Int. Journal of Math. Analysis, Vol. 3, 2009, no. 10, 457 - 471

Conditional Fourier-Feynman Transform and Convolution Product on a Function Space

Myung Jae Kim

Department of Mathematics
Kyonggi University, Suwon 443-760, Korea
mjkim@kyonggi.ac.kr

Abstract

For a partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ of the interval $[0, t]$, let $X_n(x) = (x(t_0), \cdots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), \cdots, x(t_n), x(t_{n+1}))$ on a generalized Wiener space $(C[0, t], \mathcal{B}(C[0, t]), w_\varphi)$, where $C[0, t]$ is the space of the continuous paths on $[0, t]$ and w_φ is a probability measure on the Borel class $\mathcal{B}(C[0, t])$ of $C[0, t]$.

In this paper, using simple formulas for conditional w_φ-integrals on $C[0, t]$ with the conditioning functions X_n and X_{n+1} we define the L_p analytic conditional Fourier-Feynman transform and the conditional convolution product over the generalized Wiener space. And then, we evaluate the conditional Fourier-Feynman transform and convolution product of the functions in a Banach algebra $\mathcal{S}_{w_\varphi}$ which corresponds to the Cameron and Storvick's Banach algebra $\mathcal{S}$. Finally, we show that the conditional Fourier-Feynman transform of the conditional convolution product for the functions in $\mathcal{S}_{w_\varphi}$ can be expressed as a product of the conditional Fourier-Feynman transforms of each function.

Mathematics Subject Classification: 28C20

Keywords: analogue of Wiener measure, conditional convolution product, conditional analytic Feynman integral, conditional analytic Fourier-Feynman transform, conditional analytic Wiener integral, simple formula for conditional w_φ-integral

1 Introduction and Preliminaries

It is well-known that the classical Wiener space $C_0[0, t]$ is the space of the real-valued continuous functions on the closed interval $[0, t]$ which vanish at 0. On the space, Yeh introduced an inversion formula that a conditional expectation can be found by a Fourier transform [14]. But the Yeh's inversion formula is very complicated in its applications when the conditioning function is vector-valued. For a partition $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ of $[0, t]$, let $X_n(x) = (x(t_0), \cdots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), \cdots, x(t_n), x(t_{n+1}))$ for $x \in C_0[0, t]$. In [12], Park and Skoug introduced a simple formula for conditional Wiener integrals which evaluate the conditional Wiener integral of a function given X_{n+1} as a Wiener integral of the function. In [4], Chang and Skoug defined the conditional Fourier-Feynman transform and the conditional convolution product on the classical Wiener space $C_0[0, t]$. Moreover, they evaluated conditional Fourier-Feynman transform and the conditional convolution product of the functions in a Banach algebra $\mathcal{S}$ which was introduced by Cameron and Storvick in [2]. Further works were studied by Cho [3, 5, 6] with cylinder type functions over Wiener paths in abstract Wiener space.

On the other hand, let $C[0, t]$ denote the space of the real-valued continuous functions on the interval $[0, t]$. Im and Ryu [10] introduced a probability measure w_φ on $(C[0, t], \mathcal{B}(C[0, t]))$, where $\mathcal{B}(C[0, t])$ denotes the Borel σ-algebra of $C[0, t]$ and φ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. This measure space is a generalization of the Wiener space $C_0[0, t]$. In [7, 8], Cho derived two simple formulas for the conditional w_φ-integrals with the vector-valued conditioning functions X_n and X_{n+1} which are defined on $C[0, t]$. These formulas express the conditional w_φ-integrals directly in terms of non-conditional w_φ-integrals.

In this paper, with the conditioning functions X_n and X_{n+1}, we define the L_p analytic conditional Fourier-Feynman transform and the conditional convolution product over the analogue w_φ of Wiener measure, where $1 \le p \le \infty$. Using the simple formulas over w_φ, we evaluate the L_p analytic conditional Fourier-Feynman transform and the conditional convolution product of the functions in a Banach algebra $\mathcal{S}_{w_\varphi}$ which is an analogue of the Cameron and Storvick's Banach algebra $\mathcal{S}$. Finally, we show that the L_p analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions in $\mathcal{S}_{w_\varphi}$ can be expressed as a product of the L_p analytic conditional Fourier-Feynman transforms of each function.

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of the complex numbers and that of the complex numbers with the positive real parts, respectively.

Now, we begin with introducing the probability space $(C[0, t], \mathcal{B}(C[0, t]), w_\varphi)$. For a positive real t, let $C = C[0, t]$ be the space of all real-valued continuous functions on the closed interval $[0, t]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0, t] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)).$$

For $B_j\,(j = 0, 1, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[0, t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure

φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

$$m_\varphi\left(J_{\vec{t}}^{-1}\left(\prod_{j=0}^{n} B_j\right)\right) = \left[\prod_{j=1}^{n} \frac{1}{2\pi(t_j - t_{j-1})}\right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^{n} B_j} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\} d(u_1, \cdots, u_n) d\varphi(u_0).$$

$\mathcal{B}(C[0,t])$, the Borel σ-algebra of $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $w_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [10, 13, 15].

Let $\{e_k : k = 1, 2, \cdots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_k is of bounded variation. For f in $L_2[0,t]$ and x in $C[0,t]$, we let

$$(f, x) = \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^t \langle f, e_k\rangle e_k(s) dx(s)$$

if the limit exists. Here $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. (f,x) is called the Paley-Wiener-Zygmund integral of f according to x.

Applying Theorem 3.5 in [10], we can easily prove the following theorem.

Theorem 1.1 *Let $\{h_1, h_2, \cdots, h_n\}$ be an orthonormal system of $L_2[0,t]$. For $i = 1, 2, \cdots, n$, let $Z_i(x) = (h_i, x)$ on $C[0,t]$. Then $Z_1, Z_2, \cdots, Z_n$ are independent and each Z_i has the standard normal distribution. Moreover, if $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable, then we have*

$$\int_C f(Z_1(x), Z_2(x), \cdots, Z_n(x)) dw_\varphi(x) \stackrel{*}{=} \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, u_2, \cdots, u_n) \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} u_j^2\right\} d(u_1, u_2, \cdots, u_n),$$

where $\stackrel{*}{=}$ means that if either side exists then both sides exist and they are equal.

Let $F : C[0,t] \to \mathbb{C}$ be integrable and let X be a random vector on $C[0,t]$ assuming that the value space of X is a normed space with the Borel σ-algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well-known probability theory [11]. Further, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ-a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper, let $0=t_0<t_1<\cdots<t_n<t_{n+1}=t$ be a partition of $[0,t]$ unless otherwise specified. For any x in $C[0,t]$, define the polygonal function $[x]$ on $[0,t]$ by

$$[x](t)=x(t_{j-1})+\frac{t-t_{j-1}}{t_j-t_{j-1}}(x(t_j)-x(t_{j-1})),t_{j-1}\le t\le t_j, j=1,\cdots,n+1.$$

Similarly, for $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\cdots,\xi_{n+1})\in\mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ on $[0,t]$ by

$$[\vec{\xi}_{n+1}](t)=\xi_{j-1}+\frac{t-t_j}{t_j-t_{j-1}}(\xi_j-\xi_{j-1}),t_{j-1}\le t\le t_{j-1}, j=1,\cdots,n+1.$$

In the following two theorems, we introduce simple formulas for the conditional w_φ-integrals on $C[0,t]$ [7, 8].

Theorem 1.2 *Let $F:C[0,t]\to\mathbb{C}$ be integrable and let $X_{n+1}:C[0,t]\to\mathbb{R}^{n+2}$ be given by*

$$X_{n+1}(x)=(x(t_0),x(t_1),\cdots,x(t_n),x(t_{n+1})). \tag{1}$$

Then we have for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$

$$E[F|X_{n+1}](\vec{\xi}_{n+1})=E[F(x-[x]+[\vec{\xi}_{n+1}])], \tag{2}$$

where $P_{X_{n+1}}$ is the probability distribution of X_{n+1} on $(\mathbb{R}^{n+2},\mathcal{B}(\mathbb{R}^{n+2}))$.

Next, we introduce another simple formula for the conditional w_φ-integrals removing the point $x(t)$ in the conditioning function X_{n+1} given by (1).

Theorem 1.3 *Let $X_n:C[0,t]\to\mathbb{R}^{n+1}$ be given by*

$$X_n(x)=(x(t_0),x(t_1),\cdots,x(t_n)). \tag{3}$$

Moreover let F be integrable on $C[0,t]$ and P_{X_n} be the probability distribution of X_n on $(\mathbb{R}^{n+1},\mathcal{B}(\mathbb{R}^{n+1}))$. Then we have

$$\begin{aligned} E[F|X_n](\vec{\xi}_n) &= \left[\frac{1}{2\pi(t-t_n)}\right]^{\frac{1}{2}}\int_{\mathbb{R}} E[F(x-[x]+[\vec{\xi}_{n+1}])] \\ &\quad\times\exp\left\{-\frac{(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\}d\xi_{n+1} \end{aligned} \tag{4}$$

for P_{X_n}-a.e. $\vec{\xi}_n\in\mathbb{R}^{n+1}$, where $\vec{\xi}_n=(\xi_0,\xi_1,\cdots,\xi_n)$ and $\vec{\xi}_{n+1}=(\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})$.

For a function $F : C[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ and $X^\lambda_{n+1}(x) = X_{n+1}(\lambda^{-\frac{1}{2}}x)$, $X^\lambda_n(x) = X_n(\lambda^{-\frac{1}{2}}x)$, where X_{n+1} and X_n are given by (1) and (3), respectively. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ-integral and (2), we have

$$E[F^\lambda|X^\lambda_{n+1}](\vec{\xi}_{n+1}) = E[F(\lambda^{-\frac{1}{2}}(x-[x]) + [\vec{\xi}_{n+1}])]$$

for $P_{X^\lambda_{n+1}}$-a.e. $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, where $P_{X^\lambda_{n+1}}$ is the probability distribution of X^λ_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$. For $\vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ and $\xi_{n+1} \in \mathbb{R}$, let $\vec{\xi}_{n+1}(\lambda) = (\lambda^{\frac{1}{2}}\xi_0, \lambda^{\frac{1}{2}}\xi_1, \cdots, \lambda^{\frac{1}{2}}\xi_n, \xi_{n+1})$. Then we have by (4) and the change of variable theorem

$$\begin{aligned} E[F^\lambda|X^\lambda_n](\vec{\xi}_n) &= \left[\frac{1}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F^\lambda(x-[x]+[\vec{\xi}_{n+1}(\lambda)])] \\ &\quad \times \exp\left\{-\frac{(\xi_{n+1}-\lambda^{\frac{1}{2}}\xi_n)^2}{2(t-t_n)}\right\} d\xi_{n+1} \\ &= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}_{n+1}])] \\ &\quad \times \exp\left\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\right\} d\xi_{n+1} \end{aligned} \tag{5}$$

for $P_{X^\lambda_n}$-a.e. $\vec{\xi}_n$, where $P_{X^\lambda_n}$ is the probability distribution of X^λ_n on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. If $E[F(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}_{n+1}])]$ has the analytic extension $J^*_\lambda(F)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$ as a function of λ, then it is called the conditional analytic Wiener w_φ-integral of F given X_{n+1} with parameter λ and denoted by

$$E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}) = J^*_\lambda(F)(\vec{\xi}_{n+1})$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, if for a non-zero real q, $E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F given X_{n+1} with parameter q and denoted by

$$E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda\to -iq} E^{anw_\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}).$$

Similar definitions are understood with (5) if we replace X_{n+1} by X_n.

2 Conditional Fourier-Feynman Transforms

In this section, we define an L_p analytic conditional Fourier-Feynman transform of the functions on $C[0,t]$ and evaluate the transforms of the functions in the Banach algebra $\mathcal{S}_{w_\varphi}$ which is similar to the Cameron and Storvick's Banach algebra $\mathcal{S}$ in [2].

For a given extended real number p with $1 < p \leq \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$(possibly $p' = 1$ if $p = \infty$). Let G_n and G be measurable functions such that for $\rho > 0$

$$\lim_{n\to\infty} \int_C |G_n(\rho x) - G(\rho x)|^{p'} dw_\varphi(x) = 0.$$

Then we write

$$\underset{n\to\infty}{\text{l.i.m.}}(w^{p'})(G_n) \approx G$$

and call G the limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

Now, we define an analytic conditional Fourier-Feynman transform of the functions on $C[0,t]$.

Definition 2.1 *Let F be defined on $C[0,t]$ and let X_{n+1} be given by (1). For $\lambda \in \mathbb{C}_+$ and for w_φ-a.e. $y \in C[0,t]$, let*

$$T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) = E^{anw_\lambda}[F(\cdot + y)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For a non-zero real q and for w_φ-a.e. $y \in C[0,t]$, we define the L_1 analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_{n+1}]$ of F by the formula

$$T_q^{(1)}[F|X_{n+1}](y, \vec{\xi}_{n+1}) = E^{anf_q}[F(\cdot + y)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For $1 < p \leq \infty$ we define the L_p analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_{n+1}]$ of F by the formula

$$T_q^{(p)}[F|X_{n+1}](\cdot, \vec{\xi}_{n+1}) \approx \underset{\lambda\to -iq}{\text{l.i.m.}}(w^{p'})(T_\lambda[F|X_{n+1}](\cdot, \vec{\xi}_{n+1}))$$

for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where λ approaches to $-iq$ through $\mathbb{C}_+$.

Similar definitions are understood with (5) if we replace X_{n+1} by X_n which is given by (3).

Let $\mathcal{M}(L_2[0,t])$ be the class of $\mathbb{C}$-valued Borel measures of finite variation on $L_2[0,t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of the functions F of the form for $\sigma \in \mathcal{M}(L_2[0,t])$

$$F(x) = \int_{L_2[0,t]} \exp\{i(v,x)\} d\sigma(v) \tag{6}$$

for w_φ-a.e. $x \in C[0,t]$. For $v \in L_2[0,t]$ define the sectional average $\bar{v}$ of v by letting

$$\bar{v}(s) = \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} v(u)du \tag{7}$$

on each subinterval $(t_{j-1}, t_j]$ and by letting $\bar{v}(0) = 0$ [9]. Then, we have for $v \in L_2[0,t]$ and $x \in C[0,t]$

$$(v, [x]) \;=\; \sum_{j=1}^{n+1} \bar{v}(t_j)(x(t_j) - x(t_{j-1})) = \sum_{j=1}^{n+1} \int_{t_{j-1}}^{t_j} \bar{v}(u)dx(u) = (\bar{v}, x). \tag{8}$$

We are now ready to evaluate the L_p analytic conditional Fourier-Feynman transforms of the functions in $\mathcal{S}_{w_\varphi}$.

Theorem 2.2 *Let X_{n+1} and $F \in \mathcal{S}_{w_\varphi}$ be given by (1) and (6), respectively. Moreover, let $1 \leq p \leq \infty$. Then, for $\lambda \in \mathbb{C}_+$ and for w_φ-a.e. $y \in C[0,t]$, $T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1})$ exists for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{n+2}$ and it is given by*

$$\begin{aligned} T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) \;=\; & \int_{L_2[0,t]} \exp\bigg\{ i(v, y) \\ & + i\sum_{j=1}^{n+1} \bar{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda}\|v - \bar{v}\|_2^2 \bigg\} d\sigma(v) \end{aligned} \tag{9}$$

where $\bar{v}$ is given by (7). Moreover, for a non-zero real q, $T_q^{(p)}[F|X_{n+1}](y, \vec{\xi}_{n+1})$ exists and it is given by (9) replacing λ by $-iq$.

Proof. For $\lambda > 0$ and w_φ-a.e. $y \in C[0,t]$, we have by the Fubini's theorem and (8)

$$\begin{aligned} & T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) \\ = \;& E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}] + y)] \\ = \;& \int_C \int_{L_2[0,t]} \exp\{i(v, \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}] + y)\} d\sigma(v) dw_\varphi(x) \\ = \;& \int_{L_2[0,t]} \exp\{i(v, y) + i(v, [\vec{\xi}_{n+1}])\} \int_C \exp\{i\lambda^{-\frac{1}{2}}(v - \bar{v}, x)\} dw_\varphi(x) d\sigma(v) \end{aligned}$$

for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Using the following well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\}du = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left\{-\frac{b^2}{4a}\right\} \tag{10}$$

for $a \in \mathbb{C}_+$ and any real b, we have

$$\begin{aligned} T_\lambda[F|X_{n+1}](y,\vec{\xi}_{n+1}) &= \int_{L_2[0,t]} \exp\bigg\{ i(v,y) \\ &\quad + i\sum_{j=1}^{n+1} \bar{v}(t_j)(\xi_j-\xi_{j-1}) - \frac{1}{2\lambda}\|v-\bar{v}\|_2^2 \bigg\} d\sigma(v) \end{aligned}$$

since $(v-\bar{v},\cdot)$ is mean zero Gaussian with variance $\|v-\bar{v}\|_2^2$ by Theorem 1.1. By the Morera's theorem and the dominated convergence theorem, we have the results for $p=1$. Now suppose that $1<p\le\infty$. For $\frac{1}{p}+\frac{1}{p'}=1$ and $\rho>0$, we have

$$\begin{aligned} &\int_C \bigg| \int_{L_2[0,t]} \exp\{i(v,\rho y+[\vec{\xi}_{n+1}])\} \bigg[\exp\bigg\{-\frac{1}{2\lambda}\|v-\bar{v}\|_2^2\bigg\} \\ &\quad - \exp\bigg\{\frac{1}{2qi}\|v-\bar{v}\|_2^2\bigg\}\bigg] d\sigma(v) \bigg|^{p'} dw_\varphi(y) \\ &\le \bigg[\int_{L_2[0,t]} \bigg| \exp\bigg\{-\frac{1}{2\lambda}\|v-\bar{v}\|_2^2\bigg\} - \exp\bigg\{\frac{1}{2qi}\|v-\bar{v}\|_2^2\bigg\} \bigg| d|\sigma|(v) \bigg]^{p'} \end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. Now, the proof is completed. □

Theorem 2.3 *Let X_n and $F \in \mathcal{S}_{w_\varphi}$ be given by (3) and (6), respectively. Moreover, let $1 \le p \le \infty$. Then, for $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$, $T_\lambda[F|X_n](y,\vec{\xi}_n)$ exists for P_{X_n}-a.e. $\vec{\xi}_n = (\xi_0,\xi_1,\cdots,\xi_n) \in \mathbb{R}^{n+1}$ and it is given by the formula*

$$\begin{aligned} T_\lambda[F|X_n](y,\vec{\xi}_n) &= \int_{L_2[0,t]} \exp\bigg\{ i(v,y) + i\sum_{j=1}^{n} \bar{v}(t_j)(\xi_j-\xi_{j-1}) \\ &\quad - \frac{1}{2\lambda}[\|v-\bar{v}\|_2^2 + (t-t_n)[\bar{v}(t)]^2] \bigg\} d\sigma(v) \end{aligned} \tag{11}$$

where $\bar{v}$ is given by (7). Moreover, for a non-zero real q, $T_q^{(p)}[F|X_n](y,\vec{\xi}_n)$ exists and it is given by (11) replacing λ by $-iq$.

Proof. For notational convenience, let $\vec{\xi}_n = (\xi_0,\xi_1,\cdots,\xi_n) \in \mathbb{R}^{n+1}$ and $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have by Theorems 1.3 and 2.2

$$\begin{aligned} T_\lambda[F|X_n](y,\vec{\xi}_n) &= \bigg[\frac{\lambda}{2\pi(t-t_n)}\bigg]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(\lambda^{-\frac{1}{2}}(x-[x]) + [\vec{\xi}_{n+1}] + y)] \\ &\quad \times \exp\bigg\{-\frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\bigg\} d\xi_{n+1} \end{aligned}$$

$$= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{L_2[0,t]} \exp\bigg\{ i(v,y) + i\sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda}\|v-\bar{v}\|_2^2 \bigg\} \int_{\mathbb{R}} \exp\bigg\{ i\bar{v}(t)(\xi_{n+1}-\xi_n) - \frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)} \bigg\} d\xi_{n+1} d\sigma(v)$$

by the Fubini's theorem where $\bar{v}$ is the sectional average of v given by (7). Let $u = \xi_{n+1} - \xi_n$. Then we have by the change of variable theorem

$$\begin{aligned} T_\lambda[F|X_n](y,\vec{\xi}_n) &= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{L_2[0,t]} \exp\bigg\{ i(v,y) + i\sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1}) \\ &\quad - \frac{1}{2\lambda}\|v-\bar{v}\|_2^2 \bigg\} \int_{\mathbb{R}} \exp\bigg\{ i\bar{v}(t)u - \frac{\lambda u^2}{2(t-t_n)} \bigg\} du\, d\sigma(v) \\ &= \int_{L_2[0,t]} \exp\bigg\{ i(v,y) + i\sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda}[\|v-\bar{v}\|_2^2 \\ &\quad + (t-t_n)[\bar{v}(t)]^2] \bigg\} d\sigma(v) \end{aligned}$$

by (10). By the Morera's theorem and the dominated convergence theorem, we have the results for $p = 1$. Now suppose that $1 < p \le \infty$. For $\frac{1}{p} + \frac{1}{p'} = 1$ and $\rho > 0$, we have

$$\begin{aligned} &\int_C \bigg| \int_{L_2[0,t]} \exp\bigg\{ i\Big[(v,\rho y) + \sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1})\Big]\bigg\} \Big[\exp\bigg\{ -\frac{1}{2\lambda}[\|v-\bar{v}\|_2^2 + \\ &(t-t_n)[\bar{v}(t)]^2] \bigg\} - \exp\bigg\{ \frac{1}{2qi}[\|v-\bar{v}\|_2^2 + (t-t_n)[\bar{v}(t)]^2] \bigg\} \Big] d\sigma(v) \bigg|^{p'} dw_\varphi(y) \\ \le\ &\bigg[\int_{L_2[0,t]} \bigg| \exp\bigg\{ -\frac{1}{2\lambda}[\|v-\bar{v}\|_2^2 + (t-t_n)[\bar{v}(t)]^2] \bigg\} - \exp\bigg\{ \frac{1}{2qi}[\|v-\bar{v}\|_2^2 + \\ &(t-t_n)[\bar{v}(t)]^2] \bigg\} \bigg| d|\sigma|(v) \bigg]^{p'} \end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. Now, the proof is completed. □

3 Conditional Convolution Products

We begin this section with defining the conditional convolution product over the analogue w_φ of Wiener measure and evaluate it for the functions in the Banach algebra $\mathcal{S}_{w_\varphi}$.

Definition 3.1 *Let X_{n+1} be given by (1), and let F and G be defined on $C[0,t]$. We define the conditional convolution product $[(F*G)_\lambda|X_{n+1}]$ of F and G given X_{n+1} by the formula, for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$*

$$[(F*G)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1}) = \begin{cases} E^{anw_\lambda}\left[F\left(\frac{y+\cdot}{\sqrt{2}}\right)G\left(\frac{y-\cdot}{\sqrt{2}}\right)\middle|X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda\in\mathbb{C}_+; \\ E^{anf_q}\left[F\left(\frac{y+\cdot}{\sqrt{2}}\right)G\left(\frac{y-\cdot}{\sqrt{2}}\right)\middle|X_{n+1}\right](\vec{\xi}_{n+1}), & \lambda=-iq; \quad q\in\mathbb{R}-\{0\} \end{cases}$$

if they exist for w_φ-a.e. $y\in C[0,t]$. If $\lambda=-iq$, we replace $[(F*G)_\lambda|X_{n+1}]$ by $[(F*G)_q|X_{n+1}]$.

Similar definitions are understood with (5) if we replace X_{n+1} by X_n which is given by (3).

In the following two theorems, we evaluate the conditional convolution product of the functions in $\mathcal{S}_{w_\varphi}$ with the conditioning functions X_n and X_{n+1}.

Theorem 3.2 *Let X_{n+1} be given by (1) and $F_1,F_2\in\mathcal{S}_{w_\varphi}$ by (6) replacing σ by σ_1,σ_2, respectively. Then, for $\lambda\in\mathbb{C}_+$ and w_φ-a.e. $y\in C[0,t]$, $[(F_1*F_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})$ exists for $P_{X_{n+1}}$-a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$ and it is given by*

$$[(F_1*F_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1}) = \int_{(L_2[0,t])^2} A(v_1,v_2,y,\vec{\xi}_{n+1})\exp\left\{-\frac{1}{4\lambda}\|v_1 -v_2-(\bar{v}_1-\bar{v}_2)\|_2^2\right\}d\sigma_1(v_1)d\sigma_2(v_2) \qquad (12)$$

where $A(v_1,v_2,y,\vec{\xi}_{n+1})=\exp\{\frac{i}{\sqrt{2}}[(v_1,y+[\vec{\xi}_{n+1}])+(v_2,y-[\vec{\xi}_{n+1}])]\}$. Moreover, for a non-zero real q, $[(F_1*F_2)_q|X_{n+1}](y,\vec{\xi}_{n+1})$ exists and it is given by (12) replacing λ by -iq.

Proof. For $\lambda>0$ we have by the Fubini's theorem

$$\begin{aligned} &[(F_1*F_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1}) \\ =\;& E\left[F_1\left(\frac{y+\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}_{n+1}]}{\sqrt{2}}\right)F_2\left(\frac{y-\lambda^{-\frac{1}{2}}(x-[x])-[\vec{\xi}_{n+1}]}{\sqrt{2}}\right)\right] \\ =\;& \int_{(L_2[0,t])^2} A(v_1,v_2,y,\vec{\xi}_{n+1})\int_C \exp\left\{\frac{i}{\sqrt{2\lambda}}[(v_1,x-[x])-(v_2,x-[x])]\right\} \\ & dw_\varphi(x)d\sigma_1(v_1)d\sigma_2(v_2). \end{aligned}$$

Since

$$(v_1,x-[x])-(v_2,x-[x]) = (v_1-v_2-(\bar{v}_1-\bar{v}_2),x) \qquad (13)$$

by (8), we have

$$
\begin{aligned}
&[(F_1 * F_2)_\lambda|X_{n+1}](y,\vec{\xi}_{n+1})\\
=\;& \int_{(L_2[0,t])^2} A(v_1,v_2,y,\vec{\xi}_{n+1}) \int_C \exp\Big\{\frac{i}{\sqrt{2\lambda}}(v_1-v_2-(\bar v_1-\bar v_2),x)\Big\} dw_\varphi(x)\\
& d\sigma_1(v_1)d\sigma_2(v_2)\\
=\;& \int_{(L_2[0,t])^2} A(v_1,v_2,y,\vec{\xi}_{n+1}) \exp\Big\{-\frac{\|v_1-v_2-(\bar v_1-\bar v_2)\|_2^2}{4\lambda}\Big\} d\sigma_1(v_1)d\sigma_2(v_2)
\end{aligned}
$$

by (10). By the Morera's theorem and the dominated convergence theorem, we have the results. □

Theorem 3.3 *Let the assumptions and notations be given as in Theorem 3.2 and let X_n be given by (3). Then, for $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$, $[(F_1 * F_2)_\lambda|X_n](y,\vec{\xi}_n)$ exists for P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and it is given by*

$$
\begin{aligned}
&[(F_1 * F_2)_\lambda|X_n](y,\vec{\xi}_n)\\
=\;& \int_{(L_2[0,t])^2} \exp\Big\{\frac{i}{\sqrt{2}}\Big[(v_1+v_2,y)+\sum_{j=1}^n[\bar v_1(t_j)-\bar v_2(t_j)](\xi_j-\xi_{j-1})\Big] - \frac{1}{4\lambda}[(t\\
& -t_n)[\bar v_1(t)-\bar v_2(t)]^2 + \|v_1-v_2-(\bar v_1-\bar v_2)\|_2^2]\Big\} d\sigma_1(v_1)d\sigma_2(v_2). \qquad (14)
\end{aligned}
$$

Moreover, for a non-zero real q, $[(F_1 * F_2)_q|X_{n+1}](y,\vec{\xi}_n)$ exists and it is given by (14) replacing λ by $-iq$.

Proof. For notational convenience, let $\vec{\xi}_n = (\xi_0,\xi_1,\cdots,\xi_n) \in \mathbb{R}^{n+1}$ and $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\cdots,\xi_n,\xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$ and P_{X_n}-a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have by Theorems 1.3 and 3.2

$$
\begin{aligned}
&[(F_1 * F_2)_\lambda|X_n](y,\vec{\xi}_n)\\
=\;& \Big[\frac{\lambda}{2\pi(t-t_n)}\Big]^{\frac12} \int_{(L_2[0,t])^2} \exp\Big\{-\frac{1}{4\lambda}\|v_1-v_2-(\bar v_1-\bar v_2)\|_2^2\Big\} \int_{\mathbb{R}} \exp\Big\{\frac{i}{\sqrt{2}}[(v_1,\\
& y+[\vec{\xi}_{n+1}]) + (v_2, y-[\vec{\xi}_{n+1}])] - \frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\Big\} d\xi_{n+1} d\sigma_1(v_1)d\sigma_2(v_2)\\
=\;& \Big[\frac{\lambda}{2\pi(t-t_n)}\Big]^{\frac12} \int_{(L_2[0,t])^2} \exp\Big\{-\frac{1}{4\lambda}\|v_1-v_2-(\bar v_1-\bar v_2)\|_2^2 + \frac{i}{\sqrt{2}}\Big[(v_1+v_2,\\
& y)+\sum_{j=1}^n[\bar v_1(t_j)-\bar v_2(t_j)](\xi_j-\xi_{j-1})\Big]\Big\} \int_{\mathbb{R}} \exp\Big\{\frac{i}{\sqrt{2}}[\bar v_1(t)-\bar v_2(t)](\xi_{n+1}-\\
& \xi_n) - \frac{\lambda(\xi_{n+1}-\xi_n)^2}{2(t-t_n)}\Big\} d\xi_{n+1} d\sigma_1(v_1)d\sigma_2(v_2)
\end{aligned}
$$

$$= \int_{(L_2[0,t])^2} \exp\bigg\{ \frac{i}{\sqrt{2}} \bigg[(v_1 + v_2, y) + \sum_{j=1}^{n} [\bar{v}_1(t_j) - \bar{v}_2(t_j)](\xi_j - \xi_{j-1}) \bigg] - \frac{1}{4\lambda} [(t - t_n)[\bar{v}_1(t) - \bar{v}_2(t)]^2 + \|v_1 - v_2 - (\bar{v}_1 - \bar{v}_2)\|_2^2] \bigg\} d\sigma_1(v_1) d\sigma_2(v_2)$$

by the Fubini's theorem and (10). By the Morera's theorem and the dominated convergence theorem, we have the results. □

4 Relationships Between Transforms and Convolutions

In this section, we investigate relationships between the conditional convolution products and the L_p analytic conditional Fourier-Feynman transforms of the functions in $\mathcal{S}_{w_\varphi}$. In fact, with the conditioning functions X_n and X_{n+1}, we show that the L_p analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions in $\mathcal{S}_{w_\varphi}$ can be expressed as a product of the L_p analytic conditional Fourier-Feynman transforms of each function.

Theorem 4.1 *Let $1 \le p \le \infty$. Then, under the assumptions and notations given as in Theorem 3.2, we have for $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$*

$$T_\lambda[[(F_1 * F_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1}) = \bigg[T_\lambda[F_1 | X_{n+1}] \bigg(\frac{y}{\sqrt{2}}, \frac{\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}}{\sqrt{2}} \bigg) \bigg] \bigg[T_\lambda[F_2 | X_{n+1}] \bigg(\frac{y}{\sqrt{2}}, \frac{\vec{\zeta}_{n+1} - \vec{\xi}_{n+1}}{\sqrt{2}} \bigg) \bigg]$$

for $P_{X_{n+1}}$-a.e $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$, and $T_q^{(p)}[[(F_1 * F_2)_q | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1})$ can be expressed by the right-hand side of the above equality replacing λ by q.

Proof. For $\lambda > 0$ we have by Theorem 3.2

$$T_\lambda[[(F_1 * F_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1}) = \int_C \int_{(L_2[0,t])^2} A(v_1, v_2, y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\zeta}_{n+1}], \vec{\xi}_{n+1}) \exp\bigg\{ -\frac{1}{4\lambda} \|v_1 - v_2 - (\bar{v}_1 - \bar{v}_2)\|_2^2 \bigg\} d\sigma_1(v_1) d\sigma_2(v_2) dw_\varphi(x)$$

where A is given as in Theorem 3.2. By (10), (13), Theorem 1.1 and the Fubini's theorem, we have

$$T_\lambda[[(F_1 * F_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1}) = \int_{(L_2[0,t])^2} A(v_1, v_2, y + [\vec{\zeta}_{n+1}], \vec{\xi}_{n+1}) \exp\bigg\{ -\frac{1}{4\lambda} \|v_1 - v_2 - (\bar{v}_1 - \bar{v}_2)\|_2^2 \bigg\}$$

$$
\begin{aligned}
&\times \int_C \exp\left\{ \frac{i}{\sqrt{2\lambda}}(v_1+v_2-(\bar{v}_1+\bar{v}_2),x)\right\} dw_\varphi(x)d\sigma_1(v_1)d\sigma_2(v_2)\\
= &\int_{(L_2[0,t])^2} A(v_1,v_2,y+[\vec{\zeta}_{n+1}],\vec{\xi}_{n+1})\exp\left\{-\frac{1}{4\lambda}[\|v_1-v_2-(\bar{v}_1-\bar{v}_2)\|_2^2\right.\\
&\left.+\|v_1+v_2-(\bar{v}_1+\bar{v}_2)\|_2^2]\right\} d\sigma_1(v_1)d\sigma_2(v_2)\\
= &\int_{(L_2[0,t])^2} \exp\left\{\frac{i}{\sqrt{2}}[(v_1,y+[\vec{\zeta}_{n+1}]+[\vec{\xi}_{n+1}])+(v_2,y+[\vec{\zeta}_{n+1}]-[\vec{\xi}_{n+1}])]\right.\\
&\left.-\frac{1}{2\lambda}[\|v_1-\bar{v}_1\|_2^2+\|v_2-\bar{v}_2\|_2^2]\right\} d\sigma_1(v_1)d\sigma_2(v_2)\\
= &\left[T_\lambda[F_1|X_{n+1}]\left(\frac{y}{\sqrt{2}},\frac{\vec{\zeta}_{n+1}+\vec{\xi}_{n+1}}{\sqrt{2}}\right)\right]\left[T_\lambda[F_2|X_{n+1}]\left(\frac{y}{\sqrt{2}},\frac{\vec{\zeta}_{n+1}-\vec{\xi}_{n+1}}{\sqrt{2}}\right)\right]
\end{aligned}
$$

where the last equality follows from Theorem 2.2. By the Morera's theorem and the dominated convergence theorem, we have the results. □

Theorem 4.2 *Let $1 \le p \le \infty$. Then, under the assumptions and notations given as in Theorem 3.3, we have for $\lambda \in \mathbb{C}_+$ and w_φ-a.e. $y \in C[0,t]$*

$$
\begin{aligned}
&T_\lambda[[(F_1 * F_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\
= &\left[T_\lambda[F_1|X_n]\left(\frac{y}{\sqrt{2}},\frac{\vec{\zeta}_n+\vec{\xi}_n}{\sqrt{2}}\right)\right]\left[T_\lambda[F_2|X_n]\left(\frac{y}{\sqrt{2}},\frac{\vec{\zeta}_n-\vec{\xi}_n}{\sqrt{2}}\right)\right]
\end{aligned}
$$

for P_{X_n}-a.e $\vec{\xi}_n,\vec{\zeta}_n \in \mathbb{R}^{n+1}$ and $T_q^{(p)}[[(F_1 * F_2)_q|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$ can be expressed by the right-hand side of the above equality replacing λ by q.

Proof. For notational convenience, let $\vec{\xi}_n = (\xi_0,\xi_1,\cdots,\xi_n)$, $\vec{\zeta}_n = (\zeta_0,\zeta_1,\cdots,\zeta_n) \in \mathbb{R}^{n+1}$ and $\vec{\zeta}_{n+1} = (\zeta_0,\zeta_1,\cdots,\zeta_n,\zeta_{n+1})$ for $\zeta_{n+1} \in \mathbb{R}$. For $\lambda > 0$ and P_{X_n}-a.e. $\vec{\xi}_n,\vec{\zeta}_n \in \mathbb{R}^{n+1}$, we have by Theorems 1.3 and 3.3

$$
\begin{aligned}
&T_\lambda[[(F_1 * F_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)\\
= &\left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}}\int_{\mathbb{R}}\int_C\int_{(L_2[0,t])^2}\exp\left\{\frac{i}{\sqrt{2}}\left[(v_1+v_2,y+\lambda^{-\frac{1}{2}}(x-[x])+\right.\right.\\
&\left.[\vec{\zeta}_{n+1}])+\sum_{j=1}^n[\bar{v}_1(t_j)-\bar{v}_2(t_j)](\xi_j-\xi_{j-1})\right]-\frac{1}{4\lambda}[(t-t_n)[\bar{v}_1(t)-\bar{v}_2(t)]^2\\
&\left.+\|v_1-v_2-(\bar{v}_1-\bar{v}_2)\|_2^2]-\frac{\lambda(\zeta_{n+1}-\zeta_n)^2}{2(t-t_n)}\right\} d\sigma_1(v_1)d\sigma_2(v_2)dw_\varphi(x)d\zeta_{n+1}.
\end{aligned}
$$

By (10), (13), Theorem 1.1 and the Fubini's theorem, we have

$$T_\lambda[[(F_1 * F_2)_\lambda|X_n](\cdot,\vec{\xi}_n)|X_n](y,\vec{\zeta}_n)$$

$$
\begin{aligned}
&= \left[\frac{\lambda}{2\pi(t-t_n)}\right]^{\frac{1}{2}} \int_{(L_2[0,t])^2} \exp\left\{\frac{i}{\sqrt{2}}\left[(v_1+v_2,y)+\sum_{j=1}^{n}[[\bar{v}_1(t_j)+\bar{v}_2(t_j)](\zeta_j\right.\right.\\
&\quad \left.-\zeta_{j-1})+[\bar{v}_1(t_j)-\bar{v}_2(t_j)](\xi_j-\xi_{j-1})]\right]-\frac{1}{4\lambda}[(t-t_n)[\bar{v}_1(t)-\bar{v}_2(t)]^2+\\
&\quad \left.\|v_1-v_2-(\bar{v}_1-\bar{v}_2)\|_2^2+\|v_1+v_2-(\bar{v}_1+\bar{v}_2)\|_2^2]\right\}\int_{\mathbb{R}}\exp\left\{\frac{i}{\sqrt{2}}[\bar{v}_1(t)+\right.\\
&\quad \left.\bar{v}_2(t)](\zeta_{n+1}-\zeta_n)-\frac{\lambda(\zeta_{n+1}-\zeta_n)^2}{2(t-t_n)}\right\}d\zeta_{n+1}d\sigma_1(v_1)d\sigma_2(v_2)\\
&= \int_{(L_2[0,t])^2}\exp\left\{\frac{i}{\sqrt{2}}\left[(v_1+v_2,y)+\sum_{j=1}^{n}[[\bar{v}_1(t_j)+\bar{v}_2(t_j)](\zeta_j-\zeta_{j-1})+[\bar{v}_1(t_j)\right.\right.\\
&\quad \left.-\bar{v}_2(t_j)](\xi_j-\xi_{j-1})]\right]-\frac{1}{4\lambda}[(t-t_n)[[\bar{v}_1(t)-\bar{v}_2(t)]^2+[\bar{v}_1(t)+\bar{v}_2(t)]^2]+\|\\
&\quad \left.v_1-v_2-(\bar{v}_1-\bar{v}_2)\|_2^2+\|v_1+v_2-(\bar{v}_1+\bar{v}_2)\|_2^2]\right\}d\sigma_1(v_1)d\sigma_2(v_2)\\
&= \left[\int_{L_2[0,t]}\exp\left\{\frac{i}{\sqrt{2}}\left[(v_1,y)+\sum_{j=1}^{n}\bar{v}_1(t_j)[(\zeta_j-\zeta_{j-1})+(\xi_j-\xi_{j-1})]\right]-\frac{1}{2\lambda}[(t\right.\right.\\
&\quad \left.\left.-t_n)[\bar{v}_1(t)]^2+\|v_1-\bar{v}_1\|_2^2]\right\}d\sigma_1(v_1)\right]\left[\int_{L_2[0,t]}\exp\left\{\frac{i}{\sqrt{2}}\left[(v_2,y)+\sum_{j=1}^{n}\bar{v}_2(t_j)\right.\right.\right.\\
&\quad \left.\left.\left.\times[(\zeta_j-\zeta_{j-1})-(\xi_j-\xi_{j-1})]\right]-\frac{1}{2\lambda}[(t-t_n)[\bar{v}_2(t)]^2+\|v_2-\bar{v}_2\|_2^2]\right\}d\sigma_2(v_2)\right]\\
&= \left[T_\lambda[F_1|X_n]\left(\frac{y}{\sqrt{2}},\frac{\vec{\zeta}_n+\vec{\xi}_n}{\sqrt{2}}\right)\right]\left[T_\lambda[F_2|X_n]\left(\frac{y}{\sqrt{2}},\frac{\vec{\zeta}_n-\vec{\xi}_n}{\sqrt{2}}\right)\right]
\end{aligned}
$$

where the last equality follows from Theorem 2.3. By the Morera's theorem and the dominated convergence theorem, we have the results. □

References

[1] R.H. Cameron and W.T. Martin, Transformations of Wiener integrals under translations, *Ann. Math.,* **45** (1944), 386 - 396.

[2] R.H. Cameron and D.A. Storvick, Some Banach algebras of analytic Feynman integrable functionals, *Lecture Notes in Math.,* **798**, Springer, Berlin-New York, 18 - 67, 1980.

[3] K.S. Chang, D.H. Cho, B.S. Kim, T.S. Song and I Yoo, Conditional Fourier-Feynman transform and convolution product over Wiener paths

in abstract Wiener space, *Integral Transforms Spec. Funct.,* **14** (2003), no.3, 217 - 235.

[4] S.J. Chang and D.L. Skoug, The effect of drift on conditional Fourier-Feynman transforms and conditional convolution products, *International J. Appl. Math.,* **2** (2000), no.4, 505 - 527.

[5] D.H. Cho, Conditional Fourier-Feynman transform and convolution product over Wiener paths in abstract Wiener space: An L_p theory, *J. Korean Math. Soc.,* **41** (2004), no.2, 265 - 294.

[6] D.H. Cho, Conditional Fourier-Feynman transforms of variations over Wiener paths in abstract Wiener space, *J. Korean Math. Soc.,* **43** (2006), no.5, 967 - 990.

[7] D.H. Cho, A simple formula for an analogue of conditional Wiener integrals and its applications, *Trans. Amer. Math. Soc.,* **360** (2008), no.7, 3795 - 3811.

[8] D.H. Cho, A simple formula for an analogue of conditional Wiener integrals and its applications II, *Czechoslovak Math. J.,* to appear.

[9] D.M. Chung and D.L. Skoug, Conditional analytic Feynman integrals and a related Schrödinger integral equation, *SIAM J. Math. Anal.,* **20** (1989), no.4, 950 - 965.

[10] M.K. Im and K.S. Ryu, An analogue of Wiener measure and its applications, *J. Korean Math. Soc.,* **39** (2002), no.5, 801 - 819.

[11] R.G. Laha and V.K. Rohatgi, *Probability Theory*, John Wiley & Sons, New York-Chichester-Brisbane, 358 - 384, 1979.

[12] C. Park and D.L. Skoug, A simple formula for conditional Wiener integrals with applications, *Pacific J. Math.,* **135** (1988), no.2, 381 - 394.

[13] K.S. Ryu and M.K. Im, A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula, *Trans. Amer. Math. Soc.,* **354** (2002), no.12, 4921 - 4951.

[14] J. Yeh, Inversion of conditional expectations, *Pacific J. Math.,* **52** (1974), 631 - 640.

[15] J. Yeh, *Stochastic processes and the Wiener integral*, Marcel Dekker, New York, 1973.

Received: July, 2008