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On Some Differential Equations

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Abstract

This paper investigates Cauchy and Goursat problems for partial differential operators. Successive approximation techniques for partial differential equations and the estimated results are employed to obtain the existence and the uniqueness of the solutions of such problems. An extended Darboux-Goursat-Beudon problem is studied.

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1 Introduction

The purpose of this study is to investigate some partial differential equations including Cauchy and Goursat problems and extended Darboux-Goursat-Beudon problem. The majorant and iterative methods are used.

In [2] the investigation of initial value problems of partial differential equations was initiated. In this paper, another direction is taken to study the same problems. We shall use the majorant function introduced by [9] to review some Hormander results. The following problem (P') was studied by $[6, 7]$ for the special case when the coefficients $a_{m,0,\ldots,0}$ vanishes in the hyperspace $z_0 = 0$, in other words when the operator is Fuchsian.

In this paper, we shall study the following problems. First we consider

$$(P)\begin{cases} D^{\beta}u = \sum_{|\alpha|\leq m} a_{\alpha}D^{\alpha}u + f \\ D_j^k(u-\varphi)_{|\Omega'_j} = 0, (j,k) \in \mathcal{I}_{\beta}, \end{cases}$$

for arbitrary holomorphic functions $(a_{\alpha})_{|\alpha|\leq m}$, f and φ, where

$$\mathcal{I}_{\beta} = \{(j,k) : j = 0, 1, ..., n, \text{and } k = 0, 1, ..., \beta_j - 1\}.$$

The initial values are supported by the hypersurface $z_j = 0$. Then we solve the Cauchy-Kovalevskaya problem

$$(P')\begin{cases} \sum_{|\alpha|\leq m} a_{\alpha}D^{\alpha}u = f \\ D_0^k\left(u-\varphi\right)_{|\Omega'_0} = 0, 0 \leq k \leq m-1, \end{cases}$$

and the Darboux-Goursat-Beudon problem with mixed data

$$(P'')\begin{cases} \sum_{|\alpha|\leq m} a_{\alpha}D^{\alpha}u = f \\ D_0^k\left(u-\varphi\right)_{|\Omega'_0} = 0, 0 \leq k \leq m-2 \\ \left(u-\varphi\right)_{|\Omega'_1} = 0. \end{cases}$$

We start by presenting some basic notations. Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space, $\mathbb{R}_+$ the set of real numbers ≥ 0, $\mathbb{R}_+^{n+1}$ be the set of all $r = (r_0, r_1, ..., r_n)$ with $r_j \in \mathbb{R}_+$ and $\mathcal{C}^{n+1}$ be the $(n+1)$-dimensional complex space with variables $z = (z_0, z_1, ..., z_n)$ and Ω an open subset of $\mathcal{C}^{n+1}$ containing the origin. We use the standard multi-index notation. More precisely, let $\mathbb{Z}$ be the set of integers, > 0 or ≤ 0, and $\mathbb{Z}_+$ be the set of integers ≥ 0. Then $\mathbb{Z}_+^{n+1}$ is the set of all $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$ with $\alpha_j \in \mathbb{Z}_+$ for each $j = 0, 1, ..., n$. The length of $\alpha \in \mathbb{Z}_+^{n+1}$ is $|\alpha| = \alpha_0 + \alpha_1 + ... + \alpha_n$; $\alpha \leq \beta$ means $\alpha_j \leq \beta_j$ for every $j = 0, 1, ..., n$; and $\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$. If $\alpha \in \mathbb{Z}_+^{n+1}$ and $\beta \in \mathbb{Z}_+^{n+1}$, we define the operation $+$ by

$$\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, ..., \alpha_n + \beta_n).$$

Moreover, we let $\alpha! = \alpha_0!\alpha_1!...\alpha_n!$,

$$D^{\alpha} = \left(\frac{\partial}{\partial z_0}\right)^{\alpha_0}\left(\frac{\partial}{\partial z_1}\right)^{\alpha_1}...\left(\frac{\partial}{\partial z_n}\right)^{\alpha_n},$$

and use the notation $D_j = \frac{\partial}{\partial z_j}$. Also, we let $D^{\alpha} = D_0^{\alpha_0}D_1^{\alpha_1}...D_n^{\alpha_n}$. Let u be a continuous function in Ω; by the support of u, denoted by $\sup pu$, we mean the closure in Ω of $\{z : z \in \Omega, u(z) \neq 0\}$. By $\mathcal{C}^k(\Omega)$, $k \in \mathbb{Z}_+$, $0 \leq k \leq \infty$, we denote the set of all functions u defined in Ω, whose derivatives $D^\alpha u(z)$ exist and continuous for $|\alpha| \leq k$. Using the multi-index notation, we may write the Leibnitz formula

$$D^\beta(uv) = \sum_{\alpha \leq \beta} \frac{\beta!}{\alpha!(\beta-\alpha)!} D^{\beta-\alpha} u D^\alpha v,$$

where we assume $u,\, v \in \mathcal{C}^{|\alpha|}(\Omega)$. If $u \in \mathcal{C}^\infty(\Omega)$, we may consider the Taylor expansion at the origin

$$u(z) = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} \frac{D^\alpha u(0)}{\alpha!} z^\alpha.$$

Let $\mathcal{H}(\Omega)$ denote the set of all holomorphic functions in Ω, that is functions $u(z) \in \mathcal{C}^\infty(\Omega)$ given by their Taylor expansion in some neighborhood of the origin in Ω. A linear partial differential operator $P(z; D)$ is defined by

$$P(z; D) = \sum_{|\alpha| \leq m} a_\alpha(z) D^\alpha,$$

where the coefficients $a_\alpha(z)$ are in $\mathcal{H}(\Omega)$. If for some α of length m, the coefficient $a_\alpha(z)$ does not vanish identically in Ω, m is called the order of $P(z; D)$.

2 Some General Results

We state and prove a general theorem that gives both Cauchy-Kovalevskaya and Darboux-Goursat-Beudon results.

Theorem 2.1 *Let Ω be a neighborhood of $\mathcal{C}^{n+1}$ containing the origin, β a multi-indices of $\mathbb{Z}_+^{n+1}$ such that $|\beta| = m \geq 1$, $(a_\alpha)_{|\alpha| \leq m}$, f and φ are holomorphic functions in Ω. If $\sum_{|\alpha|=m} |a_\alpha(0)| < (2e)^{-m}$, then there exists $\Omega' \subset \Omega$ connected open neighborhood of the origin such that*

$$(P) \begin{cases} D^\beta u = \sum\limits_{|\alpha| \leq m} a_\alpha D^\alpha u + f \\ D_j^k (u - \varphi)_{|\Omega'_j} = 0, (j,k) \in \mathcal{I}_\beta, \end{cases}$$

has one and only one holomorphic solution on Ω', where

$$\Omega'_j = \left\{ z : z \in \mathcal{C}^{n+1}, z_j = 0 \right\} \cap \Omega'.$$

Corollary 2.2 *Let $A = \left\{ \alpha : \alpha \in \mathbb{Z}_+^{n+1}, |\alpha| = m, a_\alpha(0) \neq 0 \right\}$. If we replace $\sum\limits_{|\alpha|=m} |a_\alpha(0)| < (2e)^{-m}$ by $\beta \notin$ convex hull of A considered as a subset of $\mathbb{R}^{n+1}$, then the conclusion of Theorem 2.1 holds.*

To prove Theorem 2.1 we need the following lemmas.

Lemma 2.3 *Let $\mathcal{D} = \{\zeta : \zeta \in \mathcal{C}, |\zeta| < 1\}$ be the unit disc centered at the origin with radius* 1. *We define the operator D^{-1} by*

$$\begin{array}{rcl} D^{-1} & : & \mathcal{H}(\Omega) \longmapsto \mathcal{H}(\Omega) \\ v(\zeta) & \longmapsto & D^{-1}v(\zeta) = \int\limits_{\gamma_\zeta} v(t)\,dt, \end{array}$$

where γ_ζ is the path that relates the origin to ζ in $\mathcal{D}$. If there are $a > 1$ and $c \geq 0$ such that

$$|v(\zeta)| \leq c(1 - |\zeta|)^{-a},$$

then

$$\left|\left(D^{-1}v\right)(\zeta)\right| \leq \frac{c}{a-1}(1-|\zeta|)^{-a}. \tag{1}$$

Proof: If $\zeta = 0$, then every path in $\mathcal{D}$ joining 0 to 0 satisfies $(D^{-1}v)(0) = 0$ and the estimate holds. If $\zeta \neq 0$, we consider $\gamma_\zeta = [0, \zeta]$. This particular path does not affect the proof of the lemma since the integral does not depend on the choice of the path. Consider

$$\gamma_\zeta = \left\{t : t \in \mathcal{C}, t = re^{i\theta}, \theta \text{ fixed and } 0 \leq r \leq |\zeta|\right\}.$$

We have

$$\begin{aligned} \left|\left(D^{-1}v\right)(\zeta)\right| &\leq \int\limits_0^{|\zeta|} v\left(re^{i\theta}\right)dr \\ &\leq \int\limits_0^{|\zeta|} (1-r)^{-a}\,dr \\ &= \frac{-c}{1-a}(1-|\zeta|)^{-a+1} + \frac{c}{1-a} \\ &\leq \frac{c}{a-1}(1-|\zeta|)^{-a+1} \\ &\leq \frac{c}{a-1}(1-|\zeta|)^{-a}, \zeta \in \mathcal{D}. \end{aligned}$$

This completes the proof.

Lemma 2.4 *Let $v \in \mathcal{H}(\mathcal{D})$ and suppose there are two constants $a \geq 0$ and $c \geq 0$ satisfying the following estimate*

$$|v(\zeta)| \leq c(1-|\zeta|)^{-a}, \forall \zeta \in \mathcal{D}.$$

Then,

$$|v'(\zeta)| \leq ce(1+a)(1-|\zeta|)^{-a-1}, \forall \zeta \in \mathcal{D}. \tag{2}$$

Proof: Set $\mathcal{D}_1 = \left\{\zeta_1 : \zeta_1 \in \mathcal{C}, \zeta_1 = \zeta + \varepsilon e^{i\theta_1}; \theta_1 \in [0, 2\pi]\right\}$ for $\varepsilon > 0$, $\varepsilon \in]0, \rho[$ and $\rho = 1 - |\zeta|$. It is trivial to show $\overline{\mathcal{D}}_1 \subset \mathcal{D}$. We have $|\zeta_1| \leq |\zeta| + \varepsilon$, so that

$$\begin{aligned} 1 - |\zeta_1| &\geq 1 - |\zeta| - \varepsilon \\ &= \rho - \varepsilon \end{aligned}$$

and

$$(1 - |\zeta_1|)^{-a} \leq (\rho - \varepsilon)^{-a}.$$

The hypothesis of Lemma 2.4 becomes

$$|v(\zeta_1)| \leq c(\rho - \varepsilon)^{-a},$$

with $\zeta_1 \in \mathcal{D}_1 \subset \mathcal{D}$. Using Cauchy's inequality on the circle with center ζ and radius ε we get

$$|v'(\zeta)| \leq \frac{c}{\varepsilon}(\rho - \varepsilon)^{-a}. \tag{3}$$

We want to minimize the right hand side of (3), namely

$$f(\varepsilon) = \frac{c}{\varepsilon}(\rho - \varepsilon)^{-a},$$

so that

$$\begin{aligned} f'(\varepsilon) &= -\frac{c}{\varepsilon^2}(\rho - \varepsilon)^{-a} + \frac{ac}{\varepsilon}(\rho - \varepsilon)^{-a-1} \\ &= \frac{c}{\varepsilon^2}(\rho - \varepsilon)^{-a-1}\left[-(\rho - \varepsilon) + a\varepsilon\right] \\ &= \frac{c}{\varepsilon^2}(\rho - \varepsilon)^{-a-1}\left[-\rho + \varepsilon(a+1)\right], \end{aligned}$$

and $f'(\varepsilon) = 0$ if and only if $\varepsilon = \frac{\rho}{a+1}$, $(\varepsilon \in]0, \rho[)$. Therefore f passes by the minimum $\left(\frac{\rho}{a+1}, f\left(\frac{\rho}{a+1}\right)\right)$. Let us compute $f\left(\frac{\rho}{a+1}\right)$. We have

$$\begin{aligned} f\left(\frac{\rho}{a+1}\right) &= c\left(\frac{a+1}{\rho}\right)\left(\rho - \frac{\rho}{a+1}\right)^{-a} \\ &= \frac{c}{\rho}(a+1)\left(\frac{\rho(a+1) - \rho}{a+1}\right)^{-a} \\ &= \frac{c}{\rho}(a+1)\rho^{-a}\left(\frac{(a+1) - 1}{a+1}\right)^{-a} \\ &= \frac{c}{\rho}(a+1)\rho^{-a}\left(\frac{a}{a+1}\right)^{-a} \\ &= \frac{c}{\rho}(a+1)\rho^{-a}\left(\frac{a+1}{a}\right)^{a} \\ &= c(a+1)\left(\frac{a+1}{a}\right)^{a}\rho^{-a-1}. \end{aligned}$$

Since $\left(\frac{a+1}{a}\right)^a \leq e$ for every $a > 0$, then

$$|v'(\zeta)| \leq c(1+a)\,e\,(1-|\zeta|)^{-a-1}, \forall \zeta \in \mathcal{D}.$$

If $a = 0$, then

$$|v'(\zeta)| \leq \frac{c}{\varepsilon}$$

by Cauchy's inequality. Since ε is arbitrary in $]0, \rho[$, we can choose it as follows $\frac{\rho}{e} \leq \varepsilon \leq \rho$ and we have

$$\frac{1}{\varepsilon} \leq \rho^{-1}e \leq e\,(1-|\zeta|)^{-1}.$$

Consequently,

$$|v'(\zeta)| \leq ce\,(1-|\zeta|)^{-1}, \forall \zeta \in \mathcal{D}.$$

This concludes the proof of Lemma 2.4.

Lemma 2.5 *Let P_r be the polydisc in $\mathcal{C}^{n+1}$ containing the origin with radius $r = (r_0, r_1, ..., r_n)$ with $r_j \in \mathbb{R}_+$, that is $P_r = \{z, z \in \mathcal{C}^{n+1}, |z_j| < r_j, r_j > 0\}$. If $g \in \mathcal{H}(P_r)$, then there is a unique solution u in $\mathcal{H}(P_r)$ for the following problem:*

$$(P)\begin{cases} D^\beta u = g \\ D_j^k u_{|_{(P_r)_j}} = 0, (j,k) \in \mathcal{I}_\beta, |\beta| = m \geq 1. \end{cases}$$

Proof: We start by showing the uniqueness of the solution u of the problem (P).

Uniqueness: Suppose that u exists and u is in $\mathcal{H}(P_r)$, then

$$u(z) = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} u_\alpha \frac{z^\alpha}{\alpha!}$$

for every $z \in P_r$. It is well known that $u_\alpha = D^\beta u(0)$ and $D^\beta u$ exists due to the fact that u is $\mathcal{C}^\infty$ function of z. Now,

$$g \in \mathcal{H}(P_r) \Rightarrow \forall z \in P_r : g(z) = \sum_{\alpha \in \mathbb{Z}_+^{n+1}} g_\alpha \frac{z^\alpha}{\alpha!}.$$

We have

$$\begin{aligned} D^\beta u(z) &= \sum_{\alpha \in \mathbb{Z}_+^{n+1}} u_{\alpha+\beta} \frac{z^\alpha}{\alpha!} \\ &= \sum_{\alpha \in \mathbb{Z}_+^{n+1}} g_\alpha \frac{z^\alpha}{\alpha!} \end{aligned}$$

and by identification $u_{\alpha+\beta} = g_\alpha, \forall \alpha \in \mathbb{Z}_+^{n+1}$. Set $\nu = \alpha + \beta$ which means $\alpha = \nu - \beta, \alpha \geq 0$. Then $u_\nu = g_{\nu-\beta}$ are determined for every $\nu \in \mathbb{Z}_+^{n+1}, \nu - \beta \geq 0$. We need to determine u_ν for which we do not have $(\nu \geq \beta)$, that is, $\exists\ j = 0, 1, ..., n$ such that $\nu_j < \beta_j$ or $(j, k) \in \mathcal{I}_\beta$, hence $u_\nu = D^\nu u(0)$, if not $\nu \geq \beta$, by the initial conditions. Finally, if the solution u exists, then it is unique and we have

$$u(z) = \sum_{\nu \geq \beta} g_{\nu-\beta} \frac{z^\nu}{\nu!}.$$

Existence:

(i) Convergence of the solution $u(z)$: We want to prove the convergence of the solution $u(z)$ using the majorant method ([6], [9]). To do so, we need to estimate $u(z)$ by a convergent series and the problem will be solved. We have

$$\begin{aligned} u(z) &= \sum_{\nu \geq \beta} g_{\nu-\beta} \frac{z^\nu}{\nu!} \\ &= \sum_{\alpha \geq 0} g_\alpha \frac{z^{\alpha+\beta}}{(\alpha+\beta)!} \\ &= z^\beta \sum_{\alpha \geq 0} g_\alpha \frac{z^{\alpha+\beta}}{(\alpha+\beta)!} \end{aligned}$$

and the problem reduces to proving that the series $\sum_{\alpha \geq 0} g_\alpha \frac{z^{\alpha+\beta}}{(\alpha+\beta)!}$ converges. First, we have the following estimate

$$\frac{|g_\alpha|}{(\alpha+\beta)!} \leq \frac{|g_\alpha|}{\alpha!}, \tag{4}$$

for every $\alpha \geq 0$. Since the term $\frac{|g_\alpha|}{\alpha!}$ of (4) is the general term of a series which converges on P_r by hypothesis, therefore u converges also on P_r.

(ii) Verification of initial conditions: Let us check that the initial conditions are satisfied, in other words $D_j^k u_{|(P_r)_j} = 0, (j, k) \in \mathcal{I}_\beta$. We have

$$\begin{aligned} u(z) &= \sum_{\alpha \geq 0} g_\alpha \frac{z^{\alpha+\beta}}{(\alpha+\beta)!} \\ &= \sum_{\alpha \geq 0} g_\alpha \frac{z_0^{\alpha_0+\beta_0} ... z_j^{\alpha_j+\beta_j} ... z_n^{\alpha_n+\beta_n}}{(\alpha_0+\beta_0)! ... (\alpha_j+\beta_j)! ... (\alpha_n+\beta_n)!}, \end{aligned}$$

and

$$D_j^k u(z) = \sum_{\alpha \geq 0} g_\alpha \frac{z_0^{\alpha_0+\beta_0} ... z_j^{\alpha_j+\beta_j-k} ... z_n^{\alpha_n+\beta_n}}{(\alpha_0+\beta_0)! ... (\alpha_j+\beta_j-k)! ... (\alpha_n+\beta_n)!}, \tag{5}$$

where $(j,k)\in\mathcal{I}_\beta$ and $\alpha_j\geq 0$, hence $\alpha_j+\beta_j-k>0$, and consequently

$$D_j^k u\left(z_0,...,z_{j-1},0,z_{j+1},...,z_n\right)=0.$$

Therefore u exists and is unique, which proves the Lemma 2.5.

Remark 2.1 We denote by $D^{-\beta}$ the inverse operator of D^β, that is,

$$D^{-\beta}:\mathcal{H}\left(P_r\right)\longmapsto\mathcal{H}\left(P_r\right)$$

and defined by

$$D^{-\beta}g=u\Leftrightarrow\begin{cases}D^\beta u=g\\ D_j^k u_{|(P_r)_j}=0,(j,k)\in\mathcal{I}_\beta.\end{cases}$$

Lemma 2.6 *Let Ω be an open subset of $\mathcal{C}^{n+1}$ containing the origin such that $\Omega\supset\overline{P}_1$, where $\overline{P}_1$ is the closed polydisc of radius $1\in\mathbb{R}_+^{n+1}$, $(a_\alpha)_{|\alpha|\leq|\beta|=m}$ and v are holomorphic functions in Ω. Then the sequence $(v_p)_{p\in\mathbb{Z}_+}\subset\mathcal{H}(P_1)$ defined by*

$$v_0=v,v_{p+1}=D^{-\beta}\left(\sum_{|\alpha|\leq m}a_\alpha D^\alpha v_p\right),\forall p\in\mathbb{Z}_+$$

satisfies the following estimate

$$\left|v_p\left(z\right)\right|\leq MC^p\prod_{j=0}^{n}\left(1-\left|z_j\right|\right)^{-mp},\forall p\in\mathbb{Z}_+,\forall z\in P_1 \tag{6}$$

with $M=\sup_{z\in\overline{P}_1}\left|v\left(z\right)\right|$, $C=A(2e)^m$ and $A=\sum_{|\alpha|\leq m}\sup_{z\in\overline{P}_1}\left|a_\alpha\left(z\right)\right|$.

Proof: We use the iterative method on p to prove the lemma as we did in [8]. For $p=0$, $v_0(z)=v(z)$ for every $z\in P_1$ and $|v(z)|\leq\sup_{z\in\overline{P}_1}\left|v\left(z\right)\right|=M$ holds. Suppose the estimate (6) is true up to the order p, that is,

$$|v_p(z)|\leq MC^p\prod_{j=0}^{n}(1-|z_j|)^{-mp},\forall z\in P_1,$$

we prove it for $p+1$. Set $V_p(\zeta)=v_p\left(\zeta,z_1,...,z_n\right)$ with $\left(\zeta,z_1,...,z_n\right)\in P_1$ and $\zeta\in\mathcal{D}=\{\zeta:\zeta\in\mathcal{C};|\zeta|<1\}$. We have

$$\left|V_p\left(\zeta\right)\right|\leq M_p\left(1-|\zeta|\right)^{-mp}, \tag{7}$$

where $M_p=MC^p\prod_{j=1}^{n}\left(1-|z_j|\right)^{-mp}$. If we apply Lemma 2.4 to (7) with $a=mp\geq 0$, $C=M_p\geq 0$, we get

$$\left|V_p'\left(\zeta\right)\right|\leq M_p e\left(1+mp\right)\left(1-|\zeta|\right)^{-mp-1},\forall\zeta\in\mathcal{D}.$$

If we repeat this process α_0 times, we obtain

$$\begin{aligned} \left|V_p^{(\alpha_0)}(\zeta)\right| &\leq M_p e^{\alpha_0}(1+mp)\ldots(\alpha_0+mp)(1-|\zeta|)^{-mp-\alpha_0} \\ &\leq M_p e^{\alpha_0}(\alpha_0+mp)^{\alpha_0}(1-|\zeta|)^{-mp-\alpha_0}, \forall\zeta\in\mathcal{D}. \end{aligned}$$

In other words,

$$|D^{\alpha_0}v_p(z_0,z_1,...,z_n)| \leq M_p e^{\alpha_0}(\alpha_0+mp)^{\alpha_0}(1-|z_0|)^{-mp-\alpha_0}, \forall z_0\in\mathcal{D}. \quad (8)$$

Apply (8) for the other components z_1, z_2,..., and z_n to get

$$|D^{\alpha}v_p(z)| \leq MC^p e^{|\alpha|}\prod_{j=0}^{n}(\alpha_j+mp)^{\alpha_j}(1-|z_j|)^{-mp-\alpha_j}, \forall z\in P_1. \quad (9)$$

For any α such that $|\alpha|\leq m$, we have from (9):

$$|D^{\alpha}v_p(z)| \leq MC^p e^m(m(p+1))^m\prod_{j=0}^{n}(1-|z_j|)^{-m(p+1)}, \forall z\in P_1. \quad (10)$$

Since the sequence $(v_p)_{p\in\mathbb{Z}_+}$ satisfies

$$D^{\beta}v_{p+1}(z) = \sum_{|\alpha|\leq m} a_{\alpha}(z)D^{\alpha}v_p(z). \quad (11)$$

hence by (10), we obtain

$$\left|D^{\beta}v_{p+1}(z)\right| \leq AMC^p e^m(m(p+1))^m\prod_{j=0}^{n}(1-|z_j|)^{-m(p+1)}, \forall z\in P_1. \quad (12)$$

where $A = \left(\sum_{|\alpha|\leq m}\sup_{z\in P_1}|a_{\alpha}(z)|D^{\alpha}v_p(z)\right)$. Suppose $m\geq 2$, then $\exists j\in\{1,...,n\}$ such that $\beta_j\geq 1$ ($|\beta|=m$). Suppose that $\beta_0\geq 1$ and set as before

$$W_{p+1}(\zeta) = D^{\beta'}v_{p+1}(\zeta,z_1,...,z_n)\,\text{for}\ \zeta\in\mathcal{D},$$

where $\beta'=(\beta_0-1,\beta_1,....,\beta_n)$. The Cauchy conditions

$$D_0^{\beta_0-1}v_{p+1}(0,z_1,...,z_n)=0,$$

allows us to write $W_{p+1}(0)=0$. As W_{p+1} is a primitive of

$$D^{\beta'}v_{p+1}(.,z_1,...,z_n) \qquad \text{that vanish at zero,}$$

the inequality (12) becomes

$$|D_0 W_{p+1}(\zeta)| \leq AMC^p (em(p+1))^m \prod_{j=1}^{n} (1-|z_j|)^{-m(p+1)} (1-|\zeta|)^{-m(p+1)}. \tag{13}$$

According to Lemma 2.3 with $a = m(p+1) \geq m \geq 2$ in (13), and

$$M_p = AMC^p (em(p+1))^m \prod_{j=1}^{n} (1-|z_j|)^{-m(p+1)},$$

we get

$$|D_0 W_{p+1}(\zeta)| \leq M_p \frac{(1-|\zeta|)^{-m(p+1)}}{m(p+1)-1}. \tag{14}$$

If we repeat this process β_0 times and also with respect to the other components, we obtain the following estimate

$$|v_{p+1}(z)| \leq AMC^p \frac{(em(p+1))^m}{(m(p+1)-1)^m} \prod_{j=0}^{n} (1-|z_j|)^{-m(p+1)}, \forall z \in P_1. \tag{15}$$

We need to check that

$$A \left[\frac{em(p+1)}{m(p+1)-1} \right]^m \leq C,$$

where $C = A(2e)^m$. In fact, $m(p+1) \leq 2(m(p+1)-1)$ because $m(p+1) \geq 1$, $(m \geq 2)$, consequently

$$|v_{p+1}(z)| \leq MC^{p+1} \prod_{j=0}^{n} (1-|z_j|)^{-m(p+1)}, \forall z \in P_1, \tag{16}$$

and the result is established for $p+1$.

The case $a = 0$, is not provided apriori in Lemma 2.3. Here our reasoning by induction, is the passage from $p = 0$ to $p = 1$.

In the case $m = 1$. We have $|\beta| = 1$, suppose for instance $\beta = (1, 0, ..., 0)$, then $C = 2eA$, $v_0 \in \mathcal{H}(P_1)$ and we have the following problem

$$(P'''')\begin{cases} D_0 v_1(z) = \sum\limits_{|\alpha| \leq 1} (a_\alpha D^\alpha v_0)(z) \\ v_1(0, z_1, ..., z_n) = 0. \end{cases}$$

We know that

$$|v_0(z)| \leq M, \forall z \in P_1, \tag{17}$$

so that

$$|v_0(\zeta, z_1, ..., z_n)| \leq M, \forall \zeta \in \mathcal{D}, \tag{18}$$

moreover $0 < 1 - |\zeta| \leq 1$ and $\zeta \in \mathcal{D}$, therefore

$$(1 - |\zeta|)^{-1} \geq 1. \tag{19}$$

Set

$$V_0(\zeta) = v_0(\zeta, z_1, ..., z_n), \tag{20}$$

then

$$|V_0(\zeta)| \leq M(1 - |\zeta|)^{-1}, \forall \zeta \in \mathcal{D}, \tag{21}$$

by (19) and (20). Using Lemma 2.4, we get

$$|V_0'(\zeta)| \leq Me(1 - |\zeta|)^{-1}, \forall \zeta \in \mathcal{D}. \tag{22}$$

Since $z \in P_1$, $0 < 1 - |z_j| \leq 1$ for every $j \in \{0, 1, ..., n\}$, then $(1 - |z_0|) \geq \prod_{j=0}^{n}(1 - |z_j|)$ and we have

$$(1 - |z_0|)^{-1} \leq \prod_{j=0}^{n}(1 - |z_j|)^{-1}. \tag{23}$$

For $z_0 \in \mathcal{D}$, the following estimate holds:

$$|D_0 v_0(z_0, z_1, ..., z_n)| \leq Me(1 - |z_0|)^{-1}. \tag{24}$$

Combining (23) and (24), we have

$$|D_0 v_0(z_0, z_1, ..., z_n)| \leq Me\prod_{j=0}^{n}(1 - |z_j|)^{-1}. \tag{25}$$

This inequality (25) is true for $D_j v_0$, $\forall\ j \in \{0, 1, ..., n\}$, that is

$$|D_j v_0(z_0, z_1, ..., z_n)| \leq Me\prod_{j=0}^{n}(1 - |z_j|)^{-1}, \forall j \in \{0, 1, ..., n\}. \tag{26}$$

If $z_0 = 0$, then

$$|v_1(0, z_1, ..., z_n)| \leq AMC\prod_{j=0}^{n}(1 - |z_j|)^{-1}. \tag{27}$$

We suppose $z \in P_1$ with $z_0 \neq 0$ and since $v_1 \in \mathcal{H}(P_1)$, (see P'''',) we have

$$v_1(z) = \int_\gamma D_0 v_1(\zeta, z_1, ..., z_n) d\zeta,$$

where γ is the path that relates the origin to $z_0 \in \mathcal{D}$. In the problem (P'''') we have $D_0 v_1(z) = \sum_{|\alpha| \leq 1} a_\alpha(z)\ D^\alpha v_0(z)$, hence

$$\begin{aligned} v_1(z) &= \int_0^{|z_0|} D_0 v_1\left(re^{i\theta}, z_1, ..., z_n\right) e^{i\theta} dr \\ &= \int_0^{|z_0|} \sum_{|\alpha| \leq 1} a_\alpha\left(re^{i\theta}, z_1, ..., z_n\right)\ D^\alpha v_0\left(re^{i\theta}, z_1, ..., z_n\right) e^{i\theta} dr, \end{aligned}$$

so that

$$|v_1(z)| \leq \int_0^{|z_0|} \left(\sum_{|\alpha| \leq 1} \left| a_\alpha\left(re^{i\theta}, z_1, ..., z_n\right) \right| \right) \max_{|\alpha| \leq 1} \left| D^\alpha v_0\left(re^{i\theta}, z_1, ..., z_n\right) \right| dr.$$

Recall that α lies on $\{(0,0), (1,0), (0,1)\}$ and using (17) and (26), we have

$$|v_1(z)| \leq A \int_0^{|z_0|} \max\left(Me \prod_{j=1}^n (1 - |z_j|)^{-1} (1 - r)^{-1}, M \right) dr.$$

However, $M \leq Me \leq Me(1 - |r|)^{-1}$ because $0 < 1 - r \leq 1$, therefore

$$|v_1(z)| \leq 2MAe \prod_{j=1}^n (1 - |z_j|)^{-1} \left[-\frac{\ln(1 - r)}{2} \right]_0^{|z_0|}$$

and $\frac{1}{2} \ln(1 - |z_0|)^{-1} \leq (1 - |z_0|)^{-1}$. If we set $1 - |z_0| = u$ and $v = \frac{1}{u} \geq 1$ then $\frac{1}{2} \ln v - v \leq 0$ and we have

$$|v_1(z)| \leq 2MAe \prod_{j=0}^n (1 - |z_j|)^{-1}, \forall z \in P_1,$$

that is,

$$|v_1(z)| \leq MC \prod_{j=0}^n (1 - |z_j|)^{-1}, \forall z \in P_1,$$

which completes the proof.

Lemma 2.7 *Let Ω be an open subset of $\mathcal{C}^{n+1}$ containing the origin, P_r a polydisc centered at the origin with radius $(r,...,r)$, $r>0$, $(a_\alpha)_{|\alpha|\leq|\beta|=m}$ and v are holomorphic functions in Ω. Set $A=\sum_{|\alpha|\leq m} r^{|\beta-\alpha|}\sup_{z\in\overline{P}_r}|a_\alpha(z)|$, $C=A(2e)^m$, and $M=\sup_{z\in\overline{P}_r}|v(z)|$, then the sequence $(v_p)_{p\in\mathbb{Z}_+}\subset\mathcal{H}(P_r)$ defined by*

$$\begin{aligned} v_0 &= v\in\mathcal{H}(P_r), \qquad (28)\\ D^\beta v_{p+1} &= \sum_{|\alpha|\leq m} a_\alpha D^\alpha v_p, \forall p\in\mathbb{Z}_+, \qquad (29)\end{aligned}$$

satisfies the following estimate

$$|v_p(z)|\leq MC^p\prod_{j=0}^{n}\left(1-\frac{|z_j|}{r}\right)^{-mp}, \forall p\in\mathbb{Z}_+, \forall z\in P_r. \qquad (30)$$

Proof: Let $\Phi: P_1\longmapsto P_r$ be a mapping defined by $\Phi(\zeta)=r\zeta$ for every $\zeta\in P_1$. We associate Φ^* to Φ, defined by

$$\begin{aligned} \Phi^* &: \mathcal{H}(P_r)\longmapsto\mathcal{H}(P_1)\\ f &\longmapsto \Phi^*(f)=f\circ\Phi.\end{aligned}$$

Now, $(v_p)_{p\in\mathbb{Z}_+}$ satisfy the hypotheses of the lemma. Set $V_p=\Phi^*(v_p)=v_p\circ\Phi$, $p\in\mathbb{Z}_+$, v_p is defined on P_r and V_p on P_1. The equation (29) becomes

$$D^\beta v_{p+1}\circ\Phi=\sum_{|\alpha|\leq m} a_\alpha\circ\Phi D^\alpha v_p\circ\Phi. \qquad (31)$$

Let $\gamma=(\gamma_0,\gamma_1,...,\gamma_n)$ be a multi-index of $\mathbb{Z}_+^{n+1}$ and let us compute $D^\gamma(v_p\circ\Phi)$. We have

$$\begin{aligned} D_0(v_p\circ\Phi)(\zeta) &= \sum_{j=0}^{n} D_j v_p(\Phi(\zeta))\circ D_0\Phi_j(\zeta)\\ &= D_0 v_p(\Phi(\zeta))\, r.\end{aligned}$$

One can check easily the following result

$$D_0^{\gamma_0}(v_p(r\zeta))=(D_0^{\gamma_0}v_p)(r\zeta)\, r^{\gamma_0}.$$

More generally, we have

$$D^\gamma(v_p\circ\Phi)(\zeta)=r^{|\gamma|}\left[(D^\gamma v_p)\circ\Phi\right](\zeta), \qquad (32)$$

or

$$(D^\gamma V_p)(\zeta)=r^{|\gamma|}\left[(D^\gamma v_p)\circ\Phi\right](\zeta), \forall\gamma\in\mathbb{Z}_+^{n+1}. \qquad (33)$$

Hence

$$r^{|\beta|}\left[\left(D^{\beta}v_{p+1}\right)\circ\Phi\right](\zeta)=\sum_{|\alpha|\le m}r^{|\beta|-|\alpha|}\left(a_{\alpha}\circ\Phi\right)(\zeta)\,r^{|\alpha|}\left(D^{\alpha}v_{p}\circ\Phi\right)(\zeta). \quad (34)$$

Now, from the relation (34), we obtain

$$D^{\beta}V_{p+1}(\zeta)=\sum_{|\alpha|\le m}r^{|\beta|-|\alpha|}\left(a_{\alpha}\circ\Phi\right)(\zeta)\,r^{|\alpha|}D^{\alpha}V_{p}(\zeta). \quad (35)$$

Set $A_{\alpha}=a_{\alpha}\circ\Phi.r^{|\beta|-|\alpha|}$, it is clear that $A_{\alpha}\in\mathcal{H}(P_1)$. The initial conditions become

$$\begin{aligned} D_j^k V_{p|_{(P_1)_j}} &= D_j^k\left(v_p\circ\Phi\right)_{|_{(P_1)_j}} \\ &= r^k D_j^k v_p\circ\Phi_{|_{(P_1)_j}} \\ &= r^k D_j^k v_{p|_{(P_r)_j}} \\ &= 0, (j,k)\in\mathcal{I}_{\beta}. \end{aligned}$$

Now, $v_0=v$, $M=\sup_{z\in\overline{P}_r}|v_0(z)|=\sup_{\zeta\in\overline{P}_1}|V_0(\zeta)|$, $A=\sum_{|\alpha|\le m}r^{|\beta-\alpha|}\sup_{z\in\overline{P}_r}|a_{\alpha}(z)|=\sum_{|\alpha|\le m}r^{|\beta-\alpha|}\sup_{\zeta\in\overline{P}_1}|A_{\alpha}(\zeta)|$, $A_{\alpha}=a_{\alpha}\circ\Phi.r^{|\beta|-|\alpha|}$, and the Lemma 2.6 allows us to write

$$|V_p(\zeta)|\le MC^p\prod_{j=0}^{n}\left(1-|z_j|\right)^{-mp},$$

$\forall p\in\mathbb{Z}_+$, and $\forall\zeta\in P_1$ or

$$|v_p(z)|\le MC^p\prod_{j=0}^{n}\left(1-|z_j|\right)^{-mp},$$

$\forall p\in\mathbb{Z}_+$, $\forall z\in P_r$, and the proof of the Lemma 2.7 is complete.

To complete the proof of Theorem 2.1, let us start with the case $\varphi=0$.

Let $(a_{\alpha})_{|\alpha|\le m}\in\mathcal{H}(\Omega)$ and $f\in\mathcal{H}(\Omega)$, Ω be a neighborhood of the origin, we are looking for u and $\Omega'\subset\Omega$ such that $u\in\mathcal{H}(\Omega')$ is a solution of the following problem:

$$\begin{aligned} D^{\beta}u &= \sum_{|\alpha|\le m}a_{\alpha}D^{\alpha}u+f \\ D_j^k\left(u-\varphi\right)_{|_{\Omega'_j}} &= 0, (j,k)\in\mathcal{I}_{\beta}, |\beta|=m. \end{aligned}$$

We know that there are $\rho>0$ and $r>0$ such that $\Omega\supset P_{\rho}\supset\overline{P_r}$. Let $u_0=0, u_{p+1}=D^{-\beta}\left(\sum_{|\alpha|\le m}a_{\alpha}D^{\alpha}u_p+f\right)$ and set $v_p=u_{p+1}-u_p$, $p\in\mathbb{Z}_+$.

Recall that $D^{-\beta} \in \mathcal{L}\left(\mathcal{H}\left(P_\rho\right), \mathcal{H}\left(P_\rho\right)\right)$, hence

$$\begin{aligned} v_{p+1} &= u_{p+2} - u_{p+1} \\ &= D^{-\beta}\left(\sum_{|\alpha|\leq m} a_\alpha D^\alpha u_p + f\right) - D^{-\beta}\left(\sum_{|\alpha|\leq m} a_\alpha D^\alpha u_p + f\right) \\ &= D^{-\beta}\left(\sum_{|\alpha|\leq m} a_\alpha D^\alpha v_p\right). \end{aligned}$$

Set $M(r) = \sup_{z\in\overline{P}_r} |u_p(z)|$, $A(r) = \sum_{|\alpha|\leq|\beta|=m} r^{|\beta-\alpha|} \sup_{z\in\overline{P}_r} |a_\alpha(z)|$ and $C(r) = (2e)^m A(r)$. According to Lemma 2.7, we have the following estimate:

$$|v_p(z)| \leq M(r)\,[C(r)]^p \prod_{j=0}^{n}\left(1 - \frac{|z_j|}{r}\right)^{-mp}, \forall p \in \mathbb{Z}_+, \forall z \in P_r. \tag{36}$$

If we denote by $R = (r_0, r_1, ..., r_n)$ with $r_j < r$, then

$$|v_p(z)| \leq M(r)\,[C(r)]^p \prod_{j=0}^{n}\left(1 - \frac{r_j}{r}\right)^{-mp}, \tag{37}$$

$\forall p \in \mathbb{Z}_+, \forall z \in P_R$. The function $r \longmapsto A(r)$ is continuous on a neighborhood of the origin and by hypothesis, we have

$$\lim_{r\to 0} C(r) = (2e)^m A(0) = (2e)^m \sum_{|\alpha|\leq m} |a_\alpha(0)| < 1. \tag{38}$$

Therefore $C(r) < 1$ is valid on a neighborhood of the origin due to the continuity of $C(r)$. We choose r_0 such that $r_j < r \leq r_0$, with r fixed. As $\lim_{r_j\to 0} \prod_{j=0}^{n}\left(1 - \frac{r_j}{r}\right) = 1$, we can always write

$$[C(r)]^{\frac{1}{m}} < \prod_{j=0}^{n}\left(1 - \frac{r_j}{r}\right) < 1.$$

Set $\gamma(r) = \gamma = C(r)\left[\prod_{j=0}^{n}\left(1 - \frac{r_j}{r}\right)\right]^{-m} < 1$. Then

$$|v_p(z)| \leq M\gamma^p, 0 \leq \gamma < 1,$$

implies the series $(v_p)_{p\in\mathbb{Z}_+}$ converges normally on P_R, that is, u_p converges uniformly to $u \in \mathcal{H}(P_R)$ on every compact P_R and $D^\alpha u_p$ converges uniformly also to $D^{\alpha}u \in \mathcal{H}(P_R)$ by Weistrass Theorem. This limit satisfies

$$\begin{aligned} D^{\beta}u &= \sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}u + f \\ D_j^k u_{|_{(P_R)_j}} &= 0, (j,k) \in \mathcal{I}_{\beta}. \end{aligned}$$

Let us prove the uniqueness of such solution. Suppose we have two solutions u and u', then

$$u - u' = D^{-\beta}\left(\sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}\left(u-u'\right)\right).$$

Set $u'' = u - u'$. It is clear that u'' satisfies

$$\begin{aligned} u'' &= D^{-\beta}\left(\sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}u''\right) \\ D_j^k u''_{|_{(P_R)_j}} &= 0, (j,k) \in \mathcal{I}_{\beta}. \end{aligned}$$

The sequence $(v_p)_{p\in\mathbb{Z}_+}$ is such that

$$\begin{aligned} v_0 &= u'', \\ v_1 &= D^{-\beta}\left(\sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}u''\right), \end{aligned}$$

and by Lemma 2.5, there is a unique v_1 on P_R, solution of the Cauchy problem and since u'' satisfies the Cauchy problem, $v_1 = u''$. Now, step by step we arrive at $v_p = u''$ for every $p \in \mathbb{Z}_+$, but

$$|v_p(z)| \leq M\gamma^p, 0 \leq \gamma < 1,$$

that is,

$$|u''(z)| \leq M\gamma^p, 0 \leq \gamma < 1,$$

hence $u'' \equiv 0$ on P_R, and we have $u = u'$ on P_R. If we choose $\Omega' \subset P_R$ connected open set containing the origin, again we use the estimates for $(v_p)_{p\in\mathbb{Z}_+}$ and apply Weistrass theorem.

2. Let us now take the case $\varphi \neq 0$; $\varphi \in \mathcal{H}(\Omega)$. Set $u = U + \varphi$, then

$$D_j^k\left(u-\varphi\right)_{|_{\Omega'_j=\left\{z:z\in\mathcal{C}^{n+1},z_j=0\right\}\cap\Omega'}} = 0 \Leftrightarrow D_j^k U_{|_{\Omega'_j}} = 0$$

for every $(j,k) \in \mathcal{I}_{\beta}$.

$$\begin{aligned} D^{\beta}u &= D^{\beta}U + D^{\beta}\varphi \\ &= \sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}U + \sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}\varphi + f. \end{aligned}$$

Set $F = -D^{\beta}\varphi + \sum\limits_{|\alpha|\leq m} a_{\alpha} D^{\alpha}\varphi + f \in \mathcal{H}(\Omega)$. We need to solve the following problem:

$$\begin{aligned} D^{\beta}U &= \sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}U + \sum_{|\alpha|\leq m} a_{\alpha} D^{\alpha}\varphi + F \\ D_j^k U_{|\Omega'_j} &= 0, (j,k) \in \mathcal{I}_{\beta}, |\beta| = m, \end{aligned}$$

it follows from the previous studies that there exists a unique U and hence $u = U + \varphi$ is the unique solution of the problem under consideration. This concludes the proof of Theorem 2.1.

Proof of Corollary 2.2: We have

$$A = \{\alpha : \alpha \in \mathbb{Z}_{+}^{n+1}, |\alpha| = m, a_{\alpha}(0) \neq 0\}.$$

If A is empty, then its convex hull is empty $\forall \alpha \in \mathbb{Z}_{+}^{n+1}$, we have $|\alpha| = m$, $a_{\alpha}(0) = 0$, hence $\sum\limits_{|\alpha|=m} |a_{\alpha}(0)| = 0 < (2e)^{-m}$ verifies the hypotheses of the Theorem 2.1.

If A is non-empty, then A is a closed subset of $\mathbb{R}^{n+1}$ because it contains finite elements.

Theorem 2.8 *(Darboux-Goursat-Beudon(DGB)) Let Ω be an open subset of $\mathcal{C}^{n+1}$ containing the origin, $(a_{\alpha})_{|\alpha|\leq m}$, f and φ be holomorphic functions on Ω. Suppose that $\alpha = (m, 0, ..., 0)$, $a_{\alpha} \equiv 0$ and if $\beta = (m-1, 1, 0, ..., 0)$, $a_{\beta}(0) \neq 0$, then there is $\Omega' \subset \Omega$ connected open neighborhood of the origin such that the problem below*

$$(DGB) \begin{cases} D^{\beta}u = \sum\limits_{|\alpha|\leq m} a_{\alpha} D^{\alpha}u + f \\ D_0^k \left(u - \varphi\right)_{|\Omega'_0} = 0, 0 \leq k < m-1 \\ \left(u - \varphi\right)_{|\Omega'_1} = 0 \end{cases}$$

admits a unique holomorphic solution on Ω'.

Remarks 2.2

1. We have $\mathcal{I}_{\beta} = \{(0,k) : 0 \leq k < m-1, (1,0)\}$,
2. The problem (DGB) (Goursat-Cauchy problem) is a problem with mixed data,
3. We can isolate β in the equation

$$\sum_{|\alpha|\leq m} a_{\alpha}(z)\, D^{\alpha}u(z) = f(z) \tag{39}$$

in the following sense and we write

$$a_\beta(z)\,D^\beta u(z)+\sum_{\substack{|\alpha|\le m\\ \alpha\ne\beta}} a_\alpha(z)\,D^\alpha u(z)=f(z), \tag{40}$$

hence

$$a_\beta(z)\,D^\beta u(z)=\sum_{\substack{|\alpha|\le m\\ \alpha\ne\beta}} -a_\alpha(z)\,D^\alpha u(z)+f(z). \tag{41}$$

As a_β does not vanish at the origin, it is not null on every neighborhood of the origin, in other words, there is $\Omega_1\subset\Omega$ such that $a_\beta(z)\ne 0$ for every $z\in\Omega_1$. Hence we have a new equation in this neighborhood

$$D^\beta u(z)=\sum_{\substack{|\alpha|\le m\\ \alpha\ne\beta}} -\frac{a_\alpha(z)}{a_\beta(z)}D^\alpha u(z)+\frac{f(z)}{a_\beta(z)}. \tag{42}$$

If we set

$$A_\alpha=-\frac{a_\alpha}{a_\beta}\text{ and }F=\frac{f}{a_\beta} \tag{43}$$

then (42) becomes

$$D^\beta u(z)=\sum_{\substack{|\alpha|\le m\\ \alpha\ne\beta}} A_\alpha(z)\,D^\alpha u(z)+F(z). \tag{44}$$

One can show that $\beta\notin$ convex hull of A (cf. [2]). The equation (44) is called the reduced equation of (42).

3 Extended Darboux-Goursat-Beudon Theorem

Let $P(z;D)=\sum_{|\alpha|\le m} a_\alpha(z)D^\alpha$, $z=(z_0,z')$ with $z'=(z_1,...,z_n)$. We shall take the first and second components of z but with the derivatives of orders $(m-k)$ and k respectively.

Theorem 3.1 *Let Ω be an open subset of $\mathcal{C}^{n+1}$ containing the origin, and let $(a_\alpha)_{|\alpha|\le m}$, f, and φ be holomorphic functions on Ω. Suppose that $\beta=(m-$*

$k, k, 0, ..., 0)$, $a_\beta(0) \neq 0$ with $k = (0, 1, ..., n)$ and $|\beta| = m \geq 1$, then there exists $\Omega' \subset \Omega$ connected open neighborhood of the origin such that the problem below

$$(P^*) \begin{cases} Pu = f \\ D_j^k(u - \varphi)_{|_{\Omega'_j}} = 0, (j, k) \in \mathcal{I}_\beta \end{cases}$$

admits a unique holomorphic solution on Ω'.

Proof: Let $A = \{\alpha : \alpha \in \mathbb{Z}_+^{n+1}, |\alpha| = m, a_\alpha(0) \neq 0\}$. According to Corollary 2.2, it suffices to prove that $\beta \notin$ convex hull of A. Since we take the two components z_0 and z_1, then we can consider the projection of this convex hull and solve the problem $\mathbb{R}^2$. For this we refine the hypotheses at maximum and discuss the position of k with respect to the center of the segment $[(0, m), (m, 0)]$. Let $E[\frac{m}{2}]$ denote the entire part of $\frac{m}{2}$. The problem is already studied for $k = 0$. Suppose now that $k \neq 0$ and $0 < k \leq m$.

a) If $(m - k) \in \{0, ..., E[\frac{m}{2}]\}$ then we take $a_{j,m-j,0,...,0}(z) \equiv 0$ for every $j : 0 \leq j \leq m - k$ and we have $\beta \notin$ convex hull of A.

b) If $(m - k) \geq E[\frac{m}{2}]$ then we take $a_{j,m-j,0,...,0}(z) \equiv 0$ for every $j : m - k \leq j \leq m$ and we have $\beta \notin$ convex hull of A. Now let us take a look at the following problem:

$$(P^{**}) \begin{cases} Pu = f \\ D_j^k u_{|_{\Omega'_j}} = u_{j,k}, (j, k) \in \mathcal{I}_\beta, \end{cases}$$

where the functions $u_{j,k}$ are holomorphic functions on Ω'_j. Let us compare the problems denoted by P^* and P^{**}. These two problems are equivalent to the condition that $u_{j,k}$ verify certain conditions of compatibility.

i) The problem P^* implies the problem P^{**}.

ii) The problem P^{**} implies the problem P^*. In fact, it is true under preservation that the functions $u_{j,k}$ verify certain conditions of compatibility between the problems.

Given the functions $u_{j,k}$, can we find a function φ such that $D_j^k \varphi_{|_{\Omega'_j}} = u_{j,k}$ for $(j, k) \in \mathcal{I}_\beta$?

Conditions in the two particular cases:

a) $\beta = (m, 0, ..., 0)$.

$$D_j^k \varphi_{|_{\Omega'_j}} = u_{j,k}, 0 \leq k \leq m - 1.$$

If we set $\varphi(z) = \sum\limits_{k=0}^{m-1} u_{0,k}(z') \frac{z_0^k}{k!}$, then φ is a candidate. In fact,

$$\varphi(z) = u_{0,0}(z') + \sum_{k=1}^{m-1} u_{0,k}(z') \frac{z_0^k}{k!}.$$

For $z_0 = 0$, we have

$$\varphi(0, z') = u_{0,0}(z') = u_{|\Omega'_0}$$

which can be written as $(u - \varphi)_{|\Omega'_0} = 0$. Now,

$$D_0^j \varphi(z) = \sum_{k=0}^{m-1} u_{0,k}(z') k(k-1)...(k-j+1) \frac{z_0^{k-j}}{k!},$$

$$D_0^{m-1} \varphi(z') = u_{0,m-1}(z'),$$

$$(D_0^j \varphi)(0, z') = u_{0,m-1}(z') = D_0^k u_{|\Omega'_0}.$$

b) $\beta = (m-1, 1, 0, ..., 0)$. If we choose $\varphi(z) = \sum_{k=0}^{m-1} V_k(z') \frac{z_0^k}{k!}$, where V_k denote the holomorphic functions on Ω_1, $\Omega_1 = \Omega' \cap \{z : z = (z_0, 0, z")\}$, and $z" = (z_2, ..., z_n)$. The function φ as defined must satisfy

$$(P^{***}) \begin{cases} D_0^k \varphi_{|\Omega'_0} = u_{0,k}, 0 \le k \le m-2. \\ \varphi_{|\Omega'_1} = u_{1,0}. \end{cases}$$

The functions $u_{0,k}$ and $u_{1,0}$ are not chosen arbitrary. There is certain dependence between them because they are the initials conditions of u; in fact:

For $k = 0$, we have

$$\varphi(0, z') = u_{0,0}(z') = V_0(z'), \tag{45}$$

$$\varphi(z_0, 0, z") = u_{1,0}(z_0, z"), \tag{46}$$

and for $z_0 = 0$,

$$\varphi(0, 0, z') = u_{1,0}(0, z"), \tag{47}$$

$$\varphi(0, 0, z") = u_{0,0}(0, z"). \tag{48}$$

Note that (47)-(48) $\Rightarrow u_{0,0}(0, z") = u_{1,0}(z_0, z")$.

For $k = 1$, we have

$$D_0^1 \varphi(0, z') = u_{0,1}(z'), \tag{49}$$

$$\varphi(z_0, 0, z") = u_{1,0}(z_0, z"). \tag{50}$$

Also, (49)-(50) $\Rightarrow D_0^1\varphi(z_0,0,z")=D_0^1u_{1,0}(z_0,z")$, and for $z_0=0$,

$$D_0^1\varphi(0,0,z")=D_0^1u_{1,0}(0,z"). \tag{51}$$

For $z_1=0$, we have

$$D_0^1\varphi(0,0,z")=u_{0,1}(0,z"), \tag{52}$$

and (51)-(52) $\Rightarrow u_{0,1}(0,z")=D_0^1u_{1,0}(0,z")$.

If we repeat this process until the order $k\leq m-2$, we find

$$u_{0,k}(0,z")=D_0^ku_{1,0}(0,z"). \tag{53}$$

We obtain a necessary condition:

$$u_{0,k}(0,z")=D_0^ku_{1,0}(0,z"),\forall k\in\mathbb{N},0\leq k\leq m-2. \tag{54}$$

Sufficient condition: Set $\varphi(z)=u_{1,0}(z_0,z")+z_1\psi(z)$, with

$$\psi(z)=\sum_{k=0}^{m-2}\frac{u_{0,k}(z_0,z")-D_0^ku_{1,0}(z_0,z")}{z_1}\frac{z_0^k}{k!},$$

then φ satisfies

$$\begin{cases} D_0^k\varphi_{|_{\Omega_0'}}=u_{0,k},0\leq k\leq m-2.\\ \varphi_{|_{\Omega_1'}}=u_{1,0}.\end{cases}$$

In fact, $\varphi_{|_{\Omega_1'}}=u_{1,0}$,

$$\varphi(z)=u_{1,0}(z_0,z")+\sum_{k=0}^{m-2}\left(u_{0,k}\left(z_1,z"\right)-D_0^ku_{1,0}\left(0,z"\right)\right)\frac{z_0^k}{k!}, \tag{55}$$

and for $z_1=0$,

$$\begin{aligned}\varphi(z_0,0,z") &= u_{1,0}(z_0,z")+\sum_{k=0}^{m-2}(u_{0,k}(0,z")-D_0^ku_{1,0}(0,z"))\frac{z_0^k}{k!}\\ &= u_{1,0}(z_0,z")\\ &= u\mid_{\Omega_1'}.\end{aligned} \tag{56}$$

To show $D_0^k\varphi_{|_{\Omega_0'}}=u_{0,k},0\leq k\leq m-2$, it suffices to apply a similar proof of a) in ii).

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