Int. Journal of Math. Analysis, Vol. 3, 2009, no. 12, 571 - 574

Investigation of Some Topological Points

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Abstract

There are some points in Analysis and Topology that every one has disjoint definition. These most important points are limit point, accumulation point, cluster point, adherent point, isolated point and condensation point.
Some sources say that the terms limit point, cluster point and accumulation point are all synonymous. Now we give a precise mathematical definition and show they are equivalent in specific topological space and investigate their difference and relation by giving some examples.

Mathematics Subject Classification: 54A20

Keywords: Topological space, limit point, cluster point, accumulation point, separating axiom

1 Recall

Definition 1.1 *Let X be a set. A topology in X is a family τ of subsets of X that satisfies:*
(1) Each union of members of τ is also a member of τ.
(2)Each finite intersection of members of τ is also a member of τ.
(3) $\emptyset$ and X are members of τ.

Definition 1.2 *T_0 axiom: if $a, b \in X$ there exist an open set $O \in \tau$ such that either $a \in O$ and $b \notin O$, or $b \in O$ and $a \notin O$.*
T_1 axiom: if $a, b \in X$, there exist open sets $O_a, O_b \in \tau$ containing a and b respectively, such that $b \notin O_a$ and $a \notin O_b$.
T_2 axiom: if $a, b \in X$, there exist disjoint open sets O_a and O_b containing a and b respectively.

Definition 1.3 *Let R be the set of real numbers. In usual introduction to analysis, a subset $G \subset R$ is called open if for each $x \in G$ there is an $r > 0$ such that the symmetric open interval $B(x;r) = \{y : |y - x| < r\} \subset G$. The family τ of sets is a topology in set R that is called the Euclidean topology of R.*

Definition 1.4 *On any set we define the discrete topology by taking all subsets of X to be open. Any subset is then both open and closed. We have three cases, the finite discrete, the countable discrete topology and the uncountable discrete topology according to whether the set X is finite, countably infinite or uncountable.*

Definition 1.5 *The topological space consisting of two points $0, 1$ with the topology $\tau = \{\emptyset, \{0\}, X\}$ is called the Sierpinski space.*

Definition 1.6 *A space X is said to have a countable basis at point x if there is a countable collection $\{U_n\}_n \in Z^+$ of neighborhoods of x such that any neighborhood U of x contains at least one of the set U_n. A space X that has a countable basis at each of its points is said to satisfy the first countability axiom.*

2 Definitions

(X, τ) be a topological space and $A \subseteq X$.

Definition 2.1 *A point $x \in X$ is called condensation point of A if each open set containing x contains uncountably many elements of A.*

Definition 2.2 *A point $x \in X$ is a accumulation point of A when every open set containing x contains infinitely many disjoint points of A.*

Definition 2.3 *A point $x \in X$ is called a cluster point of A if each neighborhood of x contains at least one point of A distinct from x.*

Definition 2.4 *A point $x \in X$ is called isolated point of A whenever it is not a cluster point of A.*

Definition 2.5 *A point $x \in X$ is a limit point of A, if there is a sequence $\{x_n\}$ in A such that every open set containing x contains all but finitely many terms of the sequence. The sequence is then said to converge to the point x.*

Definition 2.6 *A point $x \in X$ is adherent to A if each neighborhood of x contains at least one point of A(which may be x itself). The set $\overline{A}$ of all points in X adherent to A is called the closure of A.*

The sequence in definition 2.5 has three positions.
(1)The elements of sequence are disjoint which converges to x. (every one-to-one sequence of elements of X converges to x.)
(2)There is a sequence of points from A different from x converging to x.
(3)There is a sequence of elements of A (not necessarily one-to-one) which converges to x. (In particular, if x is in A, it holds trivially, take the constant sequence.)

3 Relations

Remark 3.1 *Let us remark the following:*
(i) Every limit point of type (1) implies accumulation point.
(ii) Every limit point of type (2) implies cluster point.
(iii) Every accumulation point is a cluster point.
(iv) Every limit point is a adherent point.
(v) Every condensation point is an accumulation point, a cluster point and a adherent point.
(vi) In set of A, x is a cluster point if and only if x is an adherent point of $A - \{x\}$.

It is trivial that if x be a limit point of type (1) then it will be a limit point of type (2) and type(3) and if x be a limit point of type (2) then it is a limit point of type (3). None of the implications can be reversed. For example let $A = \{x\}$ then x is a limit point of type (3).(Take $\{x_n\} = x$ for each n), but x isn't a limit point of type (2).
Accumulation point need not be condensation point, let $X = R$ with Euclidean topology, A be a set of rational numbers and x any element of X. Then x is an accumulation point of A, but not condensation point.
Adherent point isn't necessarily a cluster point , for example when x is an isolated point.
Every limit point isn't necessarily a cluster point, for example when x is an isolated point, but if x is a limit point of A and $x \notin A$ then x is a cluster point of A. Anyway, even in the case when x is a limit point of A which does not belong to A, x need not be an accumulation point of A. Take $X = \{1, 2\}$,$\tau = \{X, 0, \{1\}\}$, $A = \{1\}$, $x = 2$ and $\{x_n\} = 1$.
A limit point of A that does not belong to A need not be condensation point of A. Take $X = R$ with Euclidean topology, $A = Q$ and x be any element irrational. Condensation point isn't necessarily a limit point. Let $X = A \cup \{x\}$ where A is uncountable with discrete topology and open neighborhoods of x are complements of countable subsets of A. Then x is condensation point of A but is not the limit of any sequence of A.
Cluster point isn't necessarily an accumulation point. Let $X = \{0, 1\}$ be a Sierpinski space. Suppose $A = \{0\}$ and $x = 1$. then 1 is Cluster point but not accumulation point of A.

4 Theorems

Theorem 4.1 *In any T_1 space every cluster point is an accumulation point.*

Proof: Let X be a topological space with T_1 axiom, $A \subseteq X$ and x be a cluster point of A. Suppose x is not a accumulation of A, therefore there exist an open set U_x that only contains a finite number of elements of $A, x_1, x_2, ..., x_n$. Since X is T_1 using induction we can find a neighborhood M of x that does not intersect $\{x_1, x_2, ..., x_n\}$. Then $M \cap U_x$ is an open set that contains x which dose not contain any points of A different from x, therefore x is not a cluster point.

Corollary 4.2 *Every limit point of type (2) is accumulation point when topology of space satisfy in T_1 axiom, because a limit point of type (2) is a cluster point and base on theorem 4.1 it is an accumulation point.*

There is a sequence that it's adherent point is not limit point.[refer [5]]. Now we should get to condition that an adherent point is equivalent to limit point.

Theorem 4.3 *Every adherent point in topological space that satisfies the first axiom of countability is a limit point.*

Proof: Let x be an adherent point of $A \subseteq X$ and U_n be a countable basis at x. Suppose $U_1 \supseteq U_2 \supseteq U_3 \supseteq ... \supseteq U_n \supseteq U_n{+}1 \supseteq ...$ therefore $U_n = U_1 \cap U_2 \cap ... \cap U_n$. Since x is an adherent point, we can choose $x_n \in A \cap U_n$ for every n. we claim $\{x_n\}$ converges to the point x. Let U be an arbitrary neighborhood of x. Since X satisfies the first axiom of countability then there is n such that $U_n \subseteq U$ and for all $m \geq n$, we have $x_m \in U_m \subseteq U_n$. Therefore x is a limit point of type (3).

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Received: September, 2008