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Integral Boundary Value Problems for First Order Impulsive Differential Inclusions[1]

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Abstract

In this paper, we study the integral boundary value problem for first order impulsive differential inclusions.Using a fixed-point theorem for condensing multivalued maps , upper and lower solution method ,we establish several results.

Mathematics Subject Classifications: 34B37

Keywords: Convex valued multivalued map; Upper and lower solutions; Integral boundary conditon

1. Introduction

It is well-known that the theory of impulsive differential equations is not only richer than the corresponding theory of differential equations without impulses, but also represents a more natural framework for mathematical modeling of many real world phenomena .Example ,many real processes and phenomena studied in biology,population dynamics and mechanics are characterized by the fact at certain moments of their development the system parameters undergo rapid changes. The impulsive differential equations is a natural tool for the mathematical simulation of such processes and phenomena[1-8].In this paper, we consider the integral boundary value problem for first order impulsive differential inclusions:

$$\begin{aligned} &x'(t) \in F(t, x(t)),\ t \neq t_k,\ t \in J = [0, T], \\ &\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m, \\ &x(0) + \mu \int_0^T x(s)ds = x(T), \end{aligned} \tag{1.1}$$

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where $F : J \times R \longrightarrow 2^R$ is a compact convex valued multivalued map and $I_k \in C(R, R)(k = 1, 2, \cdots, m)$ are bounded, $\mu \leq 0, 0 < t_1 < t_2 < \cdots < t_m < T$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump of $x(t)$ at $t = t_k$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Denote $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $C(J, R)$is the Banach space of all continuous functions from J into R with the norm $||x|| = \sup\{|x(t)| : t \in J\}$.
$L^1(J, R)$is the Banach space of functions $x : J \longrightarrow R$,which are Lebesgue integrable and the norm $||x||_{L^1} = \int_0^T |x(t)|dt$.

Let $\Omega = \{u : J \to R,\ u$ is continuous for $t \in J, t \neq t_k,\ u(t_i^+),\ u(t_i^-)$ exist and $u(t_i^-) = u(t_i), i = 1, 2, \ldots, m\}$. is Banach spaces with the norms

$$||u||_{PC(J)} = \sup\{|u(t)| : t \in J\}.$$

$AC(J, R)$is the space of absolutely continuous functions $x : J \longrightarrow R$.

By a solution of (1.1) we mean a $x \in \Omega \cap AC(J, R)$ for which problem (1.1) is satisfied.

For any $x \in \Omega$,we define the set

$$S_F, x = \{v \in L^1([0, T], R) : v(t) \in F(t, x)\ a.e.\ t \in [0, T]\}$$

When $\mu \equiv 0$, equation (1.1) reduces to periodic boundary value problems for impulsive ordinary differential equation

$$\begin{aligned} & x'(t) \in F(t, x(t)),\ t \neq t_k,\ t \in J = [0, T], \\ & \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m, \\ & x(0) = x(T), \end{aligned} \tag{1.2}$$

periodic boundary problems for impulsive differential inclusions are discussed in [9].

Monotone iterative technique coupled with the method of upper and lower solutions has been widely used in the treatment of existence results of boundary value problems for nonlinear differential equations in recent years [10-13]). In this paper,first we will introduce new concept of lower and upper solutions.Then ,using the definition of lower and upper solutions α, β and monotone iterative technique,we will obtain the existence of the solutions for (1.1),with $\alpha \leq \beta$.

2. Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space.A multi-valued map $G : E \longrightarrow 2^E$ has convex(closed) values if $G(x)$ is convex(closed) for all $x \in E$.G is bounded on

bounded sets if $G(B)$ is bounded in E for each bounded set B of E.
A point $x \in E$ is called a fixed point of the multi-valued map G if $x \in G(x)$.
A multifunction G is called upper semi-continuous(u.s.c.) if for each $x_0 \in E$, the set $G(x_0)$ is a nonempty and closed subset of E, and for each open set $N \subset E$ containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subset N$.
G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq E$.

If G is completely continuous with nonempty and compact-valued,then G is u.s.c. if and only if G has a closed graph,i.e.,given sequences $\{x_n\}_{n=1}^{\infty} \longrightarrow x_0$, $\{y_n\}_{n=1}^{\infty} \longrightarrow y_0, y_n \in G(x_n)$ for every $n = 1, 2, \cdots$ imply $y_0 \in G(x_0)$.
$CC(R)$ denotes the set of all nonempty compact,convex subsets of R.
A multi-valued map $G : J \longrightarrow CC(R)$ is said to be measurable,if for each $x \in R$,the function $Y : J \longrightarrow R$ defined by

$$Y(t) = d(x, G(t)) = inf\{|x - z| : z \in G(t)\}$$

is measurable.
Definition 2.1 A multi-valued map $F : J \times R \longrightarrow 2^R$ is said to be $L^1 - Carath\acute{e}odory$ if

(i) $t \longmapsto F(t, x)$ is measurable for each $x \in R$;

(ii) $x \longmapsto F(t, x)$ is upper semi-continuous for almost all $t \in J$;

(iii) for each $r > 0$, there exists a function $h_r \in L^1(J)$ such that $\|F(t, x)\|=\sup\{|u| : u \in F(t, x)\} \leq h_r$, a.e. $t \in J$, for all $|x| \leq r$.

To apply upper and lower solutions method, we need the concept of lower solution and upper solution for equation (1.1). We say that a function $\alpha \in AC(J, R)$ is a lower solution of equation (1.1) if there exists $v_1(t) \in L^1(J, R)$ such that $v_1(t) \in F(t, \alpha(t))$ a.e. on J,

$$\begin{aligned}
&\alpha'(t) \leq v_1(t),\ t \neq t_k,\ t \in J,\\
&\Delta\alpha(t_k) \leq I_k(\alpha(t_k)),\ k = 1, 2, \ldots, m,\\
&\alpha(0) + \mu \textstyle\int_0^T \alpha(s)ds \leq \alpha(T).
\end{aligned}$$

Similarly, a function $\beta \in AC(J, R)$ is an upper solution of equation (1.1) if there exists $v_2(t) \in L^1(J, R)$ such that $v_2(t) \in F(t, \beta(t))$ a.e. on J,

$$\begin{aligned}
&\beta'(t) \geq v_2(t),\ t \neq t_k,\ t \in J,\\
&\Delta\beta(t_k) \geq I_k(\beta(t_k)),\ k = 1, 2, \cdots, m,\\
&\beta(0) + \mu \textstyle\int_0^T \beta(s)ds \geq \beta(T).
\end{aligned}$$

In what follows we define the set

$$[\alpha,\beta]=\{w\in AC(J,R):\alpha(t)\le w(t)\le\beta(t),t\in J\}$$

for $\alpha,\beta\in AC(J,R)$ and $\alpha\le\beta$.

we list the following conditions.

($\mathbf{H_1}$) $F: J\times R\longrightarrow 2^R$ is an $L^1-Carath\acute{e}odory$ multi-valued map.

($\mathbf{H_2}$) Let $\alpha,\beta\in AC(J,R)$ be lower and upper solutions for problem (1.1) such that $\alpha\le\beta$.

($\mathbf{H_3}$) $I_k\in C(R,R)(k=1,2,\cdots,m)$,are nondecreasing and bounded.

3. Main results

To obtain our main results, we need the following lemmas.

Lemma 3.1.(See[14]) Let I be a compact real interval and X be a Banach space.Let F be a multi-valued map satisfying the *Carathéodory* conditions with the set of $L^1-selections\ S_F$ nonempty,and let Γ be a linear continuous mapping from $L^1(I,X)$to $C(I,X)$.Then the operator

$$\Gamma\circ S_F: C(I,X)\longrightarrow CC(C(I,X)),y\longmapsto(\Gamma\circ S_F)(y):=\Gamma(S_F,y)$$

is a closed graph operator in $C(I,X)\times C(I,X)$

Lemma 3.2.(See[14]) Let $N: X\longrightarrow CC(X)$ be a u.s.c. and condensing map.If the set

$M:=\{x\in X: x\in\lambda N(x)\ \text{ for some }\ 0<\lambda<1\}$

is bounded ,then N has a fixed point.

Lemma 3.3. Assume that $\lambda>0$, $F: J\times R\longrightarrow 2^R$ is an $L^1-Carath\acute{e}odory$ multi-valued map, $I_k\in C(R,R)(k=1,2,\cdots,m)$,are bounded.Then

$$\begin{array}{l} x'(t)+\lambda x(t)\in F(t,x(t)),\ t\ne t_k,\ t\in J=[0,T],\\ \Delta x(t_k)=I_k(x(t_k)),\ k=1,2,\cdots,m,\\ x(0)+\mu\int_0^T x(s)ds=x(T), \end{array}\tag{3.1}$$

has at least one solution.

Proof. It is easy to see that a function x is a solution to (3.1) if and only if $x\in N(x)$ where operator $N:\Omega\longrightarrow 2^{\Omega}$ defined by

$$N(x)=\{h\in\Omega: h(t)=-\frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x(s)ds+\int_0^T G(t,s)f(s)ds+\sum_{k=1}^m G(t,t_k)I_k(x(t_k))\},$$

where $f\in S_{F,x}$, and

$$G(t,s)=\begin{cases}\frac{e^{-\lambda(t-s)}}{1-e^{-\lambda T}}, & 0\le s<t\le T,\\ \frac{e^{-\lambda(T+t-s)}}{1-e^{-\lambda T}}, & 0\le t\le s\le T.\end{cases}$$

We shall show that satisfies the assumptions of Lemma 3.2.The proof will be given in several steps.

Step1. $N(x)$ is convex for each $x \in \Omega$.
If h_1, h_2 belong to $N(x)$,then there exist $f_1, f_2 \in S_F, x$ such that for each $t \in J$ we have

$$h_i(t) = -\frac{\mu e^{-\lambda t}}{1 - e^{-\lambda T}} \int_0^T x(s)ds + \int_0^T G(t,s) f_i(s)ds + \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k)), i = 1, 2$$

Let $0 \le d \le 1$.Then,for each $t \in J$ we have

$$\begin{aligned}(dh_1 + (1-d)h_2)(t) &= -\frac{\mu e^{-\lambda t}}{1 - e^{-\lambda T}} \int_0^T x(s)ds + \int_0^T G(t,s)(df_1(s) + (1-d)f_2(s))ds \\ &\quad + \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k))\end{aligned}$$

Since S_F, x is convex,then

$$dh_1 + (1-d)h_2 \in N(x).$$

Step2. N maps bounded sets into bounded sets in Ω.
Indeed ,for given $q > 0$,if $x \in B_q = \{x \in \Omega : \|x\|_\Omega \le q\}$,then ,there are $c_k, k = 1, 2, \cdots, m$ and $\varphi \in L^1(J, R)$,such that $|I_k(x(t_k))| \le c_k, k = 1, 2, \cdots, m$ and $|F(t, x(t))| \le \varphi(t)$,since F is $L^1 - Carathéodory$ multi-valued map and $I_k(k = 1, 2, \cdots, m)$ are continuous.Thus, Letting $x \in B_q = \{x \in \Omega : \|x\|_\Omega \le q\}$ and $h \in N(x)$,then there exists $f \in S_F, x$ such that for each $t \in J$ we have

$$h(t) = -\frac{\mu e^{-\lambda t}}{1 - e^{-\lambda T}} \int_0^T x(s)ds + \int_0^T G(t,s) f(s)ds + \sum_{k=1}^{m} G(t, t_k) I_k(x(t_k))$$

Hence ,we have

$$\begin{aligned}|h(t)| &\le |-\frac{\mu e^{-\lambda t}}{1 - e^{-\lambda T}} \int_0^T x(s)ds| + \int_0^T |G(t,s) f(s)|ds + \sum_{k=1}^{m} |G(t, t_k) I_k(x(t_k))| \\ &\le \frac{|\mu|}{1 - e^{-\lambda T}} \{Tq + T\|\varphi\|_{L^1} + \sum_{k=1}^{m} c_k\} := q^*\end{aligned}$$

Step3. N maps bounded set into equicontinuous sets of Ω. Letting $\tau_1, \tau_2 \in [0,T], \tau_1 \leq \tau_2, x \in B_q, h \in N(x)$,then there exists f $\in S_{F,x}$ such that for each $t \in J$ we have

$$h(t) = -\frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x(s)ds + \int_0^T G(t,s)f(s)ds + \sum_{k=1}^m G(t,t_k)I_k(x(t_k))$$

Then,

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| \leq & \frac{|\mu|qT}{1-e^{-\lambda T}}|e^{-\lambda\tau_2} - e^{-\lambda\tau_1}| \\ & + \int_0^{\tau_1} |G(\tau_2,s) - G(\tau_1,s)||f(s)|ds + \int_{\tau_2}^T |G(\tau_2,s) - G(\tau_1,s)||f(s)|ds \\ & + \int_{\tau_1}^{\tau_2} (|G(\tau_2,s)| + |G(\tau_1,s)|)|f(s)|ds + \sum_{k=1}^m |G(\tau_2,t_k) - G(\tau_1,t_k)||I_k(x(t_k))| \end{aligned}$$

From the definition of the function $G(t,s)$ and that F is $L^1-Carath\acute{e}odory$ multi-valued map. Hence$N(B_q)$ is equicontinuous.
As a consequence of Steps 1-3 ,by the Arzela-Ascoli theorem,N is a completely continuous multi-valued map,and therefore ,a condensing map

Step4. N has a closed graph.
Let $x_n \longrightarrow x_*, h_n \in N(x_n), and\ h_n \longrightarrow h_*$.we shall prove that $h_* \in N(x_*)$.
$h_n \in N(x_n)$ means that there exists $f_n \in S_{F,x_n}$ such that for each $t \in J$

$$h_n(t) = -\frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x_n(s)ds + \int_0^T G(t,s)f_n(s)ds + \sum_{k=1}^m G(t,t_k)I_k(x_n(t_k))$$

We must prove that there exists $f_* \in S_{F,x_*}$,such that for each $t \in J$

$$h_*(t) = -\frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x_*(s)ds + \int_0^T G(t,s)f_*(s)ds + \sum_{k=1}^m G(t,t_k)I_k(x_*(t_k))$$

Since $I_k, k = 1, 2, \cdots, m$ is continuous,as $n \longrightarrow +\infty$,we have

$$\begin{aligned} \| \quad (\quad & h_n + \frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x_n(s)ds - \sum_{k=1}^m G(t,t_k)I_k(x_n(t_k))) \\ & -(h_* + \frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x_*(s)ds - \sum_{k=1}^m G(t,t_k)I_k(x_*(t_k))) \| \longrightarrow 0, \end{aligned}$$

Consider the linear continuous operator

$$\Gamma : L^1(J,R) \longrightarrow C(J,R), f \longmapsto (\Gamma f)(t) = \int_0^T G(t,s)f(s)ds.$$

From Lemma 2.1 ,$\Gamma \circ S_F$ is a closed graph operator.Since

$$h_n + \frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x_n(s)ds - \sum_{k=1}^{m} G(t,t_k)I_k(x_n(t_k)) \in \Gamma(S_{F,X_n})$$

Since $x_n \longrightarrow x_*$,it follows from Lemma2.1 that

$$h_* + \frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x_*(s)ds - \sum_{k=1}^{m} G(t,t_k)I_k(x_*(t_k)) = \int_0^T G(t,s)f_*(s)ds,$$

for some $f_* \in S_{F,x_*}$.

Step5. Now we show that the set $M := \{x \in \Omega : x \in \lambda N(x), for\ some\ 0 < \lambda < 1\}$ is bounded.
Let $x \in M$.Then,$x \in \lambda N$ for some $0 < \lambda < 1$. So,for each $t \in J$,

$$x(t) = \lambda[-\frac{\mu e^{-\lambda t}}{1-e^{-\lambda T}}\int_0^T x(s)ds + \int_0^T G(t,s)f(s)ds + \sum_{k=1}^{m} G(t,t_k)I_k(x(t_k))]$$

We have

$$\begin{aligned} |x(t)| &\leq \frac{|\mu|}{1-e^{-\lambda T}}\{\int_0^T |x(s)|ds + \int_0^T |f(s)|ds + \sum_{k=1}^{m} |I_k(x(t_k))|\} \\ &\leq \frac{|\mu|}{1-e^{-\lambda T}}\{\int_0^T |x(s)|ds + \|\varphi\|_{L^1} + \sum_{k=1}^{m} |I_k(x(t_k))|\} \\ &\leq \frac{|\mu|}{1-e^{-\lambda T}}\{\|x(s)\|_{L^1} + \|\varphi\|_{L^1} + \sum_{k=1}^{m} |I_k(x(t_k))|\} \end{aligned}$$

$I_k \in C(R,R)(k = 1,2,\cdots,m)$,are bounded,thus M is bounded.

From step1-step5,we deduce that N has a fixed point which is a solution of (3.1).

Theorem 3.1. Assume that $(H_1),(H_2)$ and (H_3) are satisfied.Then, (1.1) has at least one solution between α and β.

Proof. Consider the modified problem,

$$\begin{array}{l} x'(t) + \lambda x(t) \in F_1(t,x(t)),\ t \neq t_k,\ t \in J = [0,T], \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1,2,\cdots,m, \\ x(0) + \mu \int_0^T x(s)ds = x(T), \mu \leq 0 \end{array} \tag{3.2}$$

where

$$F_1(t, x(t)) = \begin{cases} F(t, \beta(t)) + \lambda\beta(t), & \text{if } x(t) > \beta(t), \\ F(t, \beta(t)) + \lambda x(t), & \text{if } \alpha(t) \le x(t) \le \beta(t), \\ F(t, \beta(t)) + \lambda\alpha(t), & \text{if } x(t) < \alpha(t). \end{cases}$$

Then,by similar arguments to that in Lemm3.3.,one can easily obtain that (3.2)has a solution x.Now,we will show that x satisfies $\alpha(t) \le x(t) \le \beta(t)$, for $t \in J$. We claim that $x(t) \le \beta(t)$ on $t \in J$.If this is not true,let $m(t) = x(t) - \beta(t)$,then $m(t)$ assumes a positive maximum at some $t_0 \in [t_k^+, t_{k+1}^-], k = 0, 1, \cdots, m$.

We consider two possible cases.

Case 1. $t_0 \in [t_k^+, t_{k+1}^-]$,then,there exists $t_k^* \in [t_k^+, t_{k+1}^-)$ such that

$$0 < m(t) \le m(t_0), t \in [t_k^*, t_0].$$

Consequently,we have $x'(t) + \lambda x(t) \in F(t, \beta(t)) + \lambda\beta(t)$ a.e.on $[t_k^*, t_0]$. thus,there exists $v(t) \in F(t, \beta(t))$ a.e. on $[t_k^*, t_0]$ with $v(t) \le v_2(t)$ a.e. on $[t_k^*, t_0]$.such that

$$x'(t) + \lambda x(t) = v(t) + \lambda\beta(t) \text{ a.e.on } [t_k^*, t_0].$$

Hence

$$\begin{aligned} x(t_0) - x(t_k^*) &= \int_{t_k^*}^{t_0} [v(s) - \lambda x(s) + \lambda\beta(t)]ds \\ &\le \int_{t_k^*}^{t_0} [v_2(s) - \lambda(x(s) - \beta(t))]ds \\ &\le \beta(t_0) - \beta(t_k^*) - \lambda \int_{t_k^*}^{t_0} (x(s) - \beta(t))ds. \end{aligned}$$

This yields $x(t_0) - x(t_k^*) < \beta(t_0) - \beta(t_k^*)$.Then, $m(t_0) < m(t_k^*)$.It is a contradiction.

Case 2. $t_0 = t_k^+$.From case(1),we claim $x(t_k^-) \le \beta(t_k^-)$. On the other hand, $x(t_k^+) - \beta(t_k^+) > 0$,$\Delta\beta(t_k) = \beta(t_k^+) - \beta(t_k^-), \Delta x(t_k) = x(t_k^+) - x(t_k^-)$.Hence ,we have $\Delta\beta(t_k) < \Delta x(t_k)$. Using (H_3),we obtain

$$\Delta\beta(t_k) < \Delta x(t_k) = I_k(x(t_k)) \le I_k(\beta(t_k)) \le \Delta\beta(t_k),$$

which is a contradiction.

Consequently, $x(t) \le \beta(t)$, for all $t \in J$.Similarly,we can prove that $\alpha(t) \le x(t)$ on J. The proof of the theorem is complete.

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