Fibonacci and Lucas Sums by Matrix Methods

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Abstract

In this paper, some Fibonacci and Lucas sums are derived by using the matrices

\[ S = \begin{bmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 5 \\ 1 & 0 \end{bmatrix} \]

The most notable side of this paper is our proof method, since all the identities used in the proofs of main theorems are proved previously by using the matrices \( S \) and \( K \). Although the identities we proved are known, our proofs are not encountered in the Fibonacci and Lucas numbers literature.

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1 Introduction

The Fibonacci sequence \( \{F_n\} \) is defined by the recurrence relation \( F_n = F_{n-1} + F_{n-2} \), for \( n \geq 2 \) with \( F_0 = 0 \) and \( F_1 = 1 \). The Lucas sequence \( \{L_n\} \), considered as a companion to Fibonacci sequence, is defined recursively by \( L_n = L_{n-1} + L_{n-2} \), for \( n \geq 2 \) with \( L_0 = 2 \) and \( L_1 = 1 \). It is well known that \( F_{-n} = (-1)^{n+1}F_n \) and \( L_{-n} = (-1)^nL_n \), for every \( n \in \mathbb{Z} \). For more detailed information see \([9],[10]\). This paper presents an interesting investigation about some special relations between matrices and Fibonacci, Lucas numbers. This investigation is valuable, since it provides students to use their theoretical knowledge to obtain new Fibonacci and Lucas identities by different methods. So, this paper contributes to Fibonacci and Lucas numbers literature, and encourage many researchers to investigate the properties of such number sequences.

2 Main Theorems

Lemma 1 If \( X \) is a square matrix with \( X^2 = X + I \), then \( X^n = F_nX + F_{n-1}I \) for all \( n \in \mathbb{Z} \).
Proof. If $n = 0$, then the proof is obvious. It can be shown by induction that $X^n = F_nX + F_{n-1}I$ for every $n \in \mathbb{N}$. We now show that $X^{-n} = F_{-n}X + F_{-n-1}I$ for every $n \in \mathbb{N}$. Let $Y = I - X = -X^{-1}$. Then

$$Y^2 = (I - X)^2 = I - 2X + X^2 = I - 2X + X + I = I + I - X = I + Y.$$  

This shows that $Y^n = F_nY + F_{n-1}I$. That is, $(-X^{-1})^n = F_n(I - X) + F_{n-1}I$. Therefore $(-1)^n X^{-n} = -F_nX + (F_n + F_{n-1})I = -F_nX + F_{n+1}I$. Thus

$$X^{-n} = (-1)^{n+1} F_nX + (-1)^n F_{n+1}I = F_{-n}X + F_{-n-1}I.$$  

We can give Corollary 1 from Lemma 1 easily. Also one can consult [9],[10] for more information about the matrix $Q$.

**Corollary 1** Let $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ for every $n \in \mathbb{Z}$.

**Corollary 2** Let $S = \begin{bmatrix} 1/2 & 5/2 \\ 1/2 & 1/2 \end{bmatrix}$. Then $S^n = \begin{bmatrix} L_n/2 & 5F_n/2 \\ F_n/2 & L_n/2 \end{bmatrix}$ for every $n \in \mathbb{Z}$.

Let us give the following corollary for using in the next theorems.

**Lemma 2** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $K = S + S^{-1} = \begin{bmatrix} 0 & 5 \\ 1 & 0 \end{bmatrix}$. Then $KA = \begin{bmatrix} 5c & 5d \\ a & b \end{bmatrix}$.

**Lemma 3** $L_n^2 - 5F_n^2 = 4(-1)^n$ for all $n \in \mathbb{Z}$.

**Proof.** Since $\det S = -1$, $\det(S^n) = (\det S)^n = (-1)^n$. Moreover, since $S^n = \begin{bmatrix} L_n/2 & 5F_n/2 \\ F_n/2 & L_n/2 \end{bmatrix}$, we get $\det S^n = (L_n^2 - 5F_n^2)/4$. Thus it follows that

$$L_n^2 - 5F_n^2 = 4(-1)^n \quad (2.1)$$

Let us give a different proof of one of the fundamental identities of Fibonacci and Lucas numbers, by using the matrix $S$.

**Lemma 4** $2L_{n+m} = L_nL_m + 5F_nF_m$ and $2F_{n+m} = F_nL_m + L_nF_m$ for all $n, m \in \mathbb{Z}$. 

Proof. Since
\[
\begin{bmatrix}
L_n/2 & 5F_n/2 \\
F_n/2 & L_n/2
\end{bmatrix}
\begin{bmatrix}
L_m/2 & 5F_m/2 \\
F_m/2 & L_m/2
\end{bmatrix}
= S^n S^m = S^{n+m} = \begin{bmatrix}
L_{n+m}/2 & 5F_{n+m}/2 \\
F_{n+m}/2 & L_{n+m}/2
\end{bmatrix},
\]
it is seen that
\[
\begin{bmatrix}
(L_nL_m + 5F_nF_m)/4 & 5(F_nL_m + L_nF_m)/4 \\
(F_nL_m + L_nF_m)/4 & (L_nL_m + 5F_nF_m)/4
\end{bmatrix}
= \begin{bmatrix}
L_{n+m}/2 & 5F_{n+m}/2 \\
F_{n+m}/2 & L_{n+m}/2
\end{bmatrix}.
\]
Thus it follows that
\[
2L_{n+m} = L_nL_m + 5F_nF_m \tag{2.2}
\]
\[
2F_{n+m} = F_nL_m + L_nF_m \tag{2.3}
\]

Lemma 5 \(2(-1)^mL_{n-m} = L_{m}L_{n} - 5F_{m}F_{n}\) and \(2(-1)^mF_{n-m} = F_{n}L_{m} - L_{n}F_{m}\) for all \(n, m \in \mathbb{Z}\).

Proof. Since
\[
S^{n-m} = S^n S^{-m} = S^n (S^m)^{-1} = S^n (-1)^m \begin{bmatrix}
L_m/2 & -5F_m/2 \\
-F_m/2 & L_m/2
\end{bmatrix}
\]
\[
= (-1)^m \begin{bmatrix}
L_n/2 & 5F_n/2 \\
F_n/2 & L_n/2
\end{bmatrix}
\begin{bmatrix}
L_m/2 & -5F_m/2 \\
-F_m/2 & L_m/2
\end{bmatrix}
\]
\[
= (-1)^m \begin{bmatrix}
(L_nL_m - 5F_nF_m)/4 & 5(L_nF_m - F_nL_m)/4 \\
(L_nF_m - F_nL_m)/4 & (L_nL_m - 5F_nF_m)/4
\end{bmatrix}
\]
and
\[
S^{n-m} = \begin{bmatrix}
L_{n-m}/2 & 5F_{n-m}/2 \\
F_{n-m}/2 & L_{n-m}/2
\end{bmatrix},
\]
then we get
\[
2(-1)^mL_{n-m} = L_mL_n - 5F_mF_n
\]
and
\[
2(-1)^mF_{n-m} = F_nL_m - L_nF_m \tag*{\blacksquare}
\]

Lemma 6 \(L_mL_n = L_{n+m} + (-1)^mL_{n-m}\) and \(L_mF_n = F_{n+m} + (-1)^mF_{n-m}\) for all \(n, m \in \mathbb{Z}\).
Proof. By the definition of the matrix $S^n$, it can be seen that

$$S^{n+m} + (-1)^m S^{n-m} = \begin{bmatrix} (L_{n+m} + (-1)^m L_{n-m})/2 & 5 (F_{n+m} + (-1)^m F_{n-m})/2 \\ (F_{n+m} + (-1)^m F_{n-m})/2 & (L_{n+m} + (-1)^m L_{n-m})/2 \end{bmatrix}.$$ 

On the other hand,

$$S^{n+m} + (-1)^m S^{n-m} = S^m S^n + (-1)^m S^m S^{n-m} = S^n (S^m + (-1)^m S^{-m}).$$

Then the result follows.

Theorem 2.1 Let $n \in \mathbb{N}$ and $m, k \in \mathbb{Z}$ with $m \neq 0$. Then

$$\sum_{j=0}^{n} L_{mj+k} = \frac{L_k - L_{mn+m+k} + (-1)^m (L_{mn+k} - L_{k-m})}{1 + (-1)^m - L_m};$$

and

$$\sum_{j=0}^{n} F_{mj+k} = \frac{F_k - F_{mn+m+k} + (-1)^m (F_{mn+k} - F_{k-m})}{1 + (-1)^m - L_m}.$$

Proof. It is known that

$$I - (S^m)^{n+1} = (I - S^n) \sum_{j=0}^{n} (S^n)^{j}. $$

If $\det(I - S^n) \neq 0$, then we can write

$$(I - S^n)^{-1} (I - (S^m)^{n+1}) S^k = \sum_{j=0}^{n} L_{mj+k} = \begin{bmatrix} \underline{\frac{1}{2} \sum_{j=0}^{n} L_{mj+k}} & \underline{\frac{5}{2} \sum_{j=0}^{n} F_{mj+k}} \\ \underline{\frac{5}{2} \sum_{j=0}^{n} F_{mj+k}} & \underline{\frac{1}{2} \sum_{j=0}^{n} L_{mj+k}} \end{bmatrix}.$$ \hspace{1cm} (2.4)

From Lemma 3, it is easy to see that

$$\det(I - S^n) = (1 - L_m/2)^2 - 5F_m^2/4 = 1 + (-1)^m - L_m \neq 0.$$
for \( m \neq 0 \). If we take \( D = 1 + (-1)^m - L_m \), then we get

\[
(I - S^m)^{-1} = \frac{1}{D} \begin{bmatrix}
1 - L_m/2 & 5F_m/2 \\
F_m/2 & 1 - L_m/2
\end{bmatrix} = \frac{1}{D} \left( 1 - \frac{L_m}{2} \right) I + \frac{F_m}{2} K.
\]

Thus it is seen that

\[
(I - S^m)^{-1} (S^k - S^{mn+m+k}) = \frac{1}{D} \left( 1 - \frac{L_m}{2} \right) (S^k - S^{mn+m+k}) + \frac{F_m}{2} K (S^k - S^{mn+m+k})
\]

(2.5)

By Corollary 1 and Lemma 2, we get

\[
K(S^k - S^{mn+m+k}) = \begin{bmatrix}
5(F_k - F_{mn+m+k})/2 & 5(L_k - L_{mn+m+k})/2 \\
(L_k - L_{mn+m+k})/2 & 5(F_k - F_{mn+m+k})/2
\end{bmatrix}
\]

(2.6)

Using (2.6) in (2.5), we obtain

\[
(I - S^m)^{-1} (S^k - S^{mn+m+k}) = \frac{1}{D} \left\{ \left( 1 - \frac{L_m}{2} \right) \begin{bmatrix}
(L_k - L_{mn+m+k})/2 & 5(F_k - F_{mn+m+k})/2 \\
(F_k - F_{mn+m+k})/2 & (L_k - L_{mn+m+k})/2
\end{bmatrix}
+ \frac{F_m}{2} \begin{bmatrix}
5(F_k - F_{mn+m+k})/2 & 5(L_k - L_{mn+m+k})/2 \\
(L_k - L_{mn+m+k})/2 & 5(F_k - F_{mn+m+k})/2
\end{bmatrix} \right\}
\]

Thus, from the identity (2.4), it follows that

\[
\sum_{j=0}^{n} L_{mj+k} = \frac{1}{D} \left( 1 - \frac{L_m}{2} \right) (L_k - L_{mn+m+k}) + \frac{5F_m}{2} (F_k - F_{mn+m+k})
\]

(2.7)

By Lemma 5, (2.7) becomes

\[
\sum_{j=0}^{n} L_{mj+k} = \frac{L_k - L_{mn+m+k} + (-1)^m (L_{mn+k} - L_{k-m})}{1 + (-1)^m - L_m}.
\]
On the other hand, by (2.5) and (2.6) we get

\[
\sum_{j=0}^{n} F_{mj+k} = \frac{1}{D} \left[ \left( 1 - \frac{L_m}{2} \right) (F_k - F_{mn+m+k}) + \frac{F_m}{2} (L_k - L_{mn+m+k}) \right]
\]

\[
= \frac{1}{D} \left[ F_k - F_{mn+m+k} - \frac{F_k L_m - L_k F_m}{2} + \frac{F_{mn+m+k} L_m - L_{mn+m+k} F_m}{2} \right] \tag{2.8}
\]

By Lemma 5, it follows that

\[
\sum_{j=0}^{n} F_{mj+k} = \frac{F_k - F_{mn+m+k} + (-1)^m (F_{mn+k} - F_{k-m})}{1 + (-1)^m - L_m} \quad \blacksquare
\]

**Theorem 2.2** Let \( n \in \mathbb{N} \) and \( m, k \in \mathbb{Z} \). Then

\[
\sum_{j=0}^{n} (-1)^j L_{mj+k} = \frac{L_k + L_{mn+m+k} + (-1)^m (L_{mn+k} + L_{k-m})}{1 + (-1)^m + L_m}
\]

and

\[
\sum_{j=0}^{n} (-1)^j F_{mj+k} = \frac{F_k + F_{mn+m+k} + (-1)^m (F_{mn+k} + F_{k-m})}{1 + (-1)^m + L_m}.
\]

**Proof.** We prove the theorem in two phases, by taking \( n \) as an even and odd natural number. Firstly assume that \( n \) is an even natural number. Then

\[
I + (S^m)^{n+1} = (I + S^m) \sum_{j=0}^{n} (-1)^j (S^m)^j.
\]

Since \( \det(I + S^m) = 1 + (-1)^m + L_m \neq 0 \), by Lemma 3, we can write

\[
(I + S^m)^{-1} (I + (S^m)^{n+1}) S^k = \sum_{j=0}^{n} (-1)^j S^{mj+k}
\]

\[
= \begin{bmatrix}
\frac{1}{2} \sum_{j=0}^{n} (-1)^j L_{mj+k} & \frac{5}{2} \sum_{j=0}^{n} (-1)^j F_{mj+k} \\
\frac{1}{2} \sum_{j=0}^{n} (-1)^j F_{mj+k} & \frac{1}{2} \sum_{j=0}^{n} (-1)^j L_{mj+k}
\end{bmatrix} \quad \tag{2.9}
\]

Taking \( d = 1 + (-1)^m + L_m \), we get

\[
(I + S^m)^{-1} = \frac{1}{d} \begin{bmatrix}
1 + L_m/2 & -5F_m/2 \\
-F_m/2 & 1 + L_m/2
\end{bmatrix} = \frac{1}{d} \begin{bmatrix}
(1 + L_m/2) I - \frac{F_m}{2} K
\end{bmatrix}.
\]
Thus it is seen that
\[
(I + S^m)^{-1} (S^k + S^{mn+m+k}) = \frac{1}{d} \left( 1 + \frac{L_m}{2} \right) I - \frac{F_m}{2} K \right) (S^k + S^{mn+m+k})
\] (2.10)

By Corollary2 and Lemma2, it is seen that
\[
K (S^k + S^{mn+k}) = \begin{bmatrix} 5(F_k + F_{mn+m+k})/2 & 5(L_k + L_{mn+m+k})/2 \\ (L_k + L_{mn+m+k})/2 & 5(F_k + F_{mn+m+k})/2 \end{bmatrix}
\] (2.11)

Using (2.11) in (2.10), we get
\[
(I + S^m)^{-1} (S^k + S^{mn+m+k}) = \frac{1}{d} \left\{ \left( 1 + \frac{L_m}{2} \right) \begin{bmatrix} (L_k + L_{mn+m+k})/2 & 5(F_k + F_{mn+m+k})/2 \\ (F_k + F_{mn+m+k})/2 & (L_k + L_{mn+m+k})/2 \end{bmatrix} - \frac{F_m}{2} \begin{bmatrix} 5(F_k + F_{mn+m+k})/2 & 5(L_k + L_{mn+m+k})/2 \\ (L_k + L_{mn+m+k})/2 & 5(F_k + F_{mn+m+k})/2 \end{bmatrix} \right\}
\]

Then, by the identity (2.9), it follows that
\[
\sum_{j=0}^{n} (-1)^j L_{mj+k} = \frac{1}{d} \left( 1 + \frac{L_m}{2} \right) (L_k + L_{mn+m+k}) - \frac{5F_m}{2} (F_k + F_{mn+m+k})
\]

\[
= \frac{1}{d} \left[ L_k + L_{mn+m+k} + \frac{(L_m L_k - 5F_m F_k)(L_m L_{mn+m+k} - 5F_m F_{mn+m+k})}{2} \right].
\]

By Lemma5 and the fact that \(d = 1 + (-1)^m + L_m\), we obtain
\[
\sum_{j=0}^{n} (-1)^j L_{mj+k} = \frac{L_k + L_{mn+m+k} + (-1)^m (L_{mn+k} + L_{k-m})}{1 + (-1)^m + L_m}
\] (2.12)

Similarly it can be easily seen that
\[
\sum_{j=0}^{n} (-1)^j F_{mj+k} = \frac{F_k + F_{mn+m+k} + (-1)^m (F_{mn+k} + F_{k-m})}{1 + (-1)^m + L_m}
\] (2.13)

by Lemma5.

Now assume that \(n\) is an odd natural number. Hence we get
\[
\sum_{j=0}^{n} (-1)^j L_{mj+k} = \sum_{j=0}^{n-1} (-1)^j L_{mj+k} - L_{mn+k}
\] (2.14)
Since $n$ is an odd natural number then $n - 1$ is even. Thus taking $n - 1$ in (2.12) and using it in (2.14), it follows that

$$\sum_{j=0}^{n} (-1)^{j} L_{mj+k} = \frac{L_k + L_{mn+k} + (-1)^m (L_{mn+k-m} + L_{k-m})}{1 + (-1)^m + L_m} - L_{mn+k}$$

$$= \frac{L_k + (-1)^m (L_{mn+k-m} + L_{k-m}) - (-1)^m L_{mn+k} - L_m L_{mn+k}}{1 + (-1)^m + L_m} \tag{2.15}$$

Using Lemma 6 in (2.15), we get

$$\sum_{j=0}^{n} (-1)^{j} L_{mj+k} = \frac{L_k - L_{mn+m+k} + (-1)^m (L_{k-m} - L_{mn+k})}{1 + (-1)^m + L_m}.$$ 

In a similar way, it can be seen that

$$\sum_{j=0}^{n} (-1)^{j} F_{mj+k} = \sum_{j=0}^{n-1} (-1)^{j} F_{mj+k} - F_{mn+k} \tag{2.16}$$

Then using (2.13) in (2.16), it follows that

$$\sum_{j=0}^{n} (-1)^{j} F_{mj+k} = \frac{F_k + F_{mn+k} + (-1)^m (F_{mn+k-m} + F_{k-m})}{1 + (-1)^m + L_m} - F_{mn+k}$$

$$= \frac{F_k + (-1)^m (F_{mn+km} + F_{k-m}) - (-1)^m F_{mn+k} - L_m F_{mn+k}}{1 + (-1)^m + L_m}.$$ 

Thus by Lemma 6, we obtain the expected formula

$$\sum_{j=0}^{n} (-1)^{j} F_{mj+k} = \frac{F_k - F_{mn+m+k} + (-1)^m (F_{k-m} - F_{mn+k})}{1 + (-1)^m + L_m} \blacksquare$$

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References


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