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Oscillating Criteria on Second Order Non-linear Differential Equations with Distributed Deviating Arguments and Delay Terms

Nina Shang[1], Youmin Lu[2] and Fuyi Xu[1]

[1]School of Science, Shandong University of Technology
Zibo, Shandong, 255049, P.R. China

[2]Department of Mathematics and Computer Science
Bloomsburg University, Bloomsburg, PA 17815, USA

Abstract

In this paper, we study the following nonlinear second order differential equation with distributed deviating argument and delay term:

$$\left(r(t)\psi(x(t)\chi((x(t)+p(t)x(h(t)))')\right)' + \int_a^b q(t,\xi)f(x(g(t,\xi)))d\xi = 0$$

and establish several sufficient conditions for this equation to be oscillatory. Under certain simple conditions, some of these sufficient conditions are also proved to be necessary and therefore extend some existing criteria.

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1 Introduction

In recent years, more and more attentions are paid to the oscillation criteria of nonlinear second order differential equations with distributed deviating arguments

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[2]E-mail addresses: qinhz_000@163.com(H. Qin), xfy_02@163.com(F. Xu).

and delay term. In [1, 2], the authors consider the non-linear differential equation with distributed deviating argument and delay term

$$[r(t)\psi(x(t))(x(t)+p(t)x(t-\tau))']' + \int_a^b q(t,\xi)f(x(g(t,\xi)))d\xi = 0, \tag{1.1}$$

where $0 \le p(t) \le 1$. We notice that most studies on oscillation criteria of non-linear differential equations use Riccati transformation and integral average to obtain some sufficient oscillation conditions. These authors use an H function:

$$H(x,x)=0,\ H(x,y)>0,\ h(x,y) = -\frac{\partial H(x,y)}{\partial y} \ge 0, \text{ for } (x,y)\in D, \tag{(H)}$$

where $D = \{(x,y)|x \ge y \ge x_0\}$. The criteria obtained by using these methods are very strong and sometimes are very complicated.

Jan Seman considers a simple equation in [5], but the results obtained are very simple and can be verified easily. In this paper, we apply a similar method to the following non-linear second order differential equation

$$[r(t)\psi(x(t))\chi(z'(t))]' + \int_a^b q(t,\xi)f(x(g(t,\xi)))d\xi = 0, \tag{1.2}$$

where $z(t) = x(t)+p(t)x(h(t))$. If $a=0$, $b=1$, and $g(t,\xi)$ and $q(t,\xi)$ are independent of ξ, equation (1.2) degenerates into the following equation.

$$[r(t)\psi(x(t))\chi(z'(t))]' + q(t)f(x(g(t))) = 0. \tag{1.3}$$

It is clear that equation (1.2) is a generalization of (1.1) and equation of (1.3) is a generalization of (1.3). We also generalize the conditions on functions $p(t)$, $\psi(x)$, $f(x)$ and $g(t,\xi)$.

Definition 1.1. If there exists constant $x_0 > 0$ such that $0 < \int_{\pm x_0}^{\pm\infty} \chi^{-1}(\frac{\psi(x)}{f(x)})dx < \infty$ ($0 < \int_0^{\pm x_0} \chi^{-1}(\frac{\psi(x)}{f(x)})dx < \infty$), we say that equation (1.5) is superlinear (sublinear).

In this paper, we assume that equation (1.2) satisfies the following conditions:

Condition 1 $f(x) \in C^1(R,R)$; $f'(x) \ge 0$ for all $x \in R$; $f(x)$ is an odd function with $f(x) > 0$ for all $x > 0$; $f(|xy|) \ge Cf(|x|)f(|y|)$ for some constant $C > 0$ and all $x, y \in R$; $\psi(x) \in C^1(R,R)$, and $\psi(x) > 0$ for $x \ne 0$.

Condition 2 $q, g \in C(I\times[a,b], R^+)$, p, h, $r \in C^1(I, R^+)$ for $I = [t_0,\infty)$; $0 \le p(t) < 1$, $h(t) \le t$ for all $t \in I$; there exists function $\widetilde{g}(t)$ such that $0 < \widetilde{g}(t) \le g(t,\xi) \le t$ and $\widetilde{g}'(t) \ge 0$ for all $t \in R$ and $\xi \in [a,b]$; $\lim_{t\to\infty} \widetilde{g}(t) = \lim_{t\to\infty} h(t) = \infty$.

Condition 3 χ is an odd function such that $\chi'(x) > 0$, $l_\chi\chi^{-1}(|x|)\chi^{-1}(|y|) \le \chi^{-1}(|xy|) \le L_\chi\chi^{-1}(|x|)\chi^{-1}(|y|)$ for some positive constants l_χ and L_χ and all $x, y \in R$; $\lim_{t\to\infty} R(t) = \infty$ for $R(t) = \int_{t_0}^t \chi^{-1}(\frac{1}{r(s)})ds$.

We say that a solution $x(t)$ of equation (1.2) is oscillatory if the set of all zeros of $x(t)$ is not bounded. We say that equation (1.2) is oscillatory if all its solutions are oscillatory.

2 Lemmas

If there exists constant $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \in [T_0, \infty)$, then, there exists $T_1 \geq T_0$ such that $\tilde{g}(t) \geq T_0$ and $h(t) \geq T_0$ for all $t \geq T_1$. Thus, $x(h(t)) \neq 0$, $x(\tilde{g}(t)) \neq 0$, and $x(g(t,\xi)) \neq 0$ for all $t \geq T_1$ and $\xi \in [a,b]$. To prepare for the proofs of our theorems, we first prove a few lemmas.

Lemma 2.1. Assume that $x = x(t)$ is a solution of equation (1.2). If there exists $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T_0$, then there exists $T \geq T_0$ such that, for $t \geq T$,

$$z(t)\chi(z'(t)) \geq 0, \ z(t)(r(t)\psi(x(t))\chi(z'(t)))' < 0, \tag{2.1}$$

$$|r(t)\psi(x(t))\chi(z'(t))| \geq \int_t^\infty (\int_a^b q(s,\xi)\,|f(x(g(s,\xi)))|\,d\xi)ds, \tag{2.2}$$

and

$$\int_T^\infty (\int_a^b q(s,\xi)\,|f(x(g(s,\xi)))|\,d\xi)ds < \infty, \tag{2.3}$$

where $z(t) = x(t) + p(t)x(h(t))$.

Proof. Without loss of generality, we assume that $x = x(t)$ is a positive solution of (1.5). Then, there exists constant $T \geq T_0 \geq t_0$ such that $x(h(t)) > 0$ and $x(g(t,\xi)) > 0$ for all $t \geq T$. It follows form equation (1.5) that, for $t \geq T$,

$$(r(t)\psi(x(t))\chi(z'(t)))' = -\int_a^b q(t,\xi)f(x(g(t,\xi)))d\xi < 0. \tag{2.4}$$

Thus, $\psi(x(t))r(t)\chi(z'(t))$ is decreasing and

$$r(t)\psi(x(t))\chi(z'(t)) \leq r(T)\psi(x(T))\chi(z'(T)) \text{ as } t \geq T. \tag{2.5}$$

We need to prove that $z'(t) \geq 0$ for $t \geq T$. For the sake of contradiction, we assume that $z'(T) < 0$. Then, $z'(t) < 0$ and $\chi(z'(t)) < 0$ for $t \geq T$ by (2.5). Hence, $z(t)$ and $x(t)$ are bounded over $[T, \infty)$. Let X be a constant such that $0 < x(t) < X$ for all $t \geq T$ and $L^{-1} = \max_{x\in[0,X]} \psi(x)$. By (2.5), one has

$$\begin{aligned} z'(t) &\leq \chi^{-1}\left(\frac{r(T)\psi(x(T))\chi(z'(T))}{r(t)\psi(x(t))}\right) \\ &\leq -l_\chi \chi^{-1}(Lr(T)\psi(x(T))|\chi(z'(T))|)\chi^{-1}(\frac{1}{r(t)}) < 0, \text{ for } t \geq T. \end{aligned}$$

Integrating both sides from T to $t(\geq T)$, we have

$$z(t) \leq z(T) - l_\chi \chi^{-1}(Lr(T)\psi(x(T))z'(T)|)\int_T^t \chi^{-1}(\frac{1}{r(s)})ds \to -\infty \text{ as } t \to \infty.$$

This is clearly a contradiction to $z(t) = x(t) + p(t)x(h(t)) > 0$.

Now, we integrate both sides of equation (1.2) from $t(\geq T)$ to $s(\geq t)$ and obtain

$$r(s)\psi(x(s))\chi(z'(s)) + \int_t^s \int_a^b q(\tau,\xi)f(x(g(\tau,\xi)))d\xi d\tau = r(t)\psi(x(t))\chi(z'(t)).$$

Since $r(s)\psi(x(s))\chi(z'(s)) > 0$, we have

$$\int_t^s \int_a^b q(\tau,\xi)f(x(g(\tau,\xi)))d\xi d\tau < r(t)\psi(x(t))\chi(z'(t)) \text{ for all } s \geq t \geq T.$$

Let $s \to \infty$, we obtain (2.2) and (2.3) and complete the proof of the lemma.

Lemma 2.2. If $x = x(t) \neq 0$ is a solution of equation (1.2), then there exists constant $T \geq T_0$ such that

$$1 \leq \frac{z(t)}{x(t)} \leq \frac{1}{\theta(t)} \text{ for all } t \geq T, \tag{2.5}$$

where $\theta(t) = 1 - p(t)$.

Proof Without loss of generality, we still assume that $x = x(t) > 0$ for all $t \geq T_0$. By Lemma 2.1, there exists constant $T \geq T_0$ such that $z = z(t)$ is increasing for all $t \geq T$. Hence,

$$z(t) = x(t) + p(t)x(h(t)) < x(t) + p(t)z(h(t)) \leq x(t) + p(t)z(t).$$

Moving $p(t)z(t)$ to the left side and dividing both sides by $1 - p(t)$,we complete the proof of the lemma.

3 Main Theorems and Corresponding Proofs

Assume that $x = x(t)$ is a positive solution of equation (1.2) for $t \geq T_0$. Then, by Lemma 2.1, there exists a constant $T \geq t_0$ such that $z(t)$ is positive and increasing for $t \geq T$, and $v(t) = r(t)\psi(x(t))\chi(z'(t))$ is decreasing for $t \geq T$.

Theorem 3.1. If equation (1.2) satisfies

$$\int_{t_0}^{\infty} \int_a^b q(s,\xi)f(\theta(g(s,\xi)))d\xi ds = \infty, \tag{3.1}$$

where $\theta(t) = 1 - p(t)$, then, it is oscillatory.

Proof . For the sake of contradiction, we assume that $x = x(t)$ is a positive solution of equation (1.2) for $t \geq T_0$. Then, from condition 1, Lemma 2.2 and inequality (2.2) of Lemma 2.1, we have

$$\begin{aligned} r(t)\psi(x(t))\chi(z'(t)) &\geq \textstyle\int_t^\infty \int_a^b q(s,\xi)f(x(g(s,\xi)))d\xi ds \\ &\geq \textstyle\int_t^\infty \int_a^b q(s,\xi)f(\theta(g(s,\xi))z(g(s,\xi)))d\xi ds \\ &\geq Cf(z(\widetilde{g}(T))) \textstyle\int_t^\infty \int_a^b q(s,\xi)f(\theta(g(s,\xi)))d\xi ds, \end{aligned} \tag{3.2}$$

for $t \geq T$. This is a contradiction to (3.1). We can similarly show that equation (1.2) can not have a negative solution under condition (3.1) and complete the proof of the theorem.

Now, we deal with equation (1.2) that does not satisfy condition (3.1).

Theorem 3.2. If equation (1.2) satisfies the following conditions

(A). There exist constants $x_0 > 0$ and $0 < \alpha < 1$ such that

$$0 < \int_{\pm x_0}^{\pm\infty} \chi^{-1}\left(\frac{\psi^\alpha(x)}{f(x)}\right) dx < \infty; \tag{3.3}$$

(B). There exist an increasing function f_ψ and a constant M_ψ such that $f_\psi(x) > 0$ for all $x > 0$, $\psi(\lambda x) \geq f_\psi(\frac{1}{\lambda})\psi(x)$ for all $\lambda > 1$, and $\psi(x) \leq M_\psi$ for all $x \in R$;

(C). $\int_{t_0}^{\infty} \chi^{-1}(\frac{1}{r(t)})\chi^{-1}(\int_{\widetilde{g}^{-1}(t)}^{\infty} \int_a^b q(s,\xi) f(\theta(g(s,\xi)))d\xi ds)dt = \infty$,

then equation (1.2) is oscillatory.

Proof . For contradiction, we assume that equation (1.2) satisfies conditions (A) and (B) and $x = x(t)$ is a positive solution of equation (1.2) for $t \geq T \geq t_0$ without loss of generality. Firstly, we notice that $\chi^{-1}(x)$ is an increasing function since $\chi(x)$ is increasing by condition 3. Then, noticing that $\widetilde{g}^{-1}(s) \geq s$, we have

$$\begin{aligned}
\infty &> \int_{z(T)}^{\infty} \chi^{-1}\left(\frac{\psi^\alpha(z)}{f(z)}\right) dz \geq \int_{z(T)}^{z(t)} \chi^{-1}\left(\frac{\psi^\alpha(z)}{f(z)}\right) dz = \int_T^t \chi^{-1}\left(\frac{\psi^\alpha(z(s))}{f(z(s))}\right) z'(s)ds \\
&\geq \int_T^t \chi^{-1}\left(\frac{\psi^\alpha(\frac{z(s)}{x(s)}x(s))}{f(z(s))}\right) z'(s)ds \geq \int_T^t \chi^{-1}\left(\frac{f_\psi^\alpha(\frac{x(s)}{z(s)})\psi^\alpha(x(s))}{f(z(s))}\right) \chi^{-1}(\chi(z'(s))))ds \\
&\geq L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{f_\psi^\alpha(\theta(s))\psi^\alpha(x(s))\chi(z'(s))}{f(z(s))}\right) ds = L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{f_\psi^\alpha(\theta(s))r(s)\psi(x(s))\chi(z'(s))}{r(s)\psi^{1-\alpha}(x(s))f(z(s))}\right) ds \\
&\geq L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{f_\psi^\alpha(\theta(s))r(s)\psi(x(s))\chi(z'(s))}{r(s)M_\psi^{1-\alpha}f(z(s))}\right) ds \\
&\geq L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{f_\psi^\alpha(\theta(s))}{r(s)M_\psi^{1-\alpha}f(z(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(x(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{f_\psi^\alpha(\theta(s))}{r(s)M_\psi^{1-\alpha}f(z(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi))z(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{Cf_\psi^\alpha(\theta(s))}{r(s)M_\psi^{1-\alpha}f(z(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi)))f(z(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{Cf_\psi^\alpha(\theta(s))}{r(s)M_\psi^{1-\alpha}f(z(s))} \int_{\widetilde{g}^{-1}(s)}^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi)))f(z(\widetilde{g}(\tau)))d\xi d\tau\right) ds \\
&\geq L_\chi^{-1} \int_T^t \chi^{-1}\left(\frac{Cf_\psi^\alpha(\theta(s))}{r(s)M_\psi^{1-\alpha}f(z(s))} \int_{\widetilde{g}^{-1}(s)}^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi)))f(z(s))d\xi d\tau\right) ds \\
&\geq l_\chi^2 L_\chi^{-1}\chi^{-1}(\frac{C}{M_\psi^{1-\alpha}}) \int_T^t \chi^{-1}\left(\frac{1}{r(s)}\right) \chi^{-1}\left(f_\psi^\alpha(\theta(s)) \int_{\widetilde{g}^{-1}(s)}^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi)))d\xi d\tau\right) ds.
\end{aligned}$$

This is a contradiction to condition of the theorem and complete the proof.

Theorem 3.3. If equation (1.2) is sublinear and satisfies one of the following conditions, then it is oscillatory.

(D). $\psi(x)$ is non-decreasing, $G(x) = \psi(x)\chi(x)$ is invertible, $F(x) = f(G^{-1}(x))$ is a non-decreasing function, $|F(xy)| \geq C_F\,|F(x)|\,|F(y)|$ with $C_F > 0$, $\chi^{-1}\left(\frac{\psi(\nu)}{f(\nu)}\right)F(\nu) \geq \epsilon(\nu_0) > 0$ for $\nu \in (0, \nu_0]$, and

$$\int_{t_0}^{\infty} F(\chi(R(\widetilde{g}(t)))) \int_a^b q(t,\xi) f(\theta(g(t,\xi)))d\xi dt = \infty; \tag{3.4}$$

(E). $\psi(x) \leq M_\psi$, $\chi^{-1}\left(\dfrac{\psi(\nu)}{f(\nu)}\right) f(\chi^{-1}(\nu)) \geq \epsilon(\nu_0) > 0$ for $\nu \in (0, \nu_0]$, and

$$\int_{t_0}^{\infty} f(R(\widetilde{g}(t))) \int_a^b q(t,\xi) f(\theta(g(t,\xi)))d\xi dt = \infty.$$

Proof . Without loss of generality, we still assume that $x = x(t)$, $t \geq T_0$, is a positive solution of equation (1.2) for contradiction. Then, $\nu(t) = r(t)\psi(x(t))\chi(z'(t))$ is decreasing for $t \geq T_0$. Now, one has

$$\begin{aligned}
\infty \;&> \textstyle\int_0^{\nu(T)} \chi^{-1}\left(\frac{\psi(\nu)}{f(\nu)}\right) d\nu \geq \int_{\nu(t)}^{\nu(T)} \chi^{-1}\left(\frac{\psi(\nu)}{f(\nu)}\right) d\nu \\
&= -\textstyle\int_T^t \chi^{-1}\left(\frac{\psi(\nu(s))}{f(\nu(s))}\right) \nu'(s)ds \\
&= \textstyle\int_T^t \chi^{-1}\left(\frac{\psi(\nu(s))}{f(\nu(s))}\right) \int_a^b q(s,\xi) f(x(g(s,\xi)))d\xi ds \\
&\geq \textstyle\int_T^t \chi^{-1}\left(\frac{\psi(\nu(s))}{f(\nu(s))}\right) \int_a^b q(s,\xi) f(\theta(g(s,\xi))z(g(s,\xi)))d\xi ds \\
&\geq C\textstyle\int_T^t \chi^{-1}\left(\frac{\psi(\nu(s))}{f(\nu(s))}\right) f(z(\widetilde{g}(s)) \int_a^b q(s,\xi) f(\theta(g(s,\xi))))d\xi ds,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
z(t) \;&\geq \textstyle\int_{T_0}^t z'(s)ds = \int_{T_0}^t \chi^{-1}(\frac{\nu(s)}{r(s)\psi(x(s))})ds \\
&\geq l_\chi^2 \chi^{-1}(\nu(t)) \textstyle\int_{T_0}^t \chi^{-1}\left(\frac{1}{r(s)}\right) \chi^{-1}\left(\frac{1}{\psi(x(s))}\right) ds.
\end{aligned} \tag{3.6}$$

By Condition 3, we have $\chi^{-1}(1) = \chi^{-1}\left(\frac{\psi(x(s))}{\psi(x(s))}\right) \leq L_\chi \chi^{-1}(\psi(x(s)))\chi^{-1}\left(\frac{1}{\psi(x(s))}\right)$. Thus, (3.6) becomes

$$z(t) \geq l_\chi^2 \chi^{-1}(\nu(t))\chi^{-1}(1) \int_{T_0}^t \chi^{-1}\left(\frac{1}{r(s)}\right) \frac{1}{L_\chi \chi^{-1}(\psi(x(s)))} ds. \tag{3.7}$$

Now, we consider each of the two conditions of the theorem separately.

Assume that condition (D) of the theorem is satisfied. Since $\psi(x)$ is non-decreasing in this case, (3.7) can be carried on one more step:

$$\begin{aligned}
z(t) \;&\geq \tfrac{l_\chi^2 \chi^{-1}(\nu(t))\chi^{-1}(1)}{L_\chi \chi^{-1}(\psi(z(t)))} \textstyle\int_{T_0}^t \chi^{-1}\left(\frac{1}{r(s)}\right) ds \\
&= \tfrac{l_\chi^2 \chi^{-1}(\nu(t))\chi^{-1}(1)}{L_\chi \chi^{-1}(\psi(z(t)))} (R(t) - R(T_0)) \\
&\geq \tfrac{l_\chi^2 \chi^{-1}(\nu(t))\chi^{-1}(1) R(t)}{2L_\chi \chi^{-1}(\psi(z(t)))},
\end{aligned} \tag{3.8}$$

for large t.

Thus,

$$\begin{aligned}\psi(z(t))\chi(z(t)) &\geq \chi(l_\chi\chi^{-1}(\psi(z(t)))z(t)) \geq \chi(\tfrac{l_\chi^3\chi^{-1}(1)\chi^{-1}(\nu(t))R(t)}{2L_\chi})\\ &= \chi\left(L_\chi\chi^{-1}(\nu(t))\chi^{-1}\chi(\tfrac{l_\chi^3\chi^{-1}(1)R(t)}{2L_\chi^2})\right)\\ &\geq \nu(t)\chi(\tfrac{l_\chi^3\chi^{-1}(1)R(t)}{2L_\chi^2}),\end{aligned}$$

and

$$\begin{aligned}f(z(\widetilde{g}(t))) &= f(G^{-1}(G(z(\widetilde{g}(t))))) = F(\psi(z(\widetilde{g}(t)))\chi(z(\widetilde{g}(t)))\\ &\geq F\left(\nu(\widetilde{g}(t))\chi(\tfrac{l_\chi^3\chi^{-1}(1)R(\widetilde{g}(t))}{2L_\chi^2})\right)\\ &\geq C_F F(\nu(\widetilde{g}(t)))F(\chi(\tfrac{l_\chi^3\chi^{-1}(1)R(\widetilde{g}(t))}{2L_\chi^2}))\\ &\geq C_F F(\nu(t))F(\chi(\tfrac{l_\chi^3\chi^{-1}(1)R(\widetilde{g}(t))}{2L_\chi^2})).\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.5), we have

$$\begin{aligned}\infty &> CC_F\int_T^t \chi^{-1}\left(\tfrac{\psi(\nu(s))}{f(\nu(s))}\right)F(\nu(s))F(\chi(\tfrac{l_\chi^3\chi^{-1}(1)R(\widetilde{g}(s))}{2L_\chi^2}))\int_a^b q(s,\xi)f(\theta(g(s,\xi))))d\xi ds\\ &\geq CC_F^2F(\chi(\tfrac{l_\chi^3\chi^{-1}(1)}{2L_\chi^2}))\epsilon(\nu(T))\int_T^t F(R(\widetilde{g}(s)))\int_a^b q(s,\xi)f(\theta(g(s,\xi))))d\xi ds.\end{aligned}$$

This is a contradiction to condition (3.4).

Assume that condition (E) of the theorem is satisfied. By (3.7), one has

$$\begin{aligned}z(\widetilde{g}(t)) &\geq l_\chi^2\chi^{-1}(\nu(\widetilde{g}(t)))\chi^{-1}(1)\int_{T_0}^{\widetilde{g}(t)}\chi^{-1}\left(\tfrac{1}{r(s)}\right)\tfrac{1}{L_\chi\chi^{-1}(\psi(x(s)))}ds\\ &\geq \tfrac{l_\chi^2\chi^{-1}(\nu(\widetilde{g}(t)))\chi^{-1}(1)}{L_\chi\chi^{-1}(\psi(M_\psi))}\int_{T_0}^{\widetilde{g}(t)}\chi^{-1}\left(\tfrac{1}{r(s)}\right)ds\\ &= \tfrac{l_\chi^2\chi^{-1}(\nu(\widetilde{g}(t)))\chi^{-1}(1)}{L_\chi\chi^{-1}(\psi(M_\psi))}(R(\widetilde{g}(t))-R(T_0))\\ &\geq \tfrac{l_\chi^2\chi^{-1}(\nu(t))\chi^{-1}(1)}{L_\chi\chi^{-1}(\psi(M_\psi))}(R(\widetilde{g}(t))-R(T_0))\\ &\geq \tfrac{l_\chi^2\chi^{-1}(\nu(t))\chi^{-1}(1)R(\widetilde{g}(t))}{2L_\chi\chi^{-1}(\psi(M_\psi))},\end{aligned} \tag{3.10}$$

for large t. Substituting (3.10) back to (3.5), one has

$$\begin{aligned}\infty &> C\int_T^t \chi^{-1}\left(\tfrac{\psi(\nu(s))}{f(\nu(s))}\right)f(\tfrac{l_\chi^2\chi^{-1}(\nu(s))\chi^{-1}(1)R(\widetilde{g}(s))}{2L_\chi\chi^{-1}(\psi(M_\psi))})\int_a^b q(s,\xi)f(\theta(g(s,\xi))))d\xi ds\\ &\geq C^3f(\tfrac{l_\chi^2\chi^{-1}(1)}{2L_\chi\chi^{-1}(\psi(M_\psi))})\int_T^t\chi^{-1}\left(\tfrac{\psi(\nu(s))}{f(\nu(s))}\right)f(\chi^{-1}(\nu(s)))f(R(\widetilde{g}(s)))\int_a^b q(s,\xi)f(\theta(g(s,\xi))))d\xi ds\\ &\geq C^3\epsilon(\nu(T))f(\tfrac{l_\chi^2\chi^{-1}(1)}{2L_\chi\chi^{-1}(\psi(M_\psi))})\int_T^t f(R(\widetilde{g}(s)))\int_a^b q(s,\xi)f(\theta(g(s,\xi))))d\xi ds.\end{aligned}$$

This is a contradiction to the equation in condition (E) of the theorem and completes the proof of the theorem.

Theorem 3.4. If there exists function $\varphi(x)$ and constant $n_\varphi > 0$ such that $f(x) \geq \chi(x)\psi(x)\varphi(x)$, $\varphi(x) \geq n_\varphi$, $\psi(x)$ is non-decreasing function, and

$$(R(t) - R(T_0))\chi^{-1}(\int_{\widetilde{g}^{-1}(t)}^{\infty} \int_a^b q(s,\xi) f(\theta(g(s,\xi)))d\xi ds > \frac{1}{l_\chi^3 \chi^{-1}(Cn_\varphi)}, \tag{3.11}$$

then, equation (1.2) is oscillatory

Proof . We assume that $x = x(t)$ is a positive solution of equation (1.2). Then, one has

$$\begin{aligned}
z(t) &\geq \int_{T_0}^t z'(s)ds = \int_{T_0}^t \chi^{-1}\left(\tfrac{r(s)\psi(x(s))\chi(z'(s))}{r(s)\psi(x(s))}\right) ds \geq \int_{T_0}^t \chi^{-1}\left(\tfrac{r(s)\psi(x(s))\chi(z'(s))}{r(s)\psi(z(s))}\right) ds \\
&\geq \int_{T_0}^t \chi^{-1}\left(\tfrac{1}{r(s)\psi(z(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(x(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq \int_{T_0}^t \chi^{-1}\left(\tfrac{C}{r(s)\psi(z(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(z(g(\tau,\xi)))f(\theta(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq \int_{T_0}^t \chi^{-1}\left(\tfrac{C}{r(s)\psi(z(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(z(\widetilde{g}(\tau)))f(\theta(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq \int_{T_0}^t \chi^{-1}\left(\tfrac{C}{r(s)\psi(z(s))} \int_t^\infty \int_a^b q(\tau,\xi) f(z(\widetilde{g}(\tau)))f(\theta(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq \int_{T_0}^t \chi^{-1}\left(\tfrac{C}{r(s)\psi(z(s))} \int_{\widetilde{g}^{-1}(t)}^\infty \int_a^b q(\tau,\xi) f(z(\widetilde{g}(\tau)))f(\theta(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq \int_{T_0}^t \chi^{-1}\left(\tfrac{Cf(z(t))}{r(s)\psi(z(s))} \int_{\widetilde{g}^{-1}(t)}^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq \int_{T_0}^t \chi^{-1}\left(\tfrac{Cn_\varphi\chi(z(t))\psi(z(t))}{r(s)\psi(z(s))} \int_{\widetilde{g}^{-1}(t)}^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi)))d\xi d\tau\right) ds \\
&\geq l_\chi^3 z(t)\chi(Cn_\varphi)(R(t) - R(T_0))\chi^{-1}(\int_{\widetilde{g}^{-1}(t)}^\infty \int_a^b q(\tau,\xi) f(\theta(g(\tau,\xi)))d\xi d\tau).
\end{aligned}$$

This is a contradiction to (3.11). It is similar to prove that equation (1.2) cannot have a negative solution.

In the following theorem, we establish some necessary conditions for equation (1.2) to be oscillatory.

Theorem 3.5. Assume that equation (1.2) is superlinear and $p(t) = 0$. If $\psi(x)$ and $\psi(-x)$ are decreasing functions over $[0, 1]$, then a necessary condition for equation (1.2) to be oscillatory is

$$\int_{t_0}^\infty \chi^{-1}\left(\frac{1}{r(t)} \int_t^\infty \int_a^b q(\tau,\xi)d\xi d\tau\right) dt = \infty. \tag{3.12}$$

Proof . We prove it by contradiction. Assume that (3.12) is not satisfied. Then, there exists constant $T > t_0$ such that

$$\int_{t_0}^\infty \chi^{-1}\left(\frac{1}{r(t)}\right) \chi^{-1}\left(\int_t^\infty \int_a^b q(\tau,\xi)d\xi d\tau\right) dt \leq \frac{1}{4L_\chi \chi^{-1}(\frac{f(1)}{\psi(1)})}. \tag{3.13}$$

We define a sequence $\{x_n(t)\}_{n=1}^\infty$ by using

$$x_0(t) = 1,$$

$$x_{n+1}(t) = \frac{1}{4} + \int_T^t \chi^{-1}\left(\frac{1}{r(s)\psi(x_n(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(x_n(g(\tau,\xi)))d\xi d\tau\right) ds,\ n = 0, 1, 2, \cdots, \tag{3.14}$$

By mathematical induction, it is easy to show that $x_n(t)$ is a monotonously decreasing sequence with $\frac{1}{4} \le x_n(t) \le 1$ for all $n \ge 0$. Hence, $x(t) = \lim_{n\to\infty} x_n(t)$ exists. By using Lebesgue dominated convergence theorem [6], we have

$$x(t) = \int_{t_0}^t \chi^{-1}\left(\frac{1}{r(s)\psi(x(s))} \int_t^\infty \int_a^b q(\tau,\xi) f(x(g(\tau,\xi)))d\xi d\tau\right) dt,$$

which is a positive solution of equation (1.2) and complete the proof of the Theorem.

Theorem 3.6. Assume that equation (1.2) is sublinear and $p(t) = 0$. If $\psi(x)$ and $\psi(-x)$ are decreasing functions over $[1, \infty)$, a necessary condition of equation (1.2) to be oscillatory is

$$\int_{t_0}^\infty \int_a^b q(s,\xi) f(R(g(s,\xi)))d\xi ds = \infty. \tag{3.15}$$

Proof . For the sake of contradiction, we assume that (3.15) is not true. Then, there exists constant $T \ge t_0$ such that

$$\int_T^\infty \int_a^b q(s,\xi) f(R(g(s,\xi)))d\xi ds \le C_\psi. \tag{3.16}$$

We can construct a sequence $\{x_n(t)\}_{n=0}^\infty$ by

$$\begin{aligned} x_0(t) &= R(t), \\ x_{n+1}(t) &= R(T) + \int_T^t \chi^{-1}\left(\frac{1}{r(s)\psi(x_n(s))} \int_s^\infty \int_a^b q(\tau,\xi) f(x_n(g(\tau,\xi)))d\xi d\tau\right) ds, \end{aligned} \tag{3.17}$$

for $n = 0, 1, 2, \cdots$ and complete the proof similarly.

4 Examples

Examples 4.1. Assume $\beta > 0$, $0 < d \le 1$, $\gamma \le 1$, $a \ge 0$, $r_1 \ge 0$, $r_2 \ge 0$, $r_1^2 + r_2^2 > 0$, and $\beta_3 > 0$. We consider the following equation

$$\left(\frac{(r_1 + r_2 t^\gamma) z'(t)}{1 + x^2(t)}\right)' + \int_0^1 \frac{a(t+\xi-\delta) + a_1}{t^{\beta_1}(t+\xi-\delta)^{\beta_2}} \xi^{\beta_3} f(x(g(t,\xi)))d\xi = 0, \text{for } t \ge t_0 \tag{4.1}$$

where $z(t) = x(t) + p(t)x(d(t-\sigma))$, $p(t) = p_0 + p_1 t^{-\nu}$, $p_0 \ge |p_1|$, $\nu > 0$, t_0 is an appropriate constant.

Corresponding to equation (1.2), we have here $r(t) = r_1 + r_2 t^\gamma$, $h(t) = d(t-\sigma)$, $\psi(x) = \dfrac{1}{1+x^2}$, $\chi(x) = x$, and $q(t,\xi) = \dfrac{a(t+\xi-\delta) + a_1}{t^{\beta_1}(t+\xi-\delta)^{\beta_2}} \xi^{\beta_3}$. Clearly, functions $r(t)$,

$\psi(x)$, $\chi(x)$, $q(t,\xi)$, $p(t)$, and $h(t)$ satisfy Conditions 1, 2 and 3 with $L_\chi = l_\chi = 1$. We apply Theorem 3.2 to this example. We clearly have that $\theta(t) = 1 - p_0 - p_1 t^{-\nu}$ and $f_\psi(x) = x^2$. Let $f(x) = |x|^\beta sgn(x)$. It clearly satisfies Condition 1 with $C = 1$. If $\beta + 2\alpha > 1$, condition $0 < \int_{\pm x_0}^{\pm\infty} \chi^{-1}\left(\dfrac{\psi^\alpha(x)}{f(x)}\right) dx < \infty$ is satisfied. If $g(t,\xi)$ is a function satisfying Conditions 1, 2 and 3, then condition (C) of Theorem 3.2 becomes

$$\int_{t_0}^{\infty} \frac{t^{-2\alpha\nu}}{r_1 + r_2 t^{-\gamma}} \int_{\widetilde{g}^{-1}(t)}^{\infty} \widetilde{g}^{-\beta\nu}(\tau) \left(\frac{a}{\tau^{\beta_1+\beta_2-1}} + \frac{a_1}{\tau^{\beta_1+\beta_2}}\right) d\tau dt = \infty, \text{ if } p_0 = 1 \text{ and } p_1 < 0,$$
$$\int_{t_0}^{\infty} \frac{t^{-2\alpha\nu}}{r_1 + r_2 t^{-\gamma}} \int_{\widetilde{g}^{-1}(t)}^{\infty} \left(\frac{a}{\tau^{\beta_1+\beta_2-1}} + \frac{a_1}{\tau^{\beta_1+\beta_2}}\right) d\tau dt = \infty, \text{ if } p_0 \neq 1. \tag{4.2}$$

We now can find more precise conditions for specially prescribed function $g(t,\xi)$.

Case 1. Let $g(t,\xi) = b(t + b_1 \sin\dfrac{b_2 t}{1+\xi} + \xi - \tau)$ and $\widetilde{g}(t) = b(t - |b_1| + 1 - \tau)$ with $0 < b \leq 1$. Then, $g(t,\xi)$ satisfies Conditions 1, 2 and 3, and equation (4.1) is oscillatory if one of the following eight conditions is satisfied.

1). $a > 0$, and $\varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 2$;
2). $a = 0$, $a_1 > 0$,and $\varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 1$;
3). $r_1 = 0$, $a > 0$, and $2\alpha\nu + \varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 3 - \gamma$;
4). $r_1 = 0$, $a = 0$, $a_1 > 0$,and $2\alpha\nu + \varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 2 - \gamma$;
5). $r_2 > 0$, $0 < \gamma < 1$, $a > 0$, and $2\alpha\nu + \varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 3 - \gamma$;
6). $r_2 > 0$, $0 < \gamma < 1$, $a = 0$, and $2\alpha\nu + \varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 2 - \gamma$;
7). $r_1 > 0$, $\gamma < 0$, $a > 0$,and $2\alpha\nu + \varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 3$;
8). $r_1 > 0$, $\gamma < 0$, $a = 0$,and $2\alpha\nu + \varepsilon(p_0,p_1)\nu\beta + \beta_1 + \beta_2 \leq 2$,

where

$$\varepsilon(p_0,p_1) = \begin{cases} 1, & \text{if } p_0 = 1 \text{ and } p_1 < 0, \\ 0, & \text{if } 0 \leq p_0 < 1, \end{cases}$$

Case 2. Let $g(t,\xi) = \sqrt{t+\xi}$ and $\widetilde{g}(t) = \sqrt{t}$. Then, $g(t,\xi)$ satisfies Conditions 1, 2 and 3, and equation (4.1) is oscillatory if one of the following above eight conditions is satisfied, where

$$\varepsilon(p_0,p_1) = \begin{cases} \dfrac{1}{2}, & \text{if } p_0 = 1 \text{and } p_1 < 0, \\ 0, & \text{if } 0 \leq p_0 < 1, \end{cases}$$

Examples 4.2. Assume that $\mu > 1$, $\beta > 0$, $0 < d \leq 1$, $\gamma \leq 1$, $a > 0$ and $\beta_2 > 0$. We consider the equation

$$\left(|x(t)|^\mu z'(x)\right)' + a\int_\epsilon^1 t^{\beta_1}\xi^{\beta_2-1}|x(t\xi)|^{\beta-1} x(t\xi) d\xi = 0,\ t \geq t_0 \tag{4.3}$$

where $z(t) = x(t) + p(t)x(d(t-\sigma))$, $p(t) = 1 - t^{-\nu}$, $\nu > 0$ and t_0 is an appropriate constant.

In this example, we have $r(t) = 1$, $\psi(x) = |x|^{\mu}, \chi(x) = x$, $q(t,\xi) = t^{\beta_1}\xi^{\beta_2-1}$, $f(x) = |x|^{\beta-1}x$, $g(t,\xi) = t\xi$ and $f_{\psi} = 1$. Now, we can apply different theorems to this example according different values of μ and β.

1). If $\beta > \mu+1$, $\beta_1 - \beta\nu > -1$ or $\beta_1 - \beta\nu - \nu > -1$ and $\beta_1 - \beta\nu \leq -1$, equation (4.3) is oscillatory by Theorem 3.2.

2). If $\beta < \mu + 1$, $0 < \int_0^{\pm x_0} \chi^{-1}\left(\frac{\psi(x)}{f(x)}\right) dx < \infty$. In this case, $G(x) = |x|^{\mu}x$, $G^{-1}(x) = |x|^{-\frac{\mu}{\mu+1}}x$, $F(x) = |x|^{\frac{\beta}{\mu+1}-1}x$. Thus, equation (4.3) is oscillatory by Theorem 3.3 if $\frac{\beta}{\mu+1} + \mu - \beta < 0$ and $\beta + \beta_1 - \nu > -1$.

3). If $\beta = \mu + 1$, then, $f(x) = \chi(x)\psi(x)$. By Theorem 3.4, equation (4.3) is oscillatory if either $\beta_1 - \beta\nu \geq -1$ or $\beta_1 - \beta\nu < -1$ and $\beta_1 - \beta\nu - \nu > -2$.

References

[1] Gaihua Gui, Zhiting Xu, Oscillation criteria for second-order neutral differential equations with distributed deviating arguments, Electron. J. Diff. Eqns., 10(2007), 1-11.

[2] Zhiting Xu, Peixuan Weng, Oscillation of second-order neutral differential equations with distributed argument, Journal of Computational and Applied Mathematics, Vol. 202(2)(2007), 460-477.

[3] Run Xu, Fanwei Meng, Some new oscillation criteria for second order quasi-linear neutral delay differential equations, Applied Mathematics and Computation, 182 (2006), 797-803.

[4] Run Xu, Fanwei Meng, Oscillation criteria for second order quasi-linear neutral delay differential equations, Applied Mathematics and Computation, (2007), doi:10.1016/j.amc.2007.01.108

[5] Jan Seman, On the oscillation of the second order nonlinear differential equation, Matematika Fyzika Technika 42(2003), 17-25

[6] Gabriel Klambauer, Real Analysis,Dover Publications,Pub. Date: December 2005,ISBN-13: 9780486445243

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