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Kepler’s Quartic Curve is Implemented
by a Central Force

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Abstract
In this note the authors show that a curve which Kepler supposed as the Martian orbit is realized as the orbit under some central force.

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1 Kepler’s quaric curve

Plane algebraic curves have provided various models of the orbits of planets. Many authors mentioned that Johannes Kepler supposed an "ovoid" as Mars’s orbit around the Sun before he found that an ellipse was the true orbit of Mars (cf.[1],[3],[10],[12],[13]). In [8], a quartic curve was characterized as Kepler’s "ovoid" following Fladt’s idea. In [4] he formulated Kepler’s ”ovoid” treated in [6], Ch. 30. Fladt interpreted the orbit under the assumption that the motion of a planet follows the ”area law” in place of the ”distance law” (cf. [1], [12]). We consider this quartic arc with eccentricity $0 < e < 1$. On the Euclidean plane, we shall use Cartesian coordinates $(X,Y)$ and treat normalized Kepler’s quartic curves. The Sun is located at $(0,0)$. The points $C = (1 + e,0)$, $D = (-1 + e,0)$ are respective the aphelion and the perihelion of a planet. The position $(x, y)$ of the planet is given by the equated anomaly $\theta$ as

$$x = r \cos \theta, \ y = r \sin \theta, \ r = \rho(\theta) = (1 - e^2) \sqrt{\frac{1 + e^2 + 2e \cos \theta}{1 + e^4 - 2e^2 \cos(2\theta)}}. \quad (1.1)$$
(cf. [3],[7],[8]). We assume the planet has a unit mass. Kepler found the conservation law of the angular momentum of planets. It is called his second law. Newton deduced this law from his theory of dynamics under the central forces (cf. [2]). It is worthwhile to provide explicitly the central force and the initial conditions under which the orbit of the point mass is just Kepler’s quartic curve. We show that the above orbit is realized under the central force

\[ f(r) = -\frac{r}{4(1 - e^2)^2} - \frac{3}{4r^3}, \tag{1.2} \]

with the angular momentum \( M = 1 \). This central force satisfies \( f(r) < 0 \), that is, it is attractive towards the center. The function \(-f(r)\) attains its minimum \( 1/\sqrt{3} \times (1 - e^2)^{-3/2} \) at \( \sqrt{3}(1 - e^2) \). The minimal distance \( 1 - e \) of the planet satisfies \( 1 - e < \sqrt{3}(1 - e^2) \). The maximal distance \( 1 + e \) satisfies \( 1 + e \leq \sqrt{3}(1 - e^2) \) if \( 0 < e \leq 1/2 \) and it satisfies \( 1 - e > \sqrt{3}(1 - e^2) \) if \( 1/2 < e < 1 \). The graph of the function \(-f(r)\) for \( e = 4/5 \) is given by Figure 1.

![Figure 1](image_url)

The system of differential equations

\[ x''(t) = -\frac{x(t)}{4(1 - e^2)^2} - \frac{3x(t)}{4(x(t)^2 + y(t)^2)^2}, \quad y''(t) = -\frac{y(t)}{4(1 - e^2)^2} - \frac{3y(t)}{4(x(t)^2 + y(t)^2)^2}, \tag{1.3} \]

with the initial conditions

\[ x(0) = 1 + e, \quad y(0) = 0, \quad x'(0) = 0, \quad y'(0) = \frac{1}{1 + e} \tag{1.4} \]

has the solution \((x(t), y(t))\) on the orbit represented by (1.1). For \( \pi < \theta < 2\pi \), the arc (1.1) is represented as

\[ \theta = 2\pi - \arccos\left\{ \frac{(1 + e^2)r^2 - (1 - e^2)^2}{2er^2} \right\}, \tag{1.5} \]
and hence it satisfies
\[ \frac{d\theta}{dr} = \frac{2(1 - e^2)}{r \sqrt{(r^2 - (1 - e^2))(1 + e)^2 - r^2)}. \tag{1.6} \]

The potential energy \( V(r) \) corresponding to the central force (1.2) is given by
\[ V(r) = \frac{r^2}{8(1 - e^2)^2} - \frac{3}{8r^2}. \tag{1.7} \]

The total energy \( E_0 \) of the system (1.3) with the initial conditions (1.4) is given by
\[ E_0 = \frac{1 + e^2}{4(1 - e^2)^2}. \tag{1.8} \]

Hence the first integral of the system (1.3) with the initial conditions (1.4) is given by
\[ \frac{1}{r^2} \frac{dr}{d\theta} = \frac{\sqrt{\{r^2 - (1 - e^2)^2\}\{1 + e^2 - r^2\}}}{2(1 - e^2)r}, \tag{1.9} \]

and hence \( d\theta/dr \) satisfies the equation (1.6). Thus the arc (1.1) is the unique solution of (1.3) under the initial conditions (1.4). The area of the domain surrounded by the orbit is \((1 - e^2)\pi\) and hence the period of the rotation of the planet around the Sun is \(2(1 - e^2)\pi\). For the equated anomaly \(0 \leq \theta < 2\pi\), the time \(t\) when the planet lies at \((x, y)\) given by (1.1) is
\[ t = (1 - e^2)^2 \int_0^\theta \frac{1 - e^2 + 2e \cos s}{1 + e^4 - 2e^2 \cos(2s)} ds + 2n(1 - e^2)\pi, \tag{1.9} \]

for any integer \(n\).

## 2 Cental force

The classical method due to Newton is available to compute the central force \(f(r)\) corresponding to the orbit (1.1). However exact computations to obtain the force \(f(r)\) is not so trivial since it concerns with the elimination process of the parameter \(\theta\). The orbit (1.1) is represented implicitly as

\[ g(r, \theta) = G(r, T) = 2e^2r^4T - e^8 + 2e^6(r^2 + 2) - e^4(r^4 + 2r^2 + 6) - 2e^2(r^2 - 2) - (r^2 - 1)^2 = 0, \tag{2.1} \]

where \(T = \cos(2\theta)\). The function \(g(r, \theta)\) is factorized as
\[ g(r, \theta) = -\{e^4 - 2e^2 \cos(2\theta) + 1\}\{r - \rho(\theta)\}\{r - \rho(\pi - \theta)\}\{r + \rho(\theta)\}\{r + \rho(\pi - \theta)\}, \tag{2.2} \]
for $0 \leq \theta \leq 2\pi$. Using the equation (2.1), the well known formula

$$f = f(r) = \frac{1}{r^4} \frac{dr^2}{d\theta} - \frac{2}{r^5}(\frac{dr}{d\theta})^2 - \frac{1}{r^3},$$

(2.3)

for the angular momentum $M = 1$ (cf. [5]) is rewritten in an implicit form

$h(r, \theta) = -r^5 g_3 f + r(-g_r^2 g_{\theta\theta} + 2g_{\theta r} + g_{\theta r} - g_{\theta r}^2 g_{rr}) - 2g_r g_{\theta r} - r^2 g_{rr}^2 = 0.$

(2.4)

The direct computation using (1.1) and (2.3) to obtain $f(r)$ is not so efficient. In this case this equation (2.4) is expressed as the following

$k(r, T, f) = r^3(-1+e^2+e^4-e^6+r^2+e^4 r^2-2e^2 r^2 T)^3 f - e^{18} + 3(1+r^2)e^{16} - r^2(4T+3r+6)e^{14}$

$+ \{r^4(8T+3)+8r^2 T + r^6 - 8\}e^{12} + \{r^4(-9T^2 - 8T + 2) + r^2(4T + 6) - 4r^6 T + 6\}e^{10}$

$+ \{r^4(9T^2 - 2) - 2r^2(8T + 3) + 7r^6 T^2 + 6\}e^8 + \{-2r^6(3T^3 + T) + r^4(9T^2 - 2) + r^2(4T + 6) - 8\}e^6$

$+ \{r^4(-9T^2 - 8T + 2) + 7r^6 T^2 + 8r^2 T\}e^4 - (r^2 - 1)^2(4r^2 T - 3)e^2 + (r^2 - 1)^3 = 0.$

(2.5)

Fortunately the equation (2.1) is linear in $T$. We substitute its solution into (2.5) to eliminate $T = \cos(2\theta)$. In this way, we have the following equation

$$(1-e^2)^4 e^6 r^6 ((1-e^2)^2 - (1+e^2)r^2)^3 \{(4r^2(1-e^2)^2 f + r^4 + 3(1-e^2)^2) = 0.$$

(2.6)

The equation (1.2) follows from this. In a general case (cf. [9]) we can apply the Sylvester resultant to obtain the central force $f(r)$ if the orbit lies on an algebraic curve (cf. [11]). However it is not so easy to perform this process.

### 3 Numerical experiment

Some numerical experiments using computer softwares are useful to examine our results. For instance, we assume $e = 4/5$. The following command for "Mathematica" produces a numerical approximation of the solution of the differential equations (1.3) with (1.4):

$$W = \text{NDSolve}[\{x''[t] == -x[t]/(4(1 - e^2)^2 - 3/4)x[t]/(x[t]^2 + y[t]^2)^2,$$

$$y''[t] == -y[t]/(4(1 - e^2)^2 - 3/4y[t]/(x[t]^2 + y[t]^2)^2),$$

$$x[0] == 1 + e, y[0] == 0, x'[0] == 0, y'[0] == 1/(1 + e), \{t, -1.13, 1.13\}$$

(cf. [14]). Using this approximate solution, we can produce an approximate orbit by the command

$$\text{ParametricPlot}[\{x[t], y[t]\} /. W, \{t, -1.12, 1.12\}]$$

(cf. Figure 2) or the command

$$\text{ParametricPlot}[\{x[t], y[t]\} /. W, \{t, -1.129, 1.129\}]$$

(cf. Figure 3).
Kepler’s quartic curve

Figure 2:

Figure 3:
References


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