**Int. Journal of Math. Analysis, Vol. 3, 2009, no. 11, 527 - 534**

# Indefinite $RK$-Manifolds of Constant Holomorphic Sectional Curvature

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**Abstract**

The Riemannian Curvature tensor for an $RK$-manifolds with constant holomorphic sectional curvature has been derived and then discussed the Cartan's Lemma for indefinite $RK$-manifolds.

**Mathematics Subject Classification:** 53B35, 53C55

**Keywords:** $RK$-manifolds, Cartan's lemma, Holomorphic sectional curvature

## 1 Introduction

Let $(M^{2n}, g, J)$ be an almost Hermitian manifold with almost complex structure $J$ and a metric $g$, which is $J$-Hermitian, that is, $g(JX, JY) = g(X, Y)$, where $X, Y \in \chi(M)$.

An almost Hermitian manifold whose Riemannian curvature tensor $R$ is $J$-invariant, that is,

$$R(X, Y, Z, W) = R(JX, JY, JZ, JW), \tag{1}$$

for $X, Y, Z, W \in \chi(M)$, is called an $RK$-manifold.

The metric $g$ is said to be degenerate if there exists a non-zero vector $X \in \chi(M)$

such that $g(X,Y)=0$ for all $Y \in \chi(M)$ and a vector field $X$ is a space-like, time-like or null if $g(X,X)>0$, $g(X,X)<0$ or $g(X,X)=0$ respectively for $X \neq 0$.

A plane (this will always mean a real 2-dimensional vector subspace) $P$ of $T_m(M)$ is non-degenerate (with respect to g) if and only if $P$ has a basis $\{X,Y\}$ with

$$g(X,X)g(Y,Y)-g(X,Y)^2 \neq 0, \tag{2}$$

then sectional curvature function $K$ is defined for a non-degenerate plane $P$ of $T_x(M)$ by

$$K(X,Y)=\frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}. \tag{3}$$

A holomorphic plane spanned by $\{X,JX\}$ is said to be non-degenerate if and only if it contains $X$ with $g(X,X)\neq 0$ (non null) and the restriction of $K$ to these non-degenerate holomorphic planes is called the holomorphic sectional curvature of $M$ and denoted by $H(X)$ for $K(X,JX)$. The holomorphic bisectional curvature $H(X,Y)$ is defined for a non-degenerate plane spanned by $\{X,Y\}$ of $T_x(M)$ as usual by

$$H(X,Y)=\frac{R(X,JX,Y,JY)}{g(X,X)g(Y,Y)-g(X,Y)^2}. \tag{4}$$

Let $M$ be an $RK$-manifold and $X,Y \in T_m(M), m \in M$. Then, it is known [4] that,

$$\begin{aligned} R(X,Y,X,Y) &= \frac{1}{32}\{3Q(X+JY)+3Q(X-JY)-Q(X+Y)-Q(X-Y) \\ &\quad -4Q(X)-4Q(Y)\}+\frac{5}{8}\lambda(X,Y)+\frac{1}{8}\lambda(X,JY), \end{aligned} \tag{5}$$

where $Q(X)=R(X,JX,X,JX)$ and

$$\lambda(X,Y)=R(X,Y,X,Y)-R(X,Y,JX,JY). \tag{6}$$

It is also well known that, if a Kähler manifold $M$ is of constant holomorphic sectional curvature $c(m)$, at every point $m \in M$, then the Riemannian curvature tensor $R(X,Y,Z,W)$ of $M$, is of the form [3]:

$$\begin{aligned} R(X,Y,Z,W) &= \frac{c(m)}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)+g(X,JW)g(Y,JZ) \\ &\quad -g(X,JZ)g(Y,JW)-2g(X,JY)g(Z,JW)\}, \end{aligned} \tag{7}$$

for any vectors $X,Y,Z$ and $W \in T_m(M)$.

**Definition**([1]): Let $M$ be an almost Hermitian manifold. Then $M$ is said to be of constant type at $m \in M$ provided that for all $x \in T_m(M)$ we have $\|\nabla_x(J)(y)\| = \|\nabla_x(J)(z)\|$ whenever $< x, y >=< Jx, y >=< x, z >=< Jx, z >= 0$ and $\|y\| = \|z\|$. If this hold for all $m \in M$ we say that $M$ has (pointwise) constant type. Finally, if $M$ has pointwise constant type and for $X, Y \in \chi(M)$ with $< X, Y >=< JX, Y >= 0$, the function $\|\nabla_X(J)(Y)\|$ is constant whenever $\|X\| = \|Y\| = 1$, then we say that $M$ has global constant type.

**Lemma**: Let $M$ be a nearly Kaehler manifold. Then $M$ has (pointwise) constant type if and only if there exists $\alpha$ such that

$$\lambda(X, Y) = \alpha\{g(X, X)g(Y, Y) - g(X, Y)^2 - g(X, JY)^2\}, \tag{8}$$

for all $X, Y \in \chi(M)$.

The first aim of this paper is to generalize the (7) for an $RK$-manifold when endowed with definite metric $g$. In fact, we prove the following:

**Theorem [A]**: *Let $(M^{2n}, g, J)$ be an RK-manifold of dimension $\geq 6$, with constant holomorphic sectional curvature $c(m)$, at every point $m \in M$. Then, the Riemannian curvature tensor of $M$ is of the following form:*

$$\begin{aligned} R(X, Y, Z, W) &= \frac{c(m)}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(X, JW)g(Y, JZ) \\ &\quad - g(X, JZ)g(Y, JW) - 2g(X, JY)g(Z, JW)\} \\ &\quad + \frac{3}{4}\lambda(X, Y, Z, W) + \frac{1}{4}\{R(X, Y, JZ, JW) + R(Y, Z, JX, JW) \\ &\quad + R(Z, X, JY, JW)\}, \end{aligned} \tag{9}$$

where $X, Y, Z$ and $W$ in $T_m(M)$ and $\lambda(X, Y, Z, W) = R(X, Y, Z, W) - R(X, Y, JZ, JW)$.

For a Riemannian manifold $(M, g)$ the following result, so called the Cartan‘s lemma, is well known [2]:

**Lemma [B]**: *Let $(M, g)$ be a Riemannian manifold of dimension $\geq 3$. Then, $M$ is a space of constant sectional curvature if and only if, $R(X, Y, Z, X) = 0$, for all orthonormal vectors $X$, $Y$ and $Z$ at any point of $M$.*

# 2 Proof of Theorem [A]

Since $Q(X) = R(X, JX, X, JX) = -H(X)\|X\|^4$ therefore

$$\begin{aligned} Q(X + JY) &= -H(X + JY)\|X + JY\|^4 \\ &= -H(X + JY)\{(g(X, X) + g(Y, Y))^2 + 4(g(X, X) + g(Y, Y))g(X, JY) \\ &\quad + 4g(X, JY)^2\}. \end{aligned} \tag{10}$$

Similarly, calculating $Q(X-JY)$, $Q(X+Y)$ and $Q(X-Y)$ and substituting these values in (5), we have:

$$\begin{aligned} R(X,Y,X,Y) &= -\frac{1}{32}[3H(X+JY)\{(g(X,X)+g(Y,Y))^2+4(g(X,X) \\ &+ g(Y,Y))g(X,JY)+4g(X,JY)^2\}+3H(X-JY)\{(g(X,X)+g(Y,Y))^2 \\ &- 4(g(X,X)+g(Y,Y))g(X,JY)+4g(X,JY)^2\} \\ &- H(X+Y)\{(g(X,X)+g(Y,Y))^2+4(g(X,X)+g(Y,Y))g(X,Y) \\ &+ 4g(X,Y)^2\}-H(X-Y)\{(g(X,X)+g(Y,Y))^2 \\ &- 4(g(X,X)+g(Y,Y))g(X,Y)+4g(X,Y)^2\}-4H(X)g(X,X)^2 \\ &- 4H(Y)g(Y,Y)^2]+\frac{5}{8}[R(X,Y,X,Y)-R(X,Y,JX,JY)] \\ &+ \frac{1}{8}[R(X,JY,X,JY)+R(X,JY,JX,Y)]. \end{aligned} \tag{11}$$

Hence, if $M$ is of constant holomorphic sectional curvature $c(m)$ then from above, we have

$$\begin{aligned} R(X,Y,X,Y) &= \frac{c(m)}{4}\{g(X,Y)^2-g(X,X)g(Y,Y)-3g(X,JY)^2\} \\ &+ \frac{5}{8}[R(X,Y,X,Y)-R(X,Y,JX,JY)]+\frac{1}{8}[R(X,JY,X,JY) \\ &+ R(X,JY,JX,Y)]. \end{aligned} \tag{12}$$

Replacing $Y$ by $Y+W$ in above relation, we get

$$\begin{aligned} R(X,Y,X,W) &= \frac{c(m)}{4}\{g(X,Y)g(X,W)-g(X,X).g(Y,W)-3g(X,JY)g(X,JW)\} \\ &+ \frac{5}{8}[R(X,Y,X,W)-R(X,Y,JX,JW)]+\frac{1}{8}[R(X,JY,X,JW) \\ &+ R(X,JY,JX,W)]. \end{aligned} \tag{13}$$

Again, replacing $X$ by $X+Z$ in (13), we get

$$\begin{aligned} R(X,Y,Z,W)+R(Z,Y,X,W) &= \frac{c(m)}{4}\{g(X,W)g(Y,Z)+g(X,Y)g(Z,W) \\ &- 2g(X,Z)g(Y,W)-3g(X,JW)g(Z,JY) \\ &- 3g(X,JY).g(Z,JW)\} \\ &+ \frac{5}{8}[R(X,Y,Z,W)-R(X,Y,JZ,JW)] \\ &+ \frac{1}{8}[R(X,JY,Z,JW)+R(X,JY,JZ,W)] \\ &+ \frac{5}{8}[R(Z,Y,X,W)-R(Z,Y,JX,JW)] \\ &+ \frac{1}{8}[R(Z,JY,X,JW)+R(Z,JY,JX,W)]. \end{aligned} \tag{14}$$

Interchanging $X$ and $Y$ in (14) and subtracting the equation thus obtained, we get

$$R(X,Y,Z,W)+R(Z,Y,X,W)-R(Y,X,Z,W)-R(Z,X,Y,W) \tag{15}$$

$$
\begin{aligned}
&= \frac{c(m)}{4}\{g(X,W)g(Y,Z) - g(Y,W)g(X,Z) + g(X,Y)g(Z,W) \\
&\qquad - g(Y,X)g(Z,W) - 2g(X,Z)g(Y,W) + 2g(Y,Z)g(X,W) \\
&\qquad - 3g(X,JW)g(Z,JY) + 3g(Y,JW)g(Z,JX) \\
&\qquad - 3g(X,JY)g(Z,JW) + 3g(Y,JX)g(Z,JW)\} \\
&\quad + \frac{5}{8}[R(JX,Y,JZ,W) + R(JX,Y,Z,JW) + R(JZ,Y,JX,W) \\
&\qquad + R(JZ,Y,X,JW) - R(JY,X,JZ,W) - R(JY,X,Z,JW) \\
&\qquad - R(JZ,X,JY,W) - R(JZ,X,Y,JW)] \\
&\quad + \frac{1}{8}[R(X,JY,Z,JW) + R(X,JY,JZ,W) + R(Z,JY,X,JW) \\
&\qquad + R(Z,JY,JX,W) - R(Y,JX,Z,JW) - R(Y,JX,JZ,W) \\
&\qquad - R(Z,JX,Y,JW) - R(Z,JX,JY,W)].
\end{aligned}
$$

Using Bianchi‘s first identity and the fact that

$$R(X,Y,Z,W) - R(X,Y,JZ,JW) = R(JX,Y,JZ,W) + R(JX,Y,Z,JW) \tag{16}$$

in (15), we get the required result.

**Remark [C]** *Among Riemannian manifolds, Kähler manifolds have rich geometric and topological structures because the curvature tensor of Kähler manifold satisfies a special identity, namely, the Kähler identity* $R(X,Y,Z,W) = R(X,Y,JZ,JW)$ *therefore if we consider M as a Kähler manifold then* $\lambda(X,Y,Z,W) = 0$*. Using this result with Kähler identity and Bianchi‘s first identity in (9), we get*

$$
\begin{aligned}
R(X,Y,Z,W) \;=\; & \frac{c(m)}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,JW)g(Y,JZ) \\
& - g(X,JZ)g(Y,JW) - 2g(X,JY)g(Z,JW)\},
\end{aligned} \tag{17}
$$

*which is same as (7).*

Next, we shall prove the generalization of Cartan‘s lemma for an $RK$-manifold, in the following form:

**Theorem [D]**: *Let* $(M^{2n}, g, J)$ *be an indefinite RK-manifold of constant type* $\alpha$*, with dimension* $\geq 6$*. Then, M is of constant holomorphic sectional curvature if and only if,* $R(X,Y,Z,X) = 0$*, for all orthonormal vectors X, Y and Z.*

**Proof**: Let M be an indefinite $RK$-manifold of constant type $\alpha$, and $\{X, Y, Z, JX, JY, JZ\}$ be a set of orthonormal vectors. We shall discuss the proof in two different cases.

**Case I:** When $g(X,X) = g(Y,Y)$.

Using (6), we have

$$2\alpha = \lambda(X, Y+Z) = \lambda(X,Y) + \lambda(X,Z) + 2\{R(X,Y,X,Z) - R(X,Y,JX,JZ)\}.$$

For orthonormal vectors $\{X, Y\}$, from (8) we have $\lambda(X, Y) = \alpha$. Using this in above equation, we get

$$R(X, Y, Z, X) = R(X, Y, JZ, JX). \tag{18}$$

Let $(M^{2n}, g, J)$ be of constant holomorphic sectional curvature, then from (9), replacing $W$ by $X$, we get

$$\begin{aligned} R(X,Y,Z,X) &= \frac{c(m)}{4}\{g(X,X)g(Y,Z) - g(X,Z)g(Y,X) + g(X,JX)g(Y,JZ) \\ &\quad - g(X,JZ)g(Y,JX) - 2g(X,JY)g(Z,JX)\} + \frac{3}{4}\lambda(X,Y,Z,X) \\ &\quad + \frac{1}{4}\{R(X,Y,JZ,JX) + R(Y,Z,JX,JX) + R(Z,X,JY,JX)\}, \end{aligned} \tag{19}$$

since $\{X, Y, Z, JX, JY, JZ\}$ is a set of orthonormal vectors and using Bianchi‘s first identity, we have

$$R(X, Y, Z, X) = \frac{3}{4}\{R(X, Y, Z, X) - R(X, Y, JZ, JX)\}, \tag{20}$$

using (18), we have

$$R(X, Y, Z, X) = 0. \tag{21}$$

Conversely, define $\acute{X} = X\cos\theta + Y\sin\theta$ and $\acute{Y} = -X\sin\theta + Y\cos\theta$ then $\acute{X}$, $\acute{Y}$ and $J\acute{X}$ form an orthonormal set of vectors. Using (21), we have

$$R(\acute{X}, J\acute{X}, \acute{Y}, J\acute{X}) = 0. \tag{22}$$

From this we have

$$\begin{aligned} 0 &= -\sin\theta\cos^3\theta H(X) + \sin^3\theta\cos\theta H(Y) - \sin^3\theta\cos\theta R(Y, JY, X, JX) \\ &\quad - \sin^3\theta\cos\theta R(X, JY, X, JY) - \sin^3\theta\cos\theta R(Y, JX, X, JY) \\ &\quad + \sin\theta\cos^3\theta R(X, JY, Y, JX) + \sin\theta\cos^3\theta R(Y, JX, Y, JX) \\ &\quad + \sin\theta\cos^3\theta R(X, JX, Y, JY). \end{aligned} \tag{23}$$

Choosing $\theta = \pi/4$ in above then we have

$$H(X) = H(Y). \tag{24}$$

Thus M is of constant holomorphic sectional curvature.

**Case II:** When $g(X, X) = -g(Y, Y)$.
Here define $\acute{X} = X\cosh\theta + Y\sinh\theta$ and $\acute{Y} = X\sinh\theta + Y\cosh\theta$ then $\acute{X}$, $\acute{Y}$ and $J\acute{X}$ form an orthonormal set of vectors and using (21), we have

$$R(\acute{X}, J\acute{X}, \acute{Y}, J\acute{X}) = 0, \tag{25}$$

this gives

$$\begin{aligned} 0 \;=\;& \sinh\theta\cosh^3\theta H(X) + \sinh^3\theta\cosh\theta H(Y) + \sinh^3\theta\cosh\theta R(Y, JY, X, JX) \\ &+ \sinh^3\theta\cosh\theta R(X, JY, X, JY) + \sinh^3\theta\cosh\theta R(Y, JX, X, JY) \\ &+ \sinh\theta\cosh^3\theta R(X, JY, Y, JX) + \sinh\theta\cosh^3\theta R(Y, JX, Y, JX) \\ &+ \sinh\theta\cosh^3\theta R(X, JX, Y, JY). \end{aligned} \tag{26}$$

Consequently

$$\begin{aligned} 0 \;=\;& \cosh^2\theta H(X) + \sinh^2\theta H(Y) + \sinh^2\theta R(Y, JY, X, JX) \\ &+ \sinh^2\theta R(X, JY, X, JY) + \sinh^2\theta R(Y, JX, X, JY) \\ &+ \cosh^2\theta R(X, JY, Y, JX) + \cosh^2\theta R(Y, JX, Y, JX) \\ &+ \cosh^2\theta R(X, JX, Y, JY). \end{aligned} \tag{27}$$

Or

$$\begin{aligned} 0 \;=\;& \cosh^2\theta H(X) + \sinh^2\theta H(Y) + (\cosh^2\theta + \sinh^2\theta)R(X, JX, Y, JY) \\ &+ (\cosh^2\theta + \sinh^2\theta)R(X, JY, Y, JX) + (\cosh^2\theta + \sinh^2\theta)R(X, JY, X, JY). \end{aligned} \tag{28}$$

This implies

$$\begin{aligned} 0 \;=\;& \cos^2\theta H(X) - \sin^2\theta H(Y) + (\cos^2\theta - \sin^2\theta)R(X, JX, Y, JY) \\ &+ (\cos^2\theta - \sin^2\theta)R(X, JY, Y, JX) + (\cos^2\theta - \sin^2\theta)R(X, JY, X, JY). \end{aligned} \tag{29}$$

Choosing $\theta = \pi/4$ in above then we have

$$H(X) = H(Y). \tag{30}$$

Thus $M$ is of constant holomorphic sectional curvature.

**Corollary [F]**: *Let $(M^{2n}, g, J)$ be an RK-manifold of constant type $\alpha$ with dimension $\geq 6$. Then, $M$ is of constant holomorphic sectional curvature if and only if, $R(X, Y, X, JX) = 0$, for every orthonormal set of vectors $X$, $Y$ and $JX$.*

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**Received: July, 2008**