, (1) gives $\text{deg}(L|T) \in \{-1, 0, 1\}$ for every $T \in B(X_S)$. Let $B'$ be the base locus of $L$ and $B$ the union of the irreducible components of $X_S$ contained in $B'$. Set $S_i := \{T \in B(X_S) : \text{deg}(L|T) = i\}$ and $Y_i := \cup_{T \in S_i} T$. Notice that $E_P \in S_1$ for every $P \in S$. Hence $S_1 \neq \emptyset$. If $T \in S_i$ for some $i < 0$, then $T \subseteq B$. If $T \in S_j$ for some $j \geq 0$, then either $T \subseteq B$ or $\text{length}(T \cap B') \leq j$.

(a) Here we assume $B \neq \emptyset$. Since $h^0(X_S, L) > 0$, $B \neq X$. Since $X_S$ is connected, $B \cap X_S\setminus B \neq \emptyset$. Let $A_i, 1 \leq i \leq z$, be the connected components of $X_S\setminus B$. Set $a_i := \text{deg}(L|A_i)$ and $m_i := \text{length}(A_i \cap B) > 0$. We repeat part (a) of the proof of Theorem 1. Since $\omega_{X_S}$ is not ample, we only use the inequality $p_a(A_i) \leq g - 1$, which is true because $\omega_{X_S}$ is semiample. We first get $a_i \leq m_i$ for all $i$, and then we get a contradiction unless either $a_i = m_i = 2$ and $p_a(A_i) = g - 1$ or $a_i = m_i = 1$ for all $i$.

(a1) Here we assume $a_i = m_i = 2$ and $p_a(A_i) = g - 1$. Since $E_P, P \in S$, are the only $T \in B(X_S)$ such that $\eta_T = 0$, every connected subcurve of it with arithmetic genus $g - 1$ is the complement of one of the exceptional components. Hence $z = 1$, $B$ is irreducible and $B$ is contracted by $u_S$. Hence $\text{deg}(L|B) = 1$. Hence $\text{deg}(L) = a_i + \text{deg}(L|B) = 3$, contradiction.

(a2) Here we assume $a_i = m_i = 1$ for all $i$. Since $X_S$ is connected and each connected component of $X_S\setminus B$ intersects $B$ at a unique point, $B$ must be connected. Since $\text{deg}(L|A_i) = 1$, we also see that the point $A_i \cap B$ is the only base point of $L$ contained in $A_i$, that $L|A_i \cong \mathcal{O}_{A_i}(A_i \cap B)$ and that the restriction map $\rho_i : H^0(X, L) \to H^0(A_i, L|A_i)$ has one-dimensional image. Since $h^0(X, L) \geq 2$, we get $z \geq 2$. From (2) with $a_i = m_i = 1$ we get $p_a(A_i) \geq g/2$ for all $i$. Since $B$ is connected, $p_a(B) \geq 0$. Since $z \geq 2$ and $g \geq p_a(B) + \sum_{i=1}^z p_a(A_i)$, we obtain $z = 2, g$ even, $p_a(A_1) = p_a(A_2) = g/2$ and $p_a(B) = 0$. Since $\omega_{X_S}$ is semiample, the exceptional components of $u_S$ are the only connected components $T$ of $X_S$ such that $\text{deg}(\omega_{X_S}|T) = 0$ and $z = 2$, $B$ must be one of these components, say $B = E_P$. Hence $\text{deg}(L|B) = 1$. Since $z = 2$ and $a_1 = a_2 = 1$, we get $\text{deg}(L) = 3$, contradiction.

(b) Here we assume $B = \emptyset$. Hence $S_{-1} = \emptyset$. Since $\text{deg}(L|E_P) = 1$ for all $P \in \text{Sing}(F)'$, we get $s = 1$, $Y_1 = E_P$ (where $P$ is the only point of $\text{Sing}(F)'$) and $\text{deg}(L|T) = 0$ for every $T \in B(X_S) \setminus \{E_P\}$. Thus $Y_0 = X_S\setminus E_P$ is isomorphic to the partial normalization of $X$ in which we normalize only the point $P$. Since $B = \emptyset, B' \cap Y_0 = \emptyset$. Hence a general $\sigma \in H^0(X_S, L)$ has no zero in a neighborhood of $Y_0$. Thus there is an open neighborhood $\Omega$ of $Y_0$ such that $L|\Omega \cong \mathcal{O}_\Omega$ and the trivialization is given by a global section of $L$. Since
each connected component of \( Y_0 \) intersects \( E_P \), we obtain the injectivity of the restriction map \( \rho : H^0(X_S, L) \to H^0(E_P, L|E_P) \). Since \( h^0(E_P, L|E_P) = 2 \), and \( h^0(X_S, L) \geq 2 \), \( \rho \) is bijective. Hence \( B' \cap E_P = \emptyset \). Hence \( B' = \emptyset \), i.e. \( L \) is spanned. Let \( v : C \to X \) be the partial normalization of \( X \) in which we only normalize \( P \). Set \( M := v^*(F)/\text{Tors}(v^*(F)) \). Remark 3 gives \( M \in \mathcal{S}(C, 0)' \). See \( C \) as the subcurve \( Y_0 \) of \( X_S \). With this identification \( M = L|C \) (Remark 3). Hence \( M \) is spanned and \( \deg(M|T) = 0 \) for every \( T \in \mathcal{B}(C) \). Hence \( M \cong \mathcal{O}_C \).

Since \( 1 \) for \( (1) \) the inequalities \( h^0(X_S, L) \geq 2 \) or \( h^0(X, F) \geq 2 \) gives that \( P \) is a disconnecting node of \( X \). \( F \) is obviously spanned outside \( P \). Since \( F \cong v_*(\mathcal{O}_C) \), the fiber \( F|\{P\} \) is a vector space of dimension 2. Since the two points of \( v^{-1}(P) \) are in different connected components of \( C \) and \( F \cong v_*(\mathcal{O}_C) \), \( F \) is spanned at \( P \). Thus \( F \) is spanned. Conversely, for any disconnecting node \( Q \) of \( X \) we get in this way a unique spanned sheaf \( F_{[Q]} \) with degree 1 and \( h^0(X, F_{[Q]}) = 2 \). It only remains to check that the sheaf \( F_{[P]} \) is semistable if and only if \( P \) is allowable. Let \( C_i, i = 1, 2 \), be the closure in \( X_S \) of the 2 connected components of \( X_S \setminus \{E_P\} \). Set \( g_i := p_a(C_i) \). These components are isomorphic to the closure in \( X \) of the 2 connected components of \( X \setminus \{P\} \). Notice that \( X_S \setminus C_i = C_{2-i} \cup E_P \). Since \( \deg(L|C_i) = 0 \), and length \( C_i \cap (C_{2-i} \cup E_P) \) = 1, \( C_i \) satisfies the inequality (1) for \( L \) if and only if \( 2p_a(C_i) - 1 \leq g - 1 \). Thus (1) holds for the pairs \( (L, C_1) \) and \( (L, C_2) \) if and only if \( 2 \max\{g_1, g_2\} \leq g \), i.e. \( (\text{since } g = g_1 + g_2) \) if and only if \( g \) is even and \( g_1 = g_2 = g/2 \). From now on we assume \( g \) even and \( g_1 = g_2 = g/2 \). Let \( A \) be a proper connected subcurve of \( X_S \). Since \( A \) is connected, \( q := p_a(A) \geq 0 \). Set \( x := \ell_A \). First assume \( E_P \not\subseteq A \). Hence \( A \subseteq C_i \) for some \( i \). Since \( A \) is connected, we get \( q \leq p_a(C_i) \leq g/2 \). Since \( \deg(L|A) = 0 \), the the inequality (1) for the pair \( (L, A) \) is equivalent to the inequality \( |(q - 1 + x)/2|/|g - 1| \leq x/2 \), which is always satisfied, because \( x \geq 1 \) and \( q \leq g/2 \). Now assume \( E_P \subseteq A \). Hence \( \deg(L|A) = 1 \). The pair \( (L, A) \) satisfies (1) if and only if

\[
|2g - 2q - x| \leq x(g - 1).
\]

Since \( 0 \leq q < g, g \geq 3 \) and \( x \geq 1 \), (6) is satisfied if and only if either \( x \geq 2 \) or \( x = 1 \) and \( 2q \geq g \). Assume \( x = 1 \). Hence \( A \neq E_P \). Since \( x = 1 \), the connectedness of \( X \) and \( A \) and the inclusion \( E_P \not\subseteq A \) also implies the existence of \( i \in \{1, 2\} \) such that \( C_i \subseteq A \). Thus \( q \geq p_a(C_i) \geq g/2 \). Hence \( L \) satisfies (1), i.e. \( F \in B(X, 1) \). Obviously different disconnecting nodes, say \( Q_1 \) and \( Q_2 \), give non-isomorphic sheaves, because \( \text{Sing}(F_{[Q_1]}) = \{Q_1\} \neq \{Q_2\} = \text{Sing}(F_{[Q_2]}) \), giving the injectivity of the map \( \beta : \Phi \to \Psi \). The last sentence of the statement of Theorem 2 follows from the surjectivity of \( \beta \) and that \( h^0(Y, F_{[P]}) = 2 \) and \( F_{[P]} \) is spanned.

\[\Box\]

References


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