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Periodic Solutions for Higher Order Delay Functional Differential Equation with Complex Deviating Argument

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Abstract. By using the theory of coincidence degree, the existence of periodic solutions is studied for higher order delay functional differential equation with complex deviating argument, and the existing results have been extended.

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1. Introduction and Lemma

The problem of periodic solutions of ordinary differential equation was extensively studied. In recent years, there are many results about periodic solutions to second-order scalar differential equations with deviating arguments, In this paper, we study the existence of periodic solutions for higher order delay functional differential equation with complex deviating argument as follows

$$x^{(m)}(t)+a(t)x^{(m)}(t-\tau)+f(x(t-\delta))\dot{x}(t-\delta)+b(t)g(x(x(t)))+c(t)x(t-\tau)=p(t) \quad (*)$$

and a new result to guarantee the existence of periodic solutions has been obtained by using the theory of coincidence degree.

Lemma [1] Let X,Y be Banach space, $L : D_L \subset X \to Y$ is a Fredholm mapping of index 0 $P : X \to X, Q : X \to Y$ are continuous mapping projector; Ω is an open bounded set in X and $N : \bar{\Omega} \times [0, 1] \to Y$ is L-Compact on $\bar{\Omega}$, further more suppose:
(a) $Lx \neq \lambda N(x, \lambda), \forall x \in D_L \cap \partial\Omega, \lambda \in (0, 1)$;
(b) $QN(x, 0) \neq 0, \forall x \in \ker(L) \cap \partial\Omega$;
(c) $deg(QN(x, 0), \ker(L) \cap \Omega, 0) \neq 0$.

then the equation $Lx = N(x, 1)$ has at least one solution in $\bar{\Omega}$, where deg is Brouwer degree.

2. Main Result and Proof of Theorem

Theorem : Suppose f, a, b, c, g, p are continuous for their variables respectively; There exits T , such that $p(t+T) = p(t), a(t+T) = a(t), b(t+T) = b(t) < 0, 0 < |a(t)| < a_1 < 1, c(t+T) = c(t) < 0, a_m = \max_{0\le i\le m-1}\{|a^m(t)|\} < c_0 = \min_{t\in R}\{|c(t)|\} \le c_1 \le \max_{t\in R}\{|c(t)|\} < \frac{1-a_1}{T^m}, a(t) \in C^m(R, R)$ and further more suppose are as follows,
(a) $\exists 0 < M < \frac{1-a_1-c_1T^m}{T^{m-1}}, \forall x \in R, |f(x)| \le M;$
(b) $\exists A > 0, \forall x \in R, |g(x)| \le A;$
(c) $\forall 0 \ne x \in R, xg(x) > 0$
then $Eq.(*)$ has at least one T-periodic solution.

Proof of Theorem : In order to use continuation theorem to obtain T-periodic solution of $Eq.(*)$, we firstly make some required preparations. Let

$$X = \{x \in c^{m-1}(R, R)|x(t+T) = x(t)\}, Y = \{y \in C(R, R)|y(t+T) = y(t)\}$$

and the norm of X and Y is

$$\|x\| = \max_{0\le i\le m-1}\{|x^{(i)}|_\infty\}, |x^{(i)}|_\infty = \max_{t\in R}\{|x^{(i)}(t)|\}, i = 1, 2, \cdots, m-1, \|y\| = \max_{t\in R}\{|y(t)|\}$$

then the X and Y with this norm are Banach space.
Firstly, we study the priori bound of T-periodic solution of following equation

$$x^{(m)}(t)+\lambda a(t)x^{(m)}(t-\tau)+\lambda f(x(t-\delta))\dot{x}(t-\delta)+\lambda b(t)g(x(x(t)))+\lambda c(t)x(t-\tau) = \lambda^2 p(t) \quad (1.1)$$

Suppose that $x = x(t) \in X$ is an arbitrary T-periodic solution of $Eq(1.1)$, put $x(t)$ into (1.1) and then integrate both sides of (1.1) on $[0, T]$, so yield

$$\int_0^T [a(t)x^{(m)}(t-\tau) + b(t)g(x(x(t))) + c(t)x(t-\tau)]dt = \lambda \int_0^T p(t)dt$$

We easily get:

$$(-1)^m \int_0^T a^{(m)}(t)x(t-\tau)dt+\int_0^T b(t)g(x(x(t)))dt+\int_0^T c(t)x(t-\tau)dt = \lambda \int_0^T p(t)dt$$

so there exist a $t_0 \in [0, T]$, such that

$$(-1)^m a^{(m)}(t_0)x(t_0-\tau) + b(t_0)g(x(x(t_0))) + c(t_0)x(t_0-\tau) = \lambda p(t_0)$$

i.e.

$$|c(t_0)x(t_0-\tau)| = |\lambda p(t_0) - (-1)^m a^{(m)}(t_0-\tau) - b(t_0)g(x(x(t_0)))|$$

so

$$|c_0 x(t_0-\tau)| \le p_1 + a_m|x(t_0-\tau)| + b_1 A$$

i.e.

$$|x(t_0-\tau)| \le \frac{b_1 A + p_1}{c_0 - a_m} \triangleq A_1$$

where $b_1 = \max_{t\in R}\{|b(t)|\}$, $p_1 = \max_{t\in R}\{|p(t)|\}$
let

$$t_0 - \tau = nT - t_1, n \in N, t_1 \in [0,T]$$

so

$$|x(t_1)| = |x(t_0-\tau)| \le A_1$$

In view of

$$\forall t \in [0,T], x(t) = x(t_1) + \int_{t_1}^{t} \dot{x}(s)ds$$

We have

$$|x(t)| = |x(t_1) + \int_{t_1}^{t} \dot{x}(s)ds| \le A_1 + \int_{t_1}^{t} |\dot{x}(s)|ds \le A_1 + \int_0^T |\dot{x}(t)|dt$$

i.e.

$$|x^{(0)}|_\infty = |x|_\infty \le A_1 + \int_0^T |\dot{x}(t)|dt \tag{1.2}$$

Noting $x(t) = x(t+T)$, so there must exist number $\xi_i \in (0,T)$, such that $x^{(i)}(\xi_i) = 0$, there $i = 1,2,3,\cdots,m-1$
For $\forall t \in [0,T]$

$$x^{(i)}(t) = x^{(i)}(\xi_i) + \int_{\xi_i}^{t} x^{(i+1)}(s)ds = \int_{\xi_i}^{t} x^{(i+1)}(s)ds$$

We have

$$|x^{(i)}(t)| = |\int_{\xi_i}^{t} x^{(i+1)}(s)ds| \le \int_0^T |x^{(i+1)}(t)|dt \le T\int_0^T |x^{(i+2)}(t)|dt$$

$$\le T^2 \cdot \int_0^T |x^{(i+3)}(t)|dt \le \cdots \le T^{m-(i+1)} \int_0^T |x^{(i+m-i)}(t)|dt = T^{m-(i+1)} \int_0^T |x^{(m)}(t)|dt$$

i.e.

$$|x^{(i)}|_\infty \le T^{m-(i+1)} \int_0^T |x^{(m)}(t)|dt, i = 1,2,\cdots,m-1 \tag{1.3}$$

Combining (1.2), (1.3), we get

$$|x^{(0)}|_\infty = |x|_\infty \le A_1 + T^{m-1} \int_0^T |x^{(m)}(t)|dt \tag{1.4}$$

By (1.1),we get

$$\begin{aligned}\int_0^T |x^{(m)}(t)|dt &\leq \int_0^T |-\lambda a(t)x^{(m)}(t-\tau)-\lambda f(x(t-\delta))\dot{x}(t-\delta)|dt \\ &+ \int_0^T |-\lambda b(t)g(x(x(t)))-\lambda c(t)x(t-\tau)+\lambda^2 p(t)|dt\end{aligned}$$

Combining the condition (a) and (b), we easily obtain

$$\begin{aligned}\int_0^T |x^{(m)}(t)|dt &\leq a_1\int_0^T |x^{(m)}(t-\tau)|dt + M\int_0^T |\dot{x}(t-\delta)|dt \\ &+ c_1\int_0^T |x(t-\tau)|dt + TAb_1 + Tp_1\end{aligned}$$

i.e.

$$\int_0^T |x^{(m)}(t)|dt \leq a_1\int_0^T |x^{(m)}(t)|dt + MT|\dot{x}|_\infty + c_1T|x|_\infty + TAb_1 + Tp_1 \quad (1.5)$$

Noting (1.3),(1.4) and (1.5), we observe

$$\begin{aligned}\int_0^T |x^{(m)}(t)|dt &\leq a_1\int_0^T |x^{(m)}(t)|dt + MT\cdot T^{m-2}\int_0^T |x^{(m)}(t)|dt \\ &+ c_1T(A_1 + T^{(m-1)}\int_0^T |x^{(m)}(t)|dt) + TAb_1 + Tp_1\end{aligned}$$

i.e.

$$(1-a_1-MT^{(m-1)}-c_1T^m)\int_0^T |x^{(m)}(t)|dt \leq TAb_1 + Tp_1 + c_1TA_1$$

so

$$\int_0^T |x^{(m)}(t)|dt \leq \frac{TAb_1+Tp_1+c_1TA_1}{1-a_1-MT^{(m-1)}-c_1T^m} \quad (1.6)$$

Noting (1.4), (1.5) and (1.6), we have

$$\begin{aligned}|x^{(0)}|_\infty &= |x|_\infty \leq A_1 + T^{(m-1)}\cdot\frac{TAb_1+Tp_1+c_1TA_1}{1-a_1-MT^{(m-1)}-c_1T^m} \\ &= A_1 + T^m\frac{Ab_1+p_1+c_1A_1}{1-a_1-MT^{(m-1)}-c_1T^m} \triangleq \omega_0\end{aligned}$$

$$\begin{aligned}|x^{(i)}|_\infty &\leq T^{m-(i+1)}\cdot\frac{TAb_1+Tp_1+c_1TA_1}{1-a_1-MT^{(m-1)}-c_1T^m} \\ &= T^{m-1}\frac{Ab_1+p_1+c_1A_1}{1-a_1-MT^{(m-1)}-c_1T^m} \triangleq \omega_i, i=1,2,\cdots,m-1\end{aligned}$$

Let $\omega = \max\limits_{0\le i\le m}\{\omega_i + 1\}$, and we take $\Omega = \{x|x \in X : \|x\| < \omega\}$, then Ω is an open and bounded set in X

Let

$$\begin{aligned} L &: D_L \subset X \to Y : x \to Lx = x^{(m)}(t) \\ N &: X \times I \to Y : x \to N(x,\lambda) \\ &= -a(t)x^{(m)}(t-\tau) - f(x(t-\delta))\dot{x}(t-\delta) - b(t)g(x(x(t))) - c(t)x(t-\tau) + \lambda p(t) \end{aligned}$$

then the corresponding equation of $Lx = \lambda N(x,\lambda)$ is $Eq.(1.1)$.

Now, we define projection operators as follows,

$$P : X \to Ker(L) : x \to Px = \frac{1}{T}\int_0^T x(t)dt$$

$$Q : Y \to Y/Im(L) : y \to Qy = \frac{1}{T}\int_0^T y(t)dt$$

obviously, P, Q are continuous operator, $Im(P) = R = \ker(L), \ker(Q) = Im(L)$, and it is easy to prove that L is a Fredholm mapping of index 0 and is L-Compact on $\bar{\Omega}$.

From the above discussion and the construction of Ω , we have known that $\forall x \in D_L \cap \partial\Omega, \lambda \in (0,1)$, therefore the condition (a) of lemma holds.

For arbitrary $x \in \ker(L) \cap \partial\Omega$, $\|x\| = \omega$, by the definition of Q, N, we have

$$\begin{aligned} QN(x,0) &= \frac{1}{T}\int_0^T [-a(t)x^{(m)}(t-\tau) - f(x(t-\delta))\dot{x}(t-\delta) - b(t)g(x(x(t))) - c(t)x(t-\tau)]dt \\ &= -\frac{1}{T}\int_0^T b(t)g(x(x(t)))dt - \frac{1}{T}\int_0^T c(t)x(t-\tau)dt \end{aligned}$$

so

$$\begin{aligned} xQN(x,0) &= -\frac{1}{T}x\int_0^T b(t)g(x(x(t)))dt - \frac{1}{T}\int_0^T c(t)x(t-\tau)dt \\ &= -\frac{1}{T}xg(x)\int_0^T b(t)dt - \frac{1}{T}x^2\int_0^T c(t)dt > 0 \end{aligned}$$

therefore the condition (b) of lemma holds.

Make a transformation

$$H(x,\mu) = \mu x + (1-\mu)QN(x,0), \forall x \in \partial\Omega \cap \ker(L), \mu \in [0,1]$$

We have

$$\begin{aligned} xH(x,\mu) &= \mu x^2 + x(1-\mu)QN(x,0) \\ &= \mu x^2 - (1-\mu)\frac{1}{T}g(x)x\int_0^T b(t)dt - (1-\mu)\frac{1}{T}x^2\int_0^T c(t)dt > 0 \end{aligned}$$

so $xH(x,\mu) \neq 0$, i.e.$H(x,\mu) \neq 0$ is a homotopy,$\deg(QN(x,0), \ker(L) \cap \Omega, 0) = \deg(-I, \ker(L) \cap \Omega, 0) = \deg(-I, R \cap \Omega, 0) \neq 0$, where I is identity mapping and the condition (c) of lemma holds.

From above all, the requirements of lemma are all met, so $Eq.(*)$ has at least one T-periodic solution under the condition of theorem , so far the proof of theorem is completed.

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