Some Remarks on Series in Fuzzy n-Normed Spaces

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Abstract. The purpose of this paper is to introduce the notion of the absolutely convergent series, finite convergence sequences, and obtain some important results on them.

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1. Introduction

In [7], S. Gähler introduced n-norms on a linear space. A detailed theory of n-normed linear space can be found in [9, 12, 14, 15]. In [9], H. Gunawan and M. Mashadi gave a simple way to derive an (n − 1)-norm from the n-norm in such a way that the convergence and completeness in the n-norm is related to those in the derived (n − 1)-norm. A detailed theory of fuzzy normed linear space can be found in [1, 2, 4, 5, 6, 11, 13, 18]. In [16], A. Narayanan and S. Vijayabalaji have extended the n-normed linear space to fuzzy n-normed linear space and in [20] the authors have studied the completeness of fuzzy n-normed spaces.

The main purpose of this paper is to study the results concerning infinite series (see, [3, 17, 19, 21]) in fuzzy n-normed spaces. In section 2, we quote some basic definitions of fuzzy n-normed spaces. In section 3, we consider the absolutely convergent series in fuzzy n-normed spaces and obtain some results on it. In section 4, we study the property of finite convergence sequences in fuzzy n-normed spaces.

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2. Preliminaries

Let \( n \) be a positive integer, and let \( X \) be a real vector space of dimension at least \( n \). We recall the definitions of an \( n \)-seminorm and a fuzzy \( n \)-norm [16].

**Definition 2.1.** A function \((x_1, x_2, \ldots, x_n) \mapsto \|x_1, \ldots, x_n\| \) from \( X^n \) to \([0, \infty)\) is called an \( n \)-seminorm on \( X \) if it has the following four properties:

(S1) \( \|x_1, x_2, \ldots, x_n\| = 0 \) if \( x_1, x_2, \ldots, x_n \) are linearly dependent;
(S2) \( \|x_1, x_2, \ldots, x_n\| \) is invariant under any permutation of \( x_1, x_2, \ldots, x_n \);
(S3) \( \|x_1, \ldots, x_{n-1}, cx_n\| = |c|\|x_1, \ldots, x_{n-1}, x_n\| \) for any real \( c \);
(S4) \( \|x_1, \ldots, x_{n-1}, y + z\| \leq \|x_1, \ldots, x_{n-1}, y\| + \|x_1, \ldots, x_{n-1}, z\| \).

An \( n \)-seminorm is called a \( n \)-norm if \( \|x_1, x_2, \ldots, x_n\| > 0 \) whenever \( x_1, x_2, \ldots, x_n \) are linearly independent.

**Definition 2.2.** A fuzzy subset \( N \) of \( X^n \times \mathbb{R} \) is called a fuzzy \( n \)-norm on \( X \) if and only if:

(F1) For all \( t \leq 0 \), \( N(x_1, x_2, \ldots, x_n, t) = 0 \);
(F2) For all \( t > 0 \), \( N(x_1, x_2, \ldots, x_n, t) = 1 \) if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent;
(F3) \( N(x_1, x_2, \ldots, x_n, t) \) is invariant under any permutation of \( x_1, x_2, \ldots, x_n \);
(F4) For all \( t > 0 \) and \( c \in \mathbb{R}, c \neq 0 \),

\[
N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|});
\]

(F5) For all \( s, t \in \mathbb{R} \),

\[
N(x_1, \ldots, x_{n-1}, y+z, s+t) \geq \min \{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, t)\}.
\]

(F6) \( N(x_1, x_2, \ldots, x_n, t) \) is a non-decreasing function of \( t \in \mathbb{R} \) and

\[
\lim_{t \to -\infty} N(x_1, x_2, \ldots, x_n, t) = 1.
\]

The pair \((X, N)\) will be called a fuzzy \( n \)-normed space.

**Theorem 2.1.** Let \( A \) be the family of all finite and nonempty subsets of fuzzy \( n \)-normed space \((X, N)\) and \( A \in A \). Then the system of neighborhoods

\[
\mathcal{B} = \{B(t, r, A) : t > 0, 0 < r < 1, A \in A\}
\]

where \( B(t, r, A) = \{x \in X : N(a_1, \ldots, a_{n-1}, x, t) > 1 - r, a_1, \ldots, a_{n-1} \in A\} \) is a base of the null vector \( \theta \), for a linear topology on \( X \), named \( N \)-topology generated by the fuzzy \( n \)-norm \( N \).

**Proof.** We omit the proof since it is similar to the proof of Theorem 3.6 in [8]. \( \square \)

**Definition 2.3.** A sequence \( \{x_k\} \) in a fuzzy \( n \)-normed space \((X, N)\) is said to converge to \( x \) if given \( r > 0 \), \( t > 0 \), \( 0 < r < 1 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_k - x, t) > 1 - r \) for all \( k \geq n_0 \).
Definition 2.4. A sequence \( \{x_k\} \) in a fuzzy \( n \)-normed space \((X, N)\) is said to be Cauchy sequence if given \( \epsilon > 0, \ t > 0, \ 0 < \epsilon < 1 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_m - x_k, t) > 1 - \epsilon \) for all \( m, k \geq n_0 \).

Theorem 2.2 (13). Let \( N \) be a fuzzy \( n \)-norm on \( X \). Define for \( x_1, x_2, \ldots, x_n \in X \) and \( \alpha \in (0, 1) \)

\[
\|x_1, x_2, \ldots, x_n\|_\alpha = \inf \{ t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha \}.
\]

Then the following statements hold.

(A1) for every \( \alpha \in (0, 1) \), \( \|\cdot, \cdot, \cdot, \cdot\|_\alpha \) is an \( n \)-seminorm on \( X \);

(A2) If \( 0 < \alpha < \beta < 1 \) and \( x_1, x_2, \ldots, x_n \in X \) then

\[
\|x_1, x_2, \ldots, x_n\|_\alpha \leq \|x_1, x_2, \ldots, x_n\|_\beta.
\]

Example 2.3. [10, Example 2.3]. Let \( \|\cdot, \cdot, \cdot, \cdot\| \) be a \( n \)-norm on \( X \). Then define \( N(x_1, x_2, \ldots, x_n, t) = 0 \) if \( t \leq 0 \) and, for \( t > 0 \),

\[
N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.
\]

Then the seminorms (2.1) are given by

\[
\|x_1, x_2, \ldots, x_n\|_\alpha = \frac{\alpha}{1 - \alpha}\|x_1, x_2, \ldots, x_n\|.
\]

3. Absolutely convergent series in fuzzy \( n \)-normed spaces

In this section we introduce the notion of the absolutely convergent series in a fuzzy \( n \)-normed space \((X, N)\) and give some results on it.

Definition 3.1. The series \( \sum_{k=1}^{\infty} x_k \) is called absolutely convergent in \((X, N)\) if

\[
\sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \infty
\]

for all \( a_1, \ldots, a_{n-1} \in X \) and all \( \alpha \in (0, 1) \).

Using the definition of \( \|\cdot, \cdot, \cdot, \cdot\|_\alpha \) the following lemma shows that we can express this condition directly in terms of \( N \).

Lemma 3.1. The series \( \sum_{k=1}^{\infty} x_k \) is absolutely convergent in \((X, N)\) if, for every \( a_1, \ldots, a_{n-1} \in X \) and every \( \alpha \in (0, 1) \) there are \( t_k \geq 0 \) such that \( \sum_{k=1}^{\infty} t_k < \infty \) and \( N(a_1, \ldots, a_{n-1}, x_k, t_k) \geq \alpha \) for all \( k \).
Proof. Let $\sum_{k=1}^{\infty} x_k$ be absolutely convergent in $(X, N)$. Then

$$\sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \infty$$

for every $a_1, \ldots, a_{n-1} \in X$ and every $\alpha \in (0, 1)$. Let $a_1, \ldots, a_{n-1} \in X$ and $\alpha \in (0, 1)$. For every $k$ there is $t_k \geq 0$ such that $N(a_1, \ldots, a_{n-1}, x_k, t_k) \geq \alpha$ and

$$t_k < \|a_1, \ldots, a_{n-1}, x_k\|_\alpha + \frac{1}{2^k}.$$ 

Then

$$\sum_{k=1}^{\infty} t_k < \sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$  

The other direction is even easier to show. \qed

**Definition 3.2.** A fuzzy $n$-normed space $(X, N)$ is said to be sequentially complete if every Cauchy sequence in it is convergent.

**Lemma 3.2.** Let $(X, N)$ be sequentially complete, then every absolutely convergent series $\sum_{k=1}^{\infty} x_k$ converges and

$$\left\| a_1, \ldots, a_{n-1}, \sum_{k=1}^{\infty} x_k \right\|_\alpha \leq \sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha$$

for every $a_1, \ldots, a_{n-1} \in X$ and every $\alpha \in (0, 1)$.

Proof. Let $\sum_{k=1}^{\infty} x_k$ be an infinite series such that $\sum_{k=1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \infty$ for every $a_1, \ldots, a_{n-1} \in X$ and every $\alpha \in (0, 1)$. Let $y_n = \sum_{k=1}^{n} x_k$ be a partial sum of the series. Let $a_1, \ldots, a_{n-1} \in X$ and $\alpha \in (0, 1)$ and $\epsilon > 0$. There is $N$ such that $\sum_{k=N+1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha < \epsilon$. Then, for $n > m \geq N$,

$$\|a_1, \ldots, a_{n-1}, y_n\|_\alpha - \|a_1, \ldots, a_{n-1}, y_m\|_\alpha \leq \|a_1, \ldots, a_{n-1}, y_n - y_m\|_\alpha$$

$$\leq \sum_{k=m+1}^{n} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha$$

$$\leq \sum_{k=N+1}^{\infty} \|a_1, \ldots, a_{n-1}, x_k\|_\alpha$$

$$< \epsilon.$$
This is shows that \( \{y_n\} \) is a Cauchy sequence in \((X, N)\). But since \((X, N)\) is sequentially complete, the sequence \( \{y_n\} \) converges and so the series \( \sum_{k=1}^{\infty} x_k \) converges.

**Definition 3.3.** Let \( I \) be any denumerable set. We say that the family \((x_\alpha)_{\alpha \in I}\) of elements in a complete fuzzy \( n \)-normed space \((X, N)\) is absolutely summable, if for a bijection \( \Psi \) of \( \mathbb{N}(\text{the set of all natural numbers}) \) onto \( I \) the series \( \sum_{n=1}^{\infty} x_{\Psi(n)} \) is absolutely convergent.

The following result may not be surprising but the proof requires some care.

**Theorem 3.1.** Let \((x_\alpha)_{\alpha \in I}\) be an absolutely summable family of elements in a sequentially complete fuzzy \( n \)-normed space \((X, N)\). Let \((B_n)\) be an infinite sequence of a non-empty subset of \( A \), such that \( A = \bigcup_n B_n \), \( B_i \cap B_j = \emptyset \) for \( i \neq j \), then if \( z_n = \sum_{\alpha \in B_n} x_\alpha \), the series \( \sum_{n=0}^{\infty} z_n \) is absolutely convergent and \( \sum_{n=0}^{\infty} z_n = \sum_{\alpha \in I} x_\alpha \).

**Proof.** It is easy to see that this is true for finite disjoint unions \( I = \bigcup_{n=1}^{N} B_n \). Now consider the disjoint unions \( I = \bigcup_{n=1}^{\infty} B_n \). By Lemma 3.2

\[
\sum_{n=1}^{\infty} \|a_1, \ldots, a_{n-1}, z_n\|_\alpha \leq \sum_{n=1}^{\infty} \sum_{i \in B_n} \|a_1, \ldots, a_{n-1}, x_i\|_\alpha = \sum_{i \in I} \|a_1, \ldots, a_{n-1}, x_i\|_\alpha < \infty.
\]

For every \( a_1, \ldots, a_{n-1} \in X \), and every \( \alpha \in (0,1) \). Therefore, \( \sum_{n=0}^{\infty} z_n \) is absolutely convergent. Let \( y = \sum_{i \in I} x_i \), \( z = \sum_{n=1}^{\infty} z_n \). Let \( \epsilon > 0 \), \( a_1, \ldots, a_{n-1} \in X \) and \( \alpha \in (0,1) \). There is a finite set \( J \subset I \) such that \( \sum_{i \notin J} \|a_1, \ldots, a_{n-1}, x_i\|_\alpha < \frac{\epsilon}{2} \). Choose \( N \) large enough such that \( B = \bigcup_{n=1}^{N} B_n \supset J \) and

\[
\left\| a_1, \ldots, a_{n-1}, z - \sum_{n=1}^{N} z_n \right\|_\alpha < \frac{\epsilon}{2},
\]

Then

\[
\left\| a_1, \ldots, a_{n-1}, y - \sum_{i \in B} x_i \right\|_\alpha < \frac{\epsilon}{2}.
\]
By the first part of the proof
\[ \sum_{n=1}^{N} z_n = \sum_{i \in B} x. \]
Therefore, \( \| a_1, ..., a_{n-1}, y - z \|_\alpha < \epsilon \). This is true for all \( \epsilon \) so \( \| a_1, ..., a_{n-1}, y - z \|_\alpha = 0 \). This is true for all \( a_1, ..., a_{n-1} \in X \), \( \alpha \in (0, 1) \) and \( (X, N) \) is Hausdorff see [8, Theorem 3.1]. Hence \( y = z \).

4. Finite convergent sequences in fuzzy \( n \)-normed spaces

In this section our principal goal is to show that every sequence having finite convergent property is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property in every merizable fuzzy \( n \)-normed space \( (X, N) \).

**Definition 4.1.** A sequence \( \{x_k\} \) in a fuzzy \( n \)-normed space \( (X, N) \) is said to have finite convergent property if
\[ \sum_{j=1}^{\infty} \|a_1, ..., a_{n-1}, x_j - x_{j-1}\|_\alpha < \infty \]
for all \( a_1, ..., a_{n-1} \in X \) and all \( \alpha \in (0, 1) \).

**Definition 4.2.** A fuzzy \( n \)-normed space \( (X, N) \) is said to be metrizable, if there is a metric \( d \) which generates the topology of the space.

**Theorem 4.1.** Let \( (X, N) \) be a metrizable fuzzy \( n \)-normed space, then every sequence having finite convergent property is Cauchy and every Cauchy sequence has a subsequence which has finite convergent property.

**Proof.** Since \( X \) is metrizable, there is a sequence \( \{\|a_{1,r}, ..., a_{n-1,r}, x\|_{\alpha_r}\} \) for all \( a_{1,r}, ..., a_{n-1,r} \in X \) and all \( \alpha_r \in (0, 1) \) generating the topology of \( X \). We choose an increasing sequence \( \{m_{k,1}\} \) such that
\[ \sum_{k=1}^{\infty} \|a_{1,1}, ..., a_{n-1,1}, x_{m_{k+1,1}} - x_{m_{k,1}}\|_{\alpha_1} < \infty \]
where \( a_{1,1}, ..., a_{n-1,1} \in X \) and \( \alpha_1 \in (0, 1) \). Then we choose a subsequence \( m_{k,2} \) of \( m_{k,1} \) such that
\[ \sum_{k=1}^{\infty} \|a_{1,2}, ..., a_{n-1,2}, x_{m_{k+1,2}} - x_{m_{k,2}}\|_{\alpha_2} < \infty \]
where \( a_{1,2}, ..., a_{n-1,2} \in X \) and \( \alpha_2 \in (0, 1) \). Continuing in this way we construct recursively sequences \( m_{k,r} \) such that \( m_{k,r+1} \) is a subsequence of \( m_{k,r} \) and such that
\[ \sum_{k=1}^{\infty} \|a_{1,r}, ..., a_{n-1,r}, x_{m_{k+1,r}} - x_{m_{k,r}}\|_{\alpha_r} < \infty \]
for all $a_{1,r}, \ldots, a_{n-1,r} \in X$ and all $\alpha_r \in (0, 1)$. Now consider the diagonal sequence $m_k = m_{k,k}$. Let $r \in \mathbb{N}$. The sequence $\{m_k\}_{k=r}^{\infty}$ is a subsequence of $\{m_{k,r}\}_{k=r}^{\infty}$. Let $k \geq r$. There are $u < v$ such that $m_k = m_{u,r}$ and $m_{k+1} = m_{v,r}$.

Then by the triangle inequality

$$\|a_{1,r}, \ldots, a_{n-1,r}, x_{m_{k+1}} - x_{m_k}\|_{\alpha_r} \leq \sum_{i=u}^{v-1} \|a_{1,r}, \ldots, a_{n-1,r}, x_{m_{i+1,r}} - x_{m_i,r}\|_{\alpha_r}$$

and therefore,

$$\sum_{k=r}^{\infty} \|a_{1,r}, \ldots, a_{n-1,r}, x_{m_{k+1}} - x_{m_k}\|_{\alpha} \leq \sum_{j=r}^{\infty} \|a_{1,r}, \ldots, a_{n-1,r}, x_{m_{j+1,r}} - x_{m_j,r}\|_{\alpha}$$

for all $a_{1}, \ldots, a_{n-1} \in X$ and all $\alpha \in (0, 1)$. The statement of the theorem follows.

The above theorem shows that many Cauchy sequence has a subsequence which has finite convergent. Therefore, it is natural to ask for an example of Cauchy sequence has a subsequence which has not finite convergent property.

**Example 4.2.** We consider the set $S$ consisting of all convergent real sequences. Let $X$ be the space of all functions $f : S \to \mathbb{R}$ equipped with the topology of pointwise convergence. This topology is generated by

$$\|f_{1,s}, \ldots, f_{n-1,s}, f\|_{\alpha_s} = |f(s)|,$$

for all $f_{1,s}, \ldots, f_{n-1,s}, f \in X$ and all $\alpha_s \in (0, 1)$, where $s \in S$. Then consider the sequence $f_n \in X$ defined by $f_n(s) = s_n$ where $s = (s_n) \in S$. The sequence $f_n$ is a Cauchy sequence in $X$ but there is no subsequence $f_{n_k}$ such that

$$\sum_{k=1}^{\infty} \|f_{1,s}, \ldots, f_{n-1,s}, f_{n_k+1} - f_{n_k}\|_{\alpha_s} < \infty$$

for all $s \in S$. We see this as follows. If $n_1 < n_2 < n_3 < \ldots$ is a sequence then define $s_n = (-1)^k \frac{1}{k}$ for $n_k \leq n < n_{k+1}$. Then $s = (s_n) \in S$ but

$$\sum_{k=1}^{\infty} \|f_{1,s}, \ldots, f_{n-1,s}, f_{n_k+1} - f_{n_k}\|_{\alpha_s} = \sum_{k=1}^{\infty} |s_{n_k+1} - s_{n_k}| \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

**References**


References


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