Curvature and Rigidity of Complete Space-Like Submanifolds in a de Sitter Space

Shichang Shu

Department of Mathematics, Xianyang Normal University
Xianyang, 712000, Shaanxi, P. R. China
shushichang@126.com

Juanjuan Cao

Department of Mathematics, Northwest University
Xi’an, 710069, Shaanxi, P. R. China
caojuanjuan8866@163.com

Abstract

In this paper, we characterize the \( n \)-dimensional \( (n \geq 3) \) complete space-like submanifolds \( M^n \) in a de Sitter space \( S^{n+p}_p \) with the scalar curvature \( n(n-1)R \) and the mean curvature \( H \) being linearly related. If the mean curvature \( H \geq 0 \) and obtains its maximum on \( M^n \), we show that (1) if \( H^2 < \frac{4(n-1)}{m^2} \) on \( M^n \), then \( M^n \) is totally umbilical; (2) if \( H^2 = \frac{4(n-1)}{m^2} \) on \( M^n \), then \( M^n \) is totally umbilical, or \( M^n \) is isometric to a hyperbolic cylinder \( H^1(\sinh r) \times S^{n-1}(\cosh r) \); (3) if \( \frac{4(n-1)}{m^2} < H^2 \leq 1 \) on \( M^n \) and the squared norm of the second fundamental form \( \|h\|^2 \) satisfies \( \|h\|^2 \leq nH^2 + (B^-_H(n, p, H))^2 \) or \( \|h\|^2 \geq nH^2 + (B^+_H(n, p, H))^2 \) on \( M^n \), then \( M^n \) is totally umbilical, or \( M^n \) is isometric to a hyperbolic cylinder \( H^1(\sinh r) \times S^{n-1}(\cosh r) \), where \( B^\pm_H(n, p, H) \) are the two real roots of the polynomial (1.2) and \( m^2 = (n-2)^2p + 4(n-1) \).

Mathematics Subject Classification: 53C40, 53C42

Keywords: space-like submanifold, de Sitter space, mean curvature, totally umbilical

1 Introduction

Let \( M^{n+p}_p(c) \) be an \( (n+p) \)-dimensional connected semi-Riemannian manifold of constant curvature \( c \) whose index is \( p \). It is called an indefinite space form
of index $p$ and simply a space form when $p = 0$. If $c > 0$, we call it as a de Sitter space of index $p$, and denote it by $S^{n+p}_p(c)$. If $c = 1$, we denote $S^{n+p}_p(1)$ by $S^{n+p}_p$. It was pointed out by Marsdan and Tipler [9] and Stumbles [13] that space-like hypersurfaces with constant mean curvature in arbitrary space-time get interested in the relativity theory. Therefore, space-like hypersurfaces in a de Sitter space have recently been investigated by many differential geometers in both physics and mathematical points of view, one can see [2-5, 7, 10, 12, 15]. Goddard [7] conjectured that the only complete constant mean curvature space-like hypersurfaces in a de Sitter space were the umbilical ones. Akutagawa [2] and Ramanathan [10] proved independently that a complete space-like hypersurface in a de sitter space $S^{n+p}_p$ with constant mean curvature is totally umbilical if the mean curvature $H$ satisfies $H^2 \leq 1$ when $n = 2$ and $H^2 < 4(n - 1)/n^2$ when $n \geq 3$. Cheng [4] generalized this result to general submanifolds in a de Sitter space. The well-known examples with $H^2 = 4(n - 1)/n^2$ are umbilical sphere $S^n((n - 2)/n^2)$ and the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$, $c_1 = (2 - n)$ and $c_2 = (n - 2)/(n - 1)$.

It is well-known that the investigation on space-like hypersurfaces with the scalar curvature $n(n - 1)R$ and the mean curvature $H$ being linearly related is also important and interesting. Cheng [5] and Li [8] obtained some characteristic theorems of such hypersurfaces in terms of the sectional curvature, respectively. Recently, the author [12] proved a characteristic theorem of such hypersurfaces in terms of the mean curvature $H$. The well-known complete space-like hypersurfaces with constant mean curvature are given by

$$M^n = \{ p \in S^{n+1}_1 | p^2_{k+1} + \cdots + p^2_{n+1} = \cosh^2 r \},$$

with $r \in R^1$ and $1 \leq k \leq n$, where $R^1$ is the set of all real numbers. We can prove that $M^n$ is isometric to the Riemannian product $H^k(\sinh r) \times S^{n-k}(\cosh r)$ of a $k$-dimensional hyperbolic space and a $(n - k)$-dimensional sphere of radii $\sinh r$ and $\cosh r$, respectively. $M^n$ has $k$ principal curvatures equal to $\coth r$ and $(n - k)$ principal curvatures equal to $\tanh r$, so the mean curvature is given by $nH = k \coth r + (n - k) \tanh r$. If $k = 1$, the Riemannian product $H^1(\sinh r) \times S^{n-1}(\cosh r)$ is called a hyperbolic cylinder.

**Theorem 1.1([12]).** Let $M^n$ be an $n$-dimensional $(n \geq 3)$ complete space-like hypersurface with $n(n - 1)R = k'H$ in a de Sitter space $S^{n+1}_p$, where $k'$ is a positive constant. If the mean curvature $H \geq 0$ and obtains its maximum on $M^n$, then

1. if $H^2 < \frac{4(n-1)}{n^2}$ on $M^n$, then $M^n$ is totally umbilical.
2. if $H^2 = \frac{4(n-1)}{n^2}$ on $M^n$, then $M^n$ is totally umbilical, or $M^n$ is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.
3. if $\frac{4(n-1)}{n^2} < H^2 \leq 1$ on $M^n$ and the squared norm of the second fundamental form $\|h\|^2$ satisfies $\|h\|^2 \leq nH^2 + (B_H^-)^2$ or $\|h\|^2 \geq nH^2 + (B_H^+)^2$ on
there are few results about it. In this paper, we shall prove the following:

A local field of semi-Riemannian orthonormal frames

on $M$ parallel and the mean curvature $H$ positive constant. Suppose that the normalized mean curvature vector field is that at each point of $M$ to a hyperbolic cylinder $H$.

On the other hand, it is natural and very important to study $n$-dimensional submanifolds with the scalar curvature $n(n-1)R$ and the mean curvature $H$ being linearly related and higher codimension in a de Sitter space $S^{n+p}_p$. But there are few results about it. In this paper, we shall prove the following:

**Main Theorem.** Let $M^n$ be an $n$-dimensional ($n \geq 3$) complete space-like submanifold with $n(n-1)R = k'H$ in a de Sitter space $S^{n+p}_p$, where $k'$ is a positive constant. Suppose that the normalized mean curvature vector field is parallel and the mean curvature $H$ is non-negative and obtains its maximum on $M^n$, then

1. if $H^2 < \frac{4(n-1)}{m^2}$ on $M^n$, then $M^n$ is totally umbilical.
2. if $H^2 = \frac{4(n-1)}{m^2}$ on $M^n$, then $M^n$ is totally umbilical, or $M^n$ is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$.
3. if $\frac{4(n-1)}{m^2} < H^2 \leq 1$ on $M^n$ and the squared norm of the second fundamental form $\|h\|^2$ satisfies $\|h\|^2 \leq nH^2 + (B^-_{H}(n,p,H))^2$ or $\|h\|^2 \geq nH^2 + (B^+_{H}(n,p,H))^2$ on $M^n$, then $M^n$ is totally umbilical, or $M^n$ is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$, where $m^2 = (n-2)^2p + 4(n-1)$ and $B^\pm_{H}(n,p,H)$ are the two real roots of the polynomial

$$P_H(x) = x^2 - \frac{n(n-2)H}{\sqrt{n(n-1)}}x + n(1-H^2).$$

**Remark 1.1.** Since $m^2 = (n-2)^2p + 4(n-1)$, if $p = 1$, then $m = n$, we know that Main Theorem reduces to Theorem 1.1. Thus, we generalize Theorem 1.1 to general submanifolds.

## 2 Preliminary

Let $S^{n+p}_p$ be an $(n+p)$-dimensional de Sitter space with index $p$. Let $M^n$ be an $n$-dimensional connected space-like submanifold immersed in $S^{n+p}_p$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+p}$ in $S^{n+p}_p$ such that at each point of $M^n$, $e_1, \cdots, e_n$ span the tangent space of $M^n$ and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n + p; \quad 1 \leq i, j, k, \cdots \leq n \quad n + 1 \leq \alpha, \beta, \gamma, \cdots \leq n + p.$$
Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field. Then the structure equations of $S^{n+p}_p$ are given by (see [4])

\[
d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \land \omega_B, \omega_{AB} + \omega_{BA} = 0, \tag{2.1}
\]

\[
d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \land \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \land \omega_D, \tag{2.2}
\]

\[
K_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}). \tag{2.3}
\]

The Gauss equations and the Ricci equations are

\[
R_{ijkl} = (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) - \sum_\alpha (h^\alpha_{ij} h^\alpha_{jk} - h^\alpha_{ik} h^\alpha_{jl}), \tag{2.4}
\]

\[
n(n-1)R = n(n-1) + \|h\|^2 - n^2 H^2, \tag{2.5}
\]

\[
R_{\alpha\beta kl} = -\sum_m (h^\alpha_{km} h^\beta_{ml} - h^\alpha_{lm} h^\beta_{mk}). \tag{2.6}
\]

where $R_{ijkl}, \{R_{\alpha\beta kl}\}$ and $R$ are the components of the curvature tensor, the normal curvature tensor and the normalized scalar curvature of $M^n$.

Denote by $h$ the second fundamental form of $M^n$. Then $h = \sum_{i,j,\alpha} h^\alpha_{ij} \omega_i \land \omega_j \land e_\alpha$.

Denote by $\xi$, $H$ and $\|h\|^2$ the mean curvature vector field, the mean curvature and the norm square of the second fundamental form of $M^n$, then they are defined by $\xi = \frac{1}{n} \sum_\alpha (\sum_i h^\alpha_{ii}) e_\alpha$, $H = \|\xi\| = \frac{1}{n} \sqrt{\sum_\alpha (\sum_i h^\alpha_{ii})^2}$, $\|h\|^2 = \sum_{i,j,\alpha} (h^\alpha_{ij})^2$.

Then we have the Codazzi equations and the Ricci identities

\[
h^\alpha_{ij} = h^\alpha_{jk}, \tag{2.7}
\]

\[
h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{mj} R_{mjkl} - \sum_m h^\alpha_{jm} R_{mikl} - \sum_\beta h^\beta_{ij} R_{\alpha\beta kl}. \tag{2.8}
\]

From (2.7) and (2.8), we obtain for any $\alpha, n+1 \leq \alpha \leq n+p,$

\[
\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} - \sum_{k,m} h^\alpha_{km} R_{mij} - \sum_{k,m} h^\alpha_{mk} R_{mijk} - \sum_{k,\beta} h^\beta_{ik} R_{\alpha \beta jk}. \tag{2.9}
\]

In the case of the mean curvature vector $\xi \neq 0$, we know that $e_{n+1} = \xi/H$ is a normal vector field defined globally on $M^n$. We define $\|\mu\|^2$ and $\|\tau\|^2$ by $\|\mu\|^2 = \sum_{i,j} (h_{ij}^{n+1} - H \delta_{ij})^2$, $\|\tau\|^2 = \sum_{\alpha>n+1} \sum_{i,j} (h_{ij}^\alpha)^2$, respectively. We have $\|h\|^2 = nH^2 + \|\mu\|^2 + \|\tau\|^2$. Since the normalized mean curvature vector field is parallel, we choose $e_{n+1} = \xi/H$, then

\[
\text{tr}H^{n+1} = \sum_i h_{ii}^{n+1} = nH, \quad \text{tr}H^\alpha = \sum_i h^\alpha_{ii} = 0 \ (n+2 \leq \alpha \leq n+p). \tag{2.10}
\]
From (2.4), (2.6), (2.8) and (2.10), by a direct calculation we have (see [3])

\[
\frac{1}{2} \Delta \|h\|^2 = \sum_{i,j,k,\alpha} (h_{ij}^\alpha)^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} + n\|h\|^2 - n^2 H^2
\]

\[\begin{align*}
&- nH \sum_{\alpha} \text{tr}(H_{\alpha}^2 H_{n+1}) + \sum_{\alpha,\beta} [\text{tr}(H_{\alpha} H_{\beta})]^2 \\
&+ \sum_{\alpha,\beta} N(H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha}),
\end{align*}\]

where $H_{\alpha}$ denote the matrix $(h_{ij}^\alpha)$ for all $\alpha$, $N(A) = \text{tr}(AA^t)$, for all matrix $A = (a_{ij})$.

We need the following Lemma:

**Lemma 2.1 ([11]).** Let $A, B$ be symmetric $n \times n$ matrices satisfying $AB = BA$ and $\text{tr}A = \text{tr}B = 0$. Then

\[
|\text{tr}A^2 B| \leq \frac{n-2}{\sqrt{n(n-1)}} (\text{tr}A^2)(\text{tr}B^2)^{1/2}.
\]

and the equality holds if and only if $(n-1)$ of the eigenvalues $x_i$ of $B$ and the corresponding eigenvalues $y_i$ of $A$ satisfy $|x_i| = (\text{tr}B^2)^{1/2}/\sqrt{n(n-1)}$, $x_i x_j \geq 0$, $y_i = (\text{tr}A^2)^{1/2}/\sqrt{n(n-1)}$.

### 3 Proof of Main Theorem

For a $C^2$-function $f$ defined on $M^n$, we defined its gradient and Hessian $(f_{ij})$ by

\[
\sum_i f_i \omega_i, \quad \sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji}.
\]

Let $T = \sum_{i,j} T_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor on $M^n$ defined by $T_{ij} = nH \delta_{ij} - h_{ij}^{n+1}$. Following Cheng-Yau [6], we introduce an operator $\Box$ associated to $T$ acting on $f$ by

\[
\Box f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH \delta_{ij} - h_{ij}^{n+1}) f_{ij}.
\]

By a simple calculation and from (2.5), we obtain

\[
\Box (nH) = \sum_{i,j} (nH \delta_{ij} - h_{ij}^{n+1})(nH)_{ij}
\]

\[\begin{align*}
&= \frac{1}{2} \Delta (n^2 H^2) - n^2 \|\nabla H\|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\
&= - \frac{1}{2} n(n-1) \Delta R + \frac{1}{2} \Delta \|h\|^2 - n^2 \|\nabla H\|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}.
\end{align*}\]
By making use of the similar method in [5], we may prove the following:

**Proposition 3.1.** Let $M^n$ be an $n$-dimensional space-like submanifold in a de Sitter space $S^{n+p}_p(c)$ with $n(n-1)R = k' H (k' = \text{const.} > 0)$. If the mean curvature $H \geq 0$, then the operator $L = \Box + (k'/2n) \Delta$ is elliptic and $R > 0, H > 0$.

**Proof.** Since $H \geq 0$, we have the scalar curvature $n(n-1)R \geq 0$. From (2.5), we have

\[ \|h\|^2 = k'H + n^2H^2 - n(n-1). \]  

(3.2)

If there exists a point $p$ such that $R = 0$, we have $H = 0$ at this point. (3.2) implies that $\|h\|^2 + n(n-1) = 0$ at this point. This is impossible. Therefore, we know that $R > 0$ and $H > 0$ on $M^n$.

For a fixed $\alpha$, we choose an orthonormal frame field $\{e_j\}$ at each point on $M^n$ so that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$. From $nH = \sum_i h^\alpha_{ii}$ and $\sum_i h^\alpha_{ii} = 0$ for $n+2 \leq \alpha \leq n+p$ on $M^n$, we have, for any $i$

\[
(nH - \lambda_i^{n+1} + k'/2n) = \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
+ (1/2)[\sum_{j, \alpha} (\lambda_j^{n+1})^2 - n^2H^2 + n(n-1)]/(nH) \\
\geq \sum_j \lambda_j^{n+1} - \lambda_i^{n+1} \\
+ (1/2)[\sum_j (\lambda_j^{n+1})^2 - (\sum_j \lambda_j^{n+1})^2 + n(n-1)]/(nH) \\
= [(\sum_j \lambda_j^{n+1})^2 - \lambda_i^{n+1} (\sum_j \lambda_j^{n+1}) \\
- (1/2) \sum_{l \neq j} \lambda_l^{n+1} \lambda_j^{n+1} + (1/2)n(n-1)](nH)^{-1} \\
= [\sum_j (\lambda_j^{n+1})^2 + (1/2) \sum_{l \neq j} \lambda_l^{n+1} \lambda_j^{n+1} \\
- \lambda_i^{n+1} (\sum_j \lambda_j^{n+1}) + (1/2)n(n-1)](nH)^{-1} \\
= [\sum_{j \neq i} (\lambda_j^{n+1})^2 + (1/2) \sum_{l \neq j, l \neq i} \lambda_l^{n+1} \lambda_j^{n+1} + (1/2)n(n-1)](nH)^{-1} \\
= (1/2)[\sum_{j \neq i} (\lambda_j^{n+1})^2 + (\sum_j \lambda_j^{n+1})^2 + n(n-1)](nH)^{-1} > 0.
\]
Thus, $L$ is an elliptic operator. This completes the proof of Proposition 3.1.

**Proposition 3.2.** Let $M^n$ be an $n$-dimensional space-like submanifold in a de Sitter space $S^{n+p}(c)$ with $n(n-1)R = k'H, (k' = \text{const.} > 0)$. Then $\|\nabla h\|^2 \geq n^2\|\nabla H\|^2$.

**Proof.** We choose a orthonormal frame field as in the proof of Proposition 3.1, then we have $\|h\|^2 = \sum_{i,j,\alpha}(h_{ij}^\alpha)^2 \neq 0$. In fact, if $\|h\|^2 = \sum_{i,\alpha}(\lambda_i^\alpha)^2 = 0$ at a point of $M^n$, then $\lambda_i^\alpha = 0$ for all $i$ and $\alpha$ at this point. This implies $H = 0$ and $R = 0$ at this point, from (2.5), we have $n(n-1) = 0$, this is impossible.

From (2.5) and $n(n-1)R = k'H$, we have $k'\nabla_i H = -2n^2H\nabla_i H + 2\sum_{j,k,\alpha}h_{kj}^\alpha h_{kji}^\alpha$. Thus

$$
\left(\frac{k'}{2} + n^2H\right)^2\|\nabla H\|^2 = \sum_i \left(\sum_{j,k,\alpha}h_{kj}^\alpha h_{kji}^\alpha\right)^2 \leq \sum_i \left(\sum_{j,\alpha}h_{ij}^\alpha\right)^2 \sum_i \left(\sum_{j,k}h_{ij}^\alpha\right)^2 = \|h\|^2\|\nabla h\|^2,
$$

$$
\|\nabla h\|^2 - n^2\|\nabla H\|^2 \geq \left[\left(\frac{k'}{2} + n^2H\right)^2 - n^2\|h\|^2\right]\|\nabla H\|^2 \frac{1}{\|h\|^2}.
$$

$$
= \left[\left(\frac{k'}{2}\right)^2 + n^3(n-1)\|\nabla H\|^2\right]\frac{1}{\|h\|^2} \geq 0.
$$

This completes the proof of Proposition 3.2.

**Proof of Main Theorem.** Since we assume that the normalized mean curvature vector field is parallel, we have $H \neq 0$. Set $\phi_{ij}^\alpha = h_{ij}^\alpha - \frac{1}{n}\text{tr}H^\alpha \delta_{ij}$ and consider the symmetric tensor $\phi = \sum_{i,j,\alpha} \phi_{ij}^\alpha \omega_i \omega_j e_\alpha$. From (2.11) and (3.5) in [3], we have

$$
\frac{1}{2}\Delta \|h\|^2 \geq \sum_{i,j,k,\alpha} (h_{ij}^\alpha)^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} + n(1 - H^2)\|\phi\|^2 \tag{3.4}
$$

$$
- nH \sum_\alpha \text{tr}(\Phi_\alpha^2 \Phi_{n+1}) + \sum_{\alpha,\beta} [\text{tr}(\Phi_\alpha \Phi_\beta)]^2
$$

where $\Phi_\alpha$ denotes the matrix $(\phi_{ij}^\alpha)$.

Since we choose $e_{n+1} = \xi/H$, then $\omega_{an+1} = 0$ for all $\alpha$. Consequently $R_{an+1jk} = 0$, from (2.6), we have $\sum_i h_{ij}^\alpha h_{ik}^{n+1} = \sum_i h_{ik}^\alpha h_{ij}^{n+1}$, that is, $H_\alpha H_{n+1} = H_{n+1}H_\alpha$. Thus $\Phi_\alpha \Phi_{n+1} = \Phi_{n+1} \Phi_\alpha$. Since matrices $\Phi_\alpha$ and $\Phi_{n+1}$ are traceless, by Lemma 2.1, we have

$$
\sum_\alpha \text{tr}(\Phi_\alpha^2 \Phi_{n+1}) \leq \frac{n-2}{\sqrt{n(n-1)}} \|\mu\|\|\phi\|^2 \leq \frac{n-2}{\sqrt{n(n-1)}} \|\phi\|^3, \tag{3.5}
$$
where we used the following
\[ \|\mu\|^2 \leq \|h\|^2 - nH^2 = \|\phi\|^2, \]  
(3.6)
Since \( \sum_{\alpha,\beta} [\text{tr}(\Phi_{\alpha}\Phi_{\beta})]^2 \geq \frac{1}{p}\|\phi\|^4 \), from (3.4), (3.5), we have
\[ \frac{1}{2}\Delta\|h\|^2 \geq \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij}(nH)_{ij} \]  
(3.7)
\[ + \|\phi\|^2\{n - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\phi\| + \frac{1}{p}\|\phi\|^2\}. \]

From (3.1) and (3.7), we have
\[ \Box(nH) \geq -\frac{1}{2}n(n-1)\Delta R + \|\nabla h\|^2 - n^2\|\nabla H\|^2 \]  
(3.8)
\[ + \|\phi\|^2\{n - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\phi\| + \frac{1}{p}\|\phi\|^2\}. \]

From (3.8) and Proposition 3.2, we have
\[ nLH = n[\Box H + (k'/2n)\Delta H] = \Box(nH) + (1/2)n(n-1)\Delta R \geq \|\phi\|^2\{n - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\phi\| + \frac{1}{p}\|\phi\|^2\} = \|\phi\|^2P_H(\|\phi\|), \]
where \( P_H(\|\phi\|) = n(1 - H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H\|\phi\| + \frac{1}{p}\|\phi\|^2 \). The discriminant of \( P_H(\|\phi\|) \) is \( \frac{n}{(n-1)p}[m^2H^2 - 4(n-1)] \), where \( m^2 = (n-2)p + 4(n-1) \).

(1) If \( H^2 < \frac{4(n-1)}{m^2} \) on \( M^n \), then \( P_H(\|\phi\|) > 0 \) on \( M^n \) and the right-hand side of (3.9) is nonnegative. Since the operator \( L \) is elliptic and \( H \) obtains its maximum on \( M^n \), from (3.9), we know that \( H = \text{const.} \) on \( M^n \). From (3.9) again, we get \( \|\phi\|^2P_H(\|\phi\|) = 0 \), so \( \|\phi\|^2 = 0 \) and \( M^n \) is totally umbilical.

(2) If \( H^2 = \frac{4(n-1)}{m^2} \) on \( M^n \), then \( P_H(\|\phi\|) = (\|\phi\|^2 - \frac{n(n-2)p\sqrt{n}}{m})^2 \geq 0 \) on \( M^n \). We have from (3.9) that \( \|\phi\|^2P_H(\|\phi\|) = 0 \). Thus, we know that \( \|\phi\|^2 = 0 \) and \( M^n \) is totally umbilical; or \( P_H(\|\phi\|) = 0 \). If \( P_H(\|\phi\|) = 0 \), we infer that the equalities hold in (3.9), (3.8), (3.7), (3.6) and (2.12) of Lemma 2.1. If the equality holds in (3.6), we have \( \|\mu\|^2 = \|h\|^2 - nH^2 \). This implies that \( \|\mu\| = \|\phi\| \) and \( \|\tau\| = 0 \). Since \( e_{n+1} \) is parallel on the normal bundle \( T^1(M^n) \) of \( M^n \), using the method of Yau [14], we know that \( M^n \) lies in a totally geodesic submanifold \( S^{1+n}_1 \) of \( S^{n+p}_p \). If the equality holds in Lemma 2.1, then \( M^n \) has \( n-1 \) principal curvatures equal and constant. As \( H \) is constant, the other principal curvature is constant as well, so \( M^n \) is isoparametric. Therefore, we know that \( M^n \) is isometric to a hyperbolic cylinder \( H^1(\sinh r) \times S^{n-1}(\cosh r) \) from the congruence theorem in [1].
(3) If $\frac{4(n-1)}{m^2} < H^2 \leq 1$ on $M^n$, then $P_H(\|\phi\|)$ has two real roots $B^+_H(n, p, H)$ and given by $B^+_H(n, p, H) = \frac{p}{2} \sqrt{\frac{m^2}{m^2 - 1}}(n-2)H \pm \frac{1}{\sqrt{p}} \sqrt{m^2 H^2 - 4(n - 1)}$. Clearly, $B^+_H(n, p, H)$ is always positive, and $B^-_H(n, p, H) \geq 0$ if and only if $\frac{4(n-1)}{m^2} < H^2 \leq 1$. Thus we get $\|\phi\|^2 P_H(\|\phi\|) = 0$, and so $\|\phi\|^2 = 0$ and $M^n$ is totally umbilical; or $P_H(\|\phi\|) = 0$, then we have
\[
\|\phi\| = B^-_H(n, p, H), \quad \|\phi\| = B^+_H(n, p, H),
\]
on $M^n$. If $\|\phi\| = B^-_H(n, p, H) = 0$, then we know that $M^n$ is totally umbilical. If $\|\phi\| = B^+_H(n, p, H) > 0$, by (3.10) and (3.9), the equalities hold in (3.9), (3.8), (3.7), (3.6) and (2.12) of Lemma 2.1. By making use of the same assertion as in the proof of (2) above, we infer that $M^n$ is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$. If $\|\phi\| = B^+_H(n, p, H)(> 0)$, we also have $M^n$ is isometric to a hyperbolic cylinder $H^1(\sinh r) \times S^{n-1}(\cosh r)$. This completes the proof of Main Theorem.

Acknowledgements. This work was supported by NSF of Shaanxi Province (SJ08A31) and NSF of Shaanxi Educational Committee (2008JK484).

References


*Received: June, 2009*