Conditional Fourier-Feynman Transform and Convolution Product on a Function Space

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Abstract
For a partition 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t of the interval [0, t], let
X_n(x) = (x(t_0), \cdots, x(t_n)) and X_{n+1}(x) = (x(t_0), \cdots, x(t_n), x(t_{n+1}))
on a generalized Wiener space (C[0, t], \mathcal{B}(C[0, t]), \nu), where C[0, t] is the space of the continuous paths on [0, t] and \nu is a probability measure on the Borel class \mathcal{B}(C[0, t]) of C[0, t].

In this paper, using simple formulas for conditional \nu-integrals on C[0, t] with the conditioning functions X_n and X_{n+1} we define the L^p analytic conditional Fourier-Feynman transform and the conditional convolution product over the generalized Wiener space. And then, we evaluate the conditional Fourier-Feynman transform and convolution product of the functions in a Banach algebra \mathcal{S}_\nu which corresponds to the Cameron and Storvick’s Banach algebra \mathcal{S}. Finally, we show that the conditional Fourier-Feynman transform of the conditional convolution product for the functions in \mathcal{S}_\nu can be expressed as a product of the conditional Fourier-Feynman transforms of each function.

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1 Introduction and Preliminaries

It is well-known that the classical Wiener space C_0[0, t] is the space of the real-valued continuous functions on the closed interval [0, t] which vanish at 0. On the space, Yeh introduced an inversion formula that a conditional expectation can be found by a Fourier transform [14]. But the Yeh’s inversion
formula is very complicated in its applications when the conditioning function is vector-valued. For a partition 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t of [0, t], let \( X_n(x) = (x(t_0), \ldots, x(t_n)) \) and \( X_{n+1}(x) = (x(t_0), \ldots, x(t_n), x(t_{n+1})) \) for \( x \in C_0[0, t] \). In [12], Park and Skoug introduced a simple formula for conditional Wiener integrals which evaluate the conditional Wiener integral of a function given \( X_{n+1} \) as a Wiener integral of the function. In [4], Chang and Skoug defined the conditional Fourier-Feynman transform and the conditional convolution product on the classical Wiener space \( C_0[0, t] \). Moreover, they evaluated conditional Fourier-Feynman transform and the conditional convolution product of the functions in a Banach algebra \( S \) which was introduced by Cameron and Storvick in [2]. Further works were studied by Cho [3, 5, 6] with cylinder type functions over Wiener paths in abstract Wiener space.

On the other hand, let \( C[0, t] \) denote the space of the real-valued continuous functions on the interval \([0, t]\). Im and Ryu [10] introduced a probability measure \( w_\varphi \) on \((C[0, t], \mathcal{B}(C[0, t]))\), where \( \mathcal{B}(C[0, t]) \) denotes the Borel \( \sigma \)-algebra of \( C[0, t] \) and \( \varphi \) is a probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). This measure space is a generalization of the Wiener space \( C_0[0, t] \). In [7, 8], Cho derived two simple formulas for the conditional \( w_\varphi \)-integrals with the vector-valued conditioning functions \( X_n \) and \( X_{n+1} \) which are defined on \( C[0, t] \). These formulas express the conditional \( w_\varphi \)-integrals directly in terms of non-conditional \( w_\varphi \)-integrals.

In this paper, with the conditioning functions \( X_n \) and \( X_{n+1} \), we define the \( L_p \) analytic conditional Fourier-Feynman transform and the conditional convolution product over the analogue \( w_\varphi \) of Wiener measure, where \( 1 \leq p \leq \infty \). Using the simple formulas over \( w_\varphi \), we evaluate the \( L_p \) analytic conditional Fourier-Feynman transform and the conditional convolution product of the functions in a Banach algebra \( S_{w_\varphi} \) which is an analogue of the Cameron and Storvick’s Banach algebra \( S \). Finally, we show that the \( L_p \) analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions in \( S_{w_\varphi} \) can be expressed as a product of the \( L_p \) analytic conditional Fourier-Feynman transforms of each function.

Throughout this paper, let \( \mathbb{C} \) and \( \mathbb{C}_+ \) denote the set of the complex numbers and that of the complex numbers with the positive real parts, respectively.

Now, we begin with introducing the probability space \((C[0, t], \mathcal{B}(C[0, t]), w_\varphi)\). For a positive real \( t \), let \( C = C[0, t] \) be the space of all real-valued continuous functions on the closed interval \([0, t]\) with the supremum norm. For \( \vec{t} = (t_0, t_1, \cdots, t_n) \) with \( 0 = t_0 < t_1 < \cdots < t_n \leq t \), let \( J_{\vec{t}} : C[0, t] \to \mathbb{R}^{n+1} \) be the function given by

\[
J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)).
\]

For \( B_j \) (\( j = 0, 1, \cdots, n \)) in \( \mathcal{B}(\mathbb{R}) \), the subset \( J_{\vec{t}}^{-1}\left(\prod_{j=0}^{n} B_j\right) \) of \( C[0, t] \) is called an interval and let \( \mathcal{I} \) be the set of all such intervals. For a probability measure
\[ m_\varphi \left( J_t^{-1} \left( \prod_{j=0}^n B_j \right) \right) = \left[ \prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})} \right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^n B_j} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\} d(u_1, \ldots, u_n) d\varphi(u_0). \]

\( \mathcal{B}(C[0, t]) \), the Borel \( \sigma \)-algebra of \( C[0, t] \), coincides with the smallest \( \sigma \)-algebra generated by \( \mathcal{I} \) and there exists a unique probability measure \( w_\varphi \) on \( (C[0, t], \mathcal{B}(C[0, t])) \) such that \( w_\varphi(I) = m_\varphi(I) \) for all \( I \) in \( \mathcal{I} \). This measure \( w_\varphi \) is called an analogue of the Wiener measure associated with the probability measure \( \varphi \) [10, 13, 15].

Let \( \{e_k : k = 1, 2, \ldots\} \) be a complete orthonormal subset of \( L_2[0, t] \) such that each \( e_k \) is of bounded variation. For \( f \) in \( L_2[0, t] \) and \( x \) in \( C[0, t] \), we let

\[ (f, x) = \lim_{n \to \infty} \sum_{k=1}^n \int_0^t \langle f, e_k \rangle e_k(s) dx(s) \]

if the limit exists. Here \( \langle \cdot, \cdot \rangle \) denotes the inner product over \( L_2[0, t] \). \( (f, x) \) is called the Paley-Wiener-Zygmund integral of \( f \) according to \( x \).

Applying Theorem 3.5 in [10], we can easily prove the following theorem.

**Theorem 1.1** Let \( \{h_1, h_2, \ldots, h_n\} \) be an orthonormal system of \( L_2[0, t] \). For \( i = 1, 2, \ldots, n \), let \( Z_i(x) = (h_i, x) \) on \( C[0, t] \). Then \( Z_1, Z_2, \ldots, Z_n \) are independent and each \( Z_i \) has the standard normal distribution. Moreover, if \( f : \mathbb{R}^n \to \mathbb{R} \) is Borel measurable, then we have

\[ \int_C f(Z_1(x), Z_2(x), \ldots, Z_n(x)) dw_\varphi(x) = \left( \frac{1}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, u_2, \ldots, u_n) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n u_j^2 \right\} d(u_1, u_2, \ldots, u_n), \]

where \( = \) means that if either side exists then both sides exist and they are equal.

Let \( F : C[0, t] \to \mathbb{C} \) be integrable and let \( X \) be a random vector on \( C[0, t] \) assuming that the value space of \( X \) is a normed space with the Borel \( \sigma \)-algebra. Then, we have the conditional expectation \( E[F|X] \) of \( F \) given \( X \) from a well-known probability theory [11]. Further, there exists a \( P_X \)-integrable complex-valued function \( \psi \) on the value space of \( X \) such that \( E[F|X](x) = (\psi \circ X)(x) \) for \( w_\varphi \)-a.e. \( x \in C[0, t] \), where \( P_X \) is the probability distribution of \( X \). The function \( \psi \) is called the conditional \( w_\varphi \)-integral of \( F \) given \( X \) and it is also denoted by \( E[F|X] \).
Throughout this paper, let \( 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t \) be a partition of \([0, t]\) unless otherwise specified. For any \( x \) in \( C[0, t] \), define the polygonal function \([x]\) on \([0, t]\) by

\[
[x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})), t_{j-1} \leq t \leq t_j, j = 1, \cdots, n + 1.
\]

Similarly, for \( \vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{n+2} \), define the polygonal function \([\vec{\xi}_{n+1}]\) on \([0, t]\) by

\[
[\vec{\xi}_{n+1}](t) = \xi_{j-1} + \frac{t - t_j}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}), t_{j-1} \leq t \leq t_j, j = 1, \cdots, n + 1.
\]

In the following two theorems, we introduce simple formulas for the conditional \( w_x \)-integrals on \( C[0, t] \) [7, 8].

**Theorem 1.2** Let \( F : C[0, t] \to \mathbb{C} \) be integrable and let \( X_{n+1} : C[0, t] \to \mathbb{R}^{n+2} \) be given by

\[
X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1})).
\]

Then we have for \( P_{X_{n+1}} \)-a.e. \( \vec{\xi}_{n+1} \in \mathbb{R}^{n+2} \)

\[
E[F|X_{n+1}](\vec{\xi}_{n+1}) = E[F(x - [x] + [\vec{\xi}_{n+1}])],
\]

where \( P_{X_{n+1}} \) is the probability distribution of \( X_{n+1} \) on \((\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))\).

Next, we introduce another simple formula for the conditional \( w_x \)-integrals removing the point \( x(t) \) in the conditioning function \( X_{n+1} \) given by (1).

**Theorem 1.3** Let \( X_n : C[0, t] \to \mathbb{R}^{n+1} \) be given by

\[
X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n)).
\]

Moreover let \( F \) be integrable on \( C[0, t] \) and \( P_X \) be the probability distribution of \( X_n \) on \((\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))\). Then we have

\[
E[F|X_n](\vec{\xi}_n) = \left[ \frac{1}{2\pi(t-t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(x - [x] + [\vec{\xi}_{n+1}])] \\
\times \exp\left\{ -\frac{(\xi_{n+1} - \xi_n)^2}{2(t-t_n)} \right\} d\xi_{n+1}
\]

(4)

for \( P_X \)-a.e. \( \vec{\xi}_n \in \mathbb{R}^{n+1} \), where \( \vec{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \) and \( \vec{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}) \).
For a function \( F : C[0,t] \to \mathbb{C} \) and \( \lambda > 0 \), let \( F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x) \) and \( X^\lambda_{n+1}(x) = X_{n+1}(\lambda^{-\frac{1}{2}}x) \), \( X^\lambda_n(x) = X_n(\lambda^{-\frac{1}{2}}x) \), where \( X_{n+1} \) and \( X_n \) are given by (1) and (3), respectively. Suppose that \( E[F^\lambda] \) exists for each \( \lambda > 0 \). By the definition of the conditional \( w_\varphi \)-integral and (2), we have

\[
E[F^\lambda|X^\lambda_{n+1}](\xi_{n+1}) = E[F(\lambda^{-\frac{1}{2}}(x-[x]) + [\xi_{n+1}])]
\]

for \( P_{X^\lambda_{n+1}} \)-a.e. \( \xi_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2} \), where \( P_{X^\lambda_{n+1}} \) is the probability distribution of \( X^\lambda_{n+1} \) on \( (\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2})) \). For \( \xi_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1} \) and \( \xi_{n+1} \in \mathbb{R} \), let \( \xi_{n+1}(\lambda) = (\lambda^{\frac{1}{2}}\xi_0, \lambda^{\frac{1}{2}}\xi_1, \cdots, \lambda^{\frac{1}{2}}\xi_n, \xi_{n+1}) \). Then we have by (4) and the change of variable theorem

\[
E[F^\lambda|X^\lambda_n](\xi_n) = \left[ \frac{1}{2\pi(t-t_n)} \right]^\frac{1}{2} \int_{\mathbb{R}} E[F^\lambda(x-[x]) + [\xi_{n+1}(\lambda)]] \times \exp \left\{ \frac{-((\xi_{n+1} - \lambda^{\frac{1}{2}}\xi_n)^2}{2(t-t_n)} \right\} d\xi_{n+1}
\]

\[
= \left[ \frac{\lambda}{2\pi(t-t_n)} \right]^\frac{1}{2} \int_{\mathbb{R}} E[F(\lambda^{-\frac{1}{2}}(x-[x]) + [\xi_{n+1}])] \times \exp \left\{ \frac{-\lambda(\xi_{n+1} - \xi_n)^2}{2(t-t_n)} \right\} d\xi_{n+1}
\]

(5)

for \( P_{X^\lambda_n} \)-a.e. \( \xi_n \), where \( P_{X^\lambda_n} \) is the probability distribution of \( X^\lambda_n \) on \( (\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1})) \). If \( E[F(\lambda^{-\frac{1}{2}}(x-[x]) + [\xi_{n+1}])] \) has the analytic extension \( J^\lambda_n(F)(\xi_{n+1}) \) on \( \mathbb{C}_+ \) as a function of \( \lambda \), then it is called the conditional analytic Wiener \( w_\varphi \)-integral of \( F \) given \( X_{n+1} \) with parameter \( \lambda \) and denoted by

\[
E^{anw_\lambda}[F|X_{n+1}](\xi_{n+1}) = J^\lambda_n(F)(\xi_{n+1})
\]

for \( \xi_{n+1} \in \mathbb{R}^{n+2} \). Moreover, if for a non-zero real \( q \), \( E^{anw_\lambda}[F|X_{n+1}](\xi_{n+1}) \) has a limit as \( \lambda \) approaches to \(-iq\) through \( \mathbb{C}_+ \), then it is called the conditional analytic Feynman \( w_\varphi \)-integral of \( F \) given \( X_{n+1} \) with parameter \( q \) and denoted by

\[
E^{anf_q}[F|X_{n+1}](\xi_{n+1}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_{n+1}](\xi_{n+1}).
\]

Similar definitions are understood with (5) if we replace \( X_{n+1} \) by \( X_n \).

### 2 Conditional Fourier-Feynman Transforms

In this section, we define an \( L_p \) analytic conditional Fourier-Feynman transform of the functions on \( C[0,t] \) and evaluate the transforms of the functions
in the Banach algebra $S_{w_{\varphi}}$ which is similar to the Cameron and Storvick’s Banach algebra $S$ in [2].

For a given extended real number $p$ with $1 < p \leq \infty$, suppose that $p$ and $p'$ are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let $G_n$ and $G$ be measurable functions such that for $\rho > 0$

\[
\lim_{n \to \infty} \int_C |G_n(\rho x) - G(\rho x)|^{p'} d\varphi(x) = 0.
\]

Then we write

\[
l.i.m.\left(\frac{w^{p'}}{p'}\right)(G_n) \approx G
\]

and call $G$ the limit in the mean of order $p'$. A similar definition is understood when $n$ is replaced by a continuously varying parameter.

Now, we define an analytic conditional Fourier-Feynman transform of the functions on $C[0, t]$.

**Definition 2.1** Let $F$ be defined on $C[0, t]$ and let $X_{n+1}$ be given by (1). For $\lambda \in \mathbb{C}_+$ and for $w_{\varphi}$-a.e. $y \in C[0, t]$, let

\[
T_\lambda[F|X_{n+1}](y, \xi_{n+1}) = E^{anw_{\lambda}}[F(\cdot + y)|X_{n+1}](\xi_{n+1})
\]

for $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For a non-zero real $q$ and for $w_{\varphi}$-a.e. $y \in C[0, t]$, we define the $L_1$ analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_{n+1}]$ of $F$ by the formula

\[
T_q^{(1)}[F|X_{n+1}](y, \xi_{n+1}) = E^{anf_q}[F(\cdot + y)|X_{n+1}](\xi_{n+1})
\]

for $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For $1 < p \leq \infty$ we define the $L_p$ analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_{n+1}]$ of $F$ by the formula

\[
T_q^{(p)}[F|X_{n+1}](\cdot, \xi_{n+1}) \approx l.i.m.\left(\frac{w^p}{p}\right)(T_{\lambda}^{w_{\lambda}}[F|X_{n+1}](\cdot, \xi_{n+1}))
\]

for $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$, where $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$.

Similar definitions are understood with (5) if we replace $X_{n+1}$ by $X_n$ which is given by (3).

Let $\mathcal{M}(L_2[0, t])$ be the class of $\mathbb{C}$-valued Borel measures of finite variation on $L_2[0, t]$ and let $S_{w_{\varphi}}$ be the space of the functions $F$ of the form for $\sigma \in \mathcal{M}(L_2[0, t])$

\[
F(x) = \int_{L_2[0, t]} \exp\{i(v, x)\} d\sigma(v)
\]  

(6)
for \( w_\varphi \)-a.e. \( x \in C[0,t] \). For \( v \in L_2[0,t] \) define the sectional average \( \bar{v} \) of \( v \) by letting

\[
\bar{v}(s) = \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} v(u)du
\]

on each subinterval \((t_{j-1}, t_j)\) and by letting \( \bar{v}(0) = 0 \) [9]. Then, we have for \( v \in L_2[0,t] \) and \( x \in C[0,t] \)

\[
(v, [x]) = \sum_{j=1}^{n+1} \bar{v}(t_j)(x(t_j) - x(t_{j-1})) = \sum_{j=1}^{n+1} \int_{t_{j-1}}^{t_j} \bar{v}(u)dx(u) = (\bar{v}, x). \tag{8}
\]

We are now ready to evaluate the \( L_p \) analytic conditional Fourier-Feynman transforms of the functions in \( S_{w_\varphi} \).

**Theorem 2.2** Let \( X_{n+1} \) and \( F \in S_{w_\varphi} \) be given by (1) and (6), respectively. Moreover, let \( 1 \leq p \leq \infty \). Then, for \( \lambda \in \mathbb{C}_+ \) and for \( w_\varphi \)-a.e. \( y \in C[0,t] \), \( T_\lambda[F|X_{n+1}](y, \bar{\xi}_{n+1}) \) exists for \( P_{X_{n+1}} \)-a.e. \( \bar{\xi}_{n+1} = (\xi_0, \xi_1, \ldots , \xi_{n+1}) \in \mathbb{R}^{n+2} \) and it is given by

\[
T_\lambda[F|X_{n+1}](y, \bar{\xi}_{n+1}) = \int_{L_2[0,t]} \exp\left\{ i(v, y) + i\sum_{j=1}^{n+1} \bar{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda} \|v - \bar{v}\|^2 \right\} d\sigma(v) \tag{9}
\]

where \( \bar{v} \) is given by (7). Moreover, for a non-zero real \( q \), \( T_\lambda^{(p)}[F|X_{n+1}](y, \bar{\xi}_{n+1}) \) exists and it is given by (9) replacing \( \lambda \) by \( -iq \).

**Proof.** For \( \lambda > 0 \) and \( w_\varphi \)-a.e. \( y \in C[0,t] \), we have by the Fubini’s theorem and (8)

\[
T_\lambda[F|X_{n+1}](y, \bar{\xi}_{n+1}) = \int_{C} \int_{L_2[0,t]} \exp\left\{ i(v, x - [x]) + [\bar{\xi}_{n+1}] + y \right\} d\sigma(v)dw_\varphi(x)
\]

\[
= \int_{L_2[0,t]} \exp\{i(v, y) + i(v, [\bar{\xi}_{n+1}])\} \int_{C} \exp\{i\lambda^{-\frac{1}{2}}(v - \bar{v}, x)\} dw_\varphi(x)d\sigma(v)
\]

for a.e. \( \bar{\xi}_{n+1} \in \mathbb{R}^{n+2} \). Using the following well-known integration formula

\[
\int_{\mathbb{R}} \exp\{-au^2 + ibu\}du = \left( \frac{\pi}{a} \right)^{\frac{1}{2}} \exp\left\{ -\frac{b^2}{4a} \right\} \tag{10}
\]
for $a \in \mathbb{C}_+$ and any real $b$, we have

$$T_\lambda[F|X_{n+1}](y, \bar{\xi}_{n+1}) = \int_{L_2[0,t]} \exp \left\{ i(v, y) + i \sum_{j=1}^{n+1} \bar{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda} ||v - \bar{v}||_2^2 \right\} d\sigma(v)$$

since $(v - \bar{v}, \cdot)$ is mean zero Gaussian with variance $||v - \bar{v}||_2^2$ by Theorem 1.1. By the Morera’s theorem and the dominated convergence theorem, we have the results for $p = 1$. Now suppose that $1 < p \leq \infty$. For $\frac{1}{p} + \frac{1}{p'} = 1$ and $\rho > 0$, we have

$$\int_C \left| \int_{L_2[0,t]} \exp \left\{ i(v, \rho y + [\bar{\xi}_{n+1}]) \right\} \left[ \exp \left\{ -\frac{1}{2\lambda} ||v - \bar{v}||_2^2 \right\} - \exp \left\{ \frac{1}{2\rho} ||v - \bar{v}||_2^2 \right\} \right] d\sigma(v) \right|^p d\omega(y)$$

which converges to 0 as $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. Now, the proof is completed. \qed

**Theorem 2.3** Let $X_n$ and $F \in S_{w_\phi}$ be given by (3) and (6), respectively. Moreover, let $1 \leq p \leq \infty$. Then, for $\lambda \in \mathbb{C}_+$ and $w_\phi$-a.e. $y \in C[0,t]$, $T_\lambda[F|X_n](y, \bar{\xi}_n)$ exists for $P_{X_n}$-a.e. $\bar{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ and it is given by the formula

$$T_\lambda[F|X_n](y, \bar{\xi}_n) = \int_{L_2[0,t]} \exp \left\{ i(v, y) + i \sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1}) - \frac{1}{2\lambda} [||v - \bar{v}||_2^2 + (t - t_n)[\bar{v}(t)]^2] \right\} d\sigma(v)$$

where $\bar{v}$ is given by (7). Moreover, for a non-zero real $q$, $T_q^{(p)}[F|X_n](y, \bar{\xi}_n)$ exists and it is given by (11) replacing $\lambda$ by $-iq$.

**Proof.** For notational convenience, let $\bar{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$ and $\bar{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$ and $P_{X_n}$-a.e. $\bar{\xi}_n \in \mathbb{R}^{n+1}$, we have by Theorems 1.3 and 2.2

$$T_\lambda[F|X_n](y, \bar{\xi}_n) = \left[ \frac{\lambda}{2\pi(t - t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\bar{\xi}_{n+1}] + y)] \times \exp \left\{ -\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t - t_n)} \right\} d\xi_{n+1}$$

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\[
= \left[ \frac{\lambda}{2\pi(t-t_n)} \right]^\frac{1}{2} \int_{L_2[0,t]} \exp \left\{ \frac{i(v,y) + i \sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1})}{\frac{\lambda}{2} \|v - \bar{v}\|^2} \right\} \frac{d\xi_{n+1}}{d\sigma(v)}
\]

by the Fubini's theorem where \(\bar{v}\) is the sectional average of \(v\) given by (7). Let \(u = \xi_{n+1} - \xi_n\). Then we have by the change of variable theorem

\[
\begin{align*}
T_\lambda[F|X_n](y, \xi_n) &= \left[ \frac{\lambda}{2\pi(t-t_n)} \right]^\frac{1}{2} \int_{L_2[0,t]} \exp \left\{ i(v,y) + i \sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1}) \right\} \frac{d\xi_{n+1}}{d\sigma(v)}
\end{align*}
\]

by (10). By the Morera's theorem and the dominated convergence theorem, we have the results for \(p = 1\). Now suppose that \(1 < p \leq \infty\). For \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(\rho > 0\), we have

\[
\left| \int_{C} \int_{L_2[0,t]} \exp \left\{ i \left[ (v, \rho y) + \sum_{j=1}^{n} \bar{v}(t_j)(\xi_j - \xi_{j-1}) \right] \right\} \frac{d\xi_{n+1}}{d\sigma(v)} \right|^{p'} \leq \int_{L_2[0,t]} \left| \exp \left\{ \frac{1}{2q} \|v - \bar{v}\|^2 + (t-t_n)[\bar{v}(t)]^2 \right\} \right|^{p'} d\sigma(v)
\]

which converges to 0 as \(\lambda\) approaches to \(-iq\) through \(\mathbb{C}_+\) by the dominated convergence theorem. Now, the proof is completed. \(\square\)

3 Conditional Convolution Products

We begin this section with defining the conditional convolution product over the analogue \(w_\varphi\) of Wiener measure and evaluate it for the functions in the Banach algebra \(S_{w_\varphi}\).
Definition 3.1 Let $X_{n+1}$ be given by (1), and let $F$ and $G$ be defined on $C[0,t]$. We define the conditional convolution product $[(F \ast G)_\lambda X_{n+1}]$ of $F$ and $G$ given $X_{n+1}$ by the formula, for $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$

\[
[(F \ast G)_\lambda X_{n+1}](y, \xi_{n+1}) = \begin{cases} 
E_{w}\left[F\left(y + \frac{\cdot}{\sqrt{2}}\right)G\left(y - \frac{\cdot}{\sqrt{2}}\right)\right]X_{n+1}(\xi_{n+1}), & \lambda \in \mathbb{C}_+; \\
E_{q}\left[F\left(y + \frac{\cdot}{\sqrt{2}}\right)G\left(y - \frac{\cdot}{\sqrt{2}}\right)\right]X_{n+1}(\xi_{n+1}), & \lambda = -iq; \ q \in \mathbb{R} - \{0\}
\end{cases}
\]

if they exist for $w_\varphi$-a.e. $y \in C[0,t]$. If $\lambda = -iq$, we replace $[(F \ast G)_\lambda X_{n+1}]$ by $[(F \ast G)_q X_{n+1}]$.

Similar definitions are understood with (5) if we replace $X_{n+1}$ by $X_n$ which is given by (3).

In the following two theorems, we evaluate the conditional convolution product of the functions in $S_{w_\varphi}$ with the conditioning functions $X_n$ and $X_{n+1}$.

Theorem 3.2 Let $X_{n+1}$ be given by (1) and $F_1, F_2 \in S_{w_\varphi}$ by (6) replacing $\sigma$ by $\sigma_1, \sigma_2$, respectively. Then, for $\lambda \in \mathbb{C}_+$ and $w_\varphi$-a.e. $y \in C[0,t], [(F_1 \ast F_2)_\lambda X_{n+1}](y, \xi_{n+1})$ exists for $P_{X_{n+1}}$-a.e. $\xi_{n+1} \in \mathbb{R}^{n+2}$ and it is given by

\[
[(F_1 \ast F_2)_\lambda X_{n+1}](y, \xi_{n+1}) = \int_{(L_2[0,t])^2} A(v_1, v_2, y, \xi_{n+1}) \exp\left\{-\frac{1}{4\lambda} ||v_1 - v_2||^2 - (\bar{v}_1 - \bar{v}_2)||^2 2\right\} d\sigma_1(v_1) d\sigma_2(v_2) \quad (12)
\]

where $A(v_1, v_2, y, \xi_{n+1}) = \exp\left\{\frac{1}{\sqrt{2\lambda}} [(v_1, y + \xi_{n+1})] + (v_2, y - \xi_{n+1})]\right\}$. Moreover, for a non-zero real $q$, $[(F_1 \ast F_2)_q X_{n+1}](y, \xi_{n+1})$ exists and it is given by (12) replacing $\lambda$ by $-iq$.

Proof. For $\lambda > 0$ we have by the Fubini’s theorem

\[
[(F_1 \ast F_2)_\lambda X_{n+1}](y, \xi_{n+1}) = \int_{(L_2[0,t])^2} A(v_1, v_2, y, \xi_{n+1}) \exp\left\{\frac{i}{\sqrt{2\lambda}} [(v_1, x - [x]) - (v_2, x - [x])]\right\} dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2).
\]

Since

\[
(v_1, x - [x]) - (v_2, x - [x]) = (v_1 - v_2 - (\bar{v}_1 - \bar{v}_2), x) \quad (13)
\]
by (8), we have

\[
[(F_1 * F_2)_\lambda|X_{n+1})(y, \bar{\xi}_{n+1}) = \int_{(L_2[0,t])^2} A(v_1, v_2, y, \bar{\xi}_{n+1}) \exp \left\{ \frac{i}{\sqrt{2\lambda}} (v_1 - v_2 - (\bar{v}_1 - \bar{v}_2), x) \right\} d\sigma_1(v_1) d\sigma_2(v_2)
\]

by (14) replacing \( \lambda \) by \( -\alpha q \).

**Theorem 3.3** Let the assumptions and notations be given as in Theorem 3.2 and let \( X_n \) be given by (3). Then, for \( \lambda \in \mathbb{C}_+ \) and \( w_v \)-a.e. \( y \in C[0,t] \), \([(F_1 * F_2)_\lambda|X_n])y, \bar{\xi}_n \) exists for \( P_{X_n}\)-a.e. \( \xi_n \in \mathbb{R}^{n+1} \) and it is given by

\[
[(F_1 * F_2)_\lambda|X_n](y, \bar{\xi}_n) = \int_{(L_2[0,t])^2} \exp \left\{ \frac{i}{\sqrt{2}} \left[ (v_1 + v_2, y) + \sum_{j=1}^n [\bar{v}_1(t_j) - \bar{v}_2(t_j)](\xi_j - \xi_{j-1}) \right] - \frac{1}{4\lambda} \left[ (t - t_n)[\bar{v}_1(t) - \bar{v}_2(t)]^2 + ||v_1 - v_2 - (\bar{v}_1 - \bar{v}_2)||^2_2 \right] \right\} d\sigma_1(v_1) d\sigma_2(v_2).
\]

Moreover, for a non-zero real \( q \), \([(F_1 * F_2)_q|X_{n+1})(y, \bar{\xi}_n) \) exists and it is given by (14) replacing \( \lambda \) by \(-i\lambda q\).

**Proof.** For notational convenience, let \( \bar{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1} \) and \( \bar{\xi}_{n+1} = (\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}) \) for \( \xi_{n+1} \in \mathbb{R} \). For \( \lambda > 0 \) and \( P_{X_n} \)-a.e. \( \xi_n \in \mathbb{R}^{n+1} \), we have by Theorems 1.3 and 3.2

\[
[(F_1 * F_2)_\lambda|X_n](y, \bar{\xi}_n) = \left[ \frac{\lambda}{2\pi(t - t_n)} \right]^\frac{1}{2} \int_{(L_2[0,t])^2} \exp \left\{ -\frac{1}{2\lambda} ||v_1 - v_2 - (\bar{v}_1 - \bar{v}_2)||^2_2 \right\} d\sigma_1(v_1) d\sigma_2(v_2)
\]

Moreover, for a non-zero real \( q \), \([(F_1 * F_2)_q|X_{n+1})(y, \bar{\xi}_n) \) exists and it is given by (14) replacing \( \lambda \) by \(-i\lambda q\).
for \( P \) can be expressed by the right-hand side of the above equality replacing □ convergence theorem, we have the results.

\[ \int_{(L^2[0,t])^2} \exp \left\{ \frac{i}{\sqrt{2}} \left[ (v_1 + v_2, y) + \sum_{j=1}^{n} (\tilde{v}_1(t_j) - \tilde{v}_2(t_j))(\xi_j - \xi_{j-1}) \right] \right\} 
- \frac{1}{4\lambda} \left[ (t - t_n)(\tilde{v}_1(t) - \tilde{v}_2(t))^2 + \|v_1 - v_2 - (\tilde{v}_1 - \tilde{v}_2)\|_2^2 \right] \text{d}\sigma_1(v_1)\text{d}\sigma_2(v_2) \]

by the Fubini’s theorem and (10). By the Morera’s theorem and the dominated convergence theorem, we have the results. \( \square \)

4 Relationships Between Transforms and Convolutions

In this section, we investigate relationships between the conditional convolution products and the \( L_p \) analytic conditional Fourier-Feynman transforms of the functions in \( S_{w_p} \). In fact, with the conditioning functions \( X_n \) and \( X_{n+1} \), we show that the \( L_p \) analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions in \( S_{w_p} \) can be expressed as a product of the \( L_p \) analytic conditional Fourier-Feynman transforms of each function.

**Theorem 4.1** Let \( 1 \leq p \leq \infty \). Then, under the assumptions and notations given as in Theorem 3.2, we have for \( \lambda \in \mathbb{C}_+ \) and \( w_{\varphi} \)-a.e. \( y \in C[0,t] \)

\[ T_\lambda \left[ [(F_1 * F_2)_\lambda |X_{n+1}] (\cdot, \xi_{n+1}) |X_{n+1}] (y, \xi_{n+1}) \right] 
= \left[ T_\lambda [F_1 |X_{n+1}] \left( \frac{y}{\sqrt{2}}, \frac{\xi_{n+1} + \xi_{n+1}}{\sqrt{2}} \right) \right] \left[ T_\lambda [F_2 |X_{n+1}] \left( \frac{y}{\sqrt{2}}, \frac{\xi_{n+1} - \xi_{n+1}}{\sqrt{2}} \right) \right] \]

for \( P_{X_{n+1}} \)-a.e \( \tilde{\xi}_{n+1}, \tilde{\xi}_{n+1} \in \mathbb{R}^{n+2} \), and \( T^{(p)} \left[ [(F_1 * F_2)_q |X_{n+1}] (\cdot, \xi_{n+1}) |X_{n+1}] (y, \xi_{n+1}) \right] \)

can be expressed by the right-hand side of the above equality replacing \( \lambda \) by \( q \).

**Proof.** For \( \lambda > 0 \) we have by Theorem 3.2

\[ T_\lambda \left[ [(F_1 * F_2)_\lambda |X_{n+1}] (\cdot, \tilde{\xi}_{n+1}) |X_{n+1}] (y, \tilde{\xi}_{n+1}) \right] 
= \int_{C} \int_{(L^2[0,t])^2} \text{A}(v_1, v_2, y + \lambda^{-\frac{1}{2}}(x - [x]) + [\tilde{\xi}_{n+1}, \tilde{\xi}_{n+1}, \tilde{\xi}_{n+1}, \text{exp} \left\{ -\frac{1}{4\lambda} \|v_1 - v_2 \right. 
- (\tilde{v}_1 - \tilde{v}_2)\|_2^2 \right\} \text{d}\sigma_1(v_1)\text{d}\sigma_2(v_2)\text{d}w_{\varphi}(x) \]

where \( A \) is given as in Theorem 3.2. By (10), (13), Theorem 1.1 and the Fubini’s theorem, we have

\[ T_\lambda \left[ [(F_1 * F_2)_\lambda |X_{n+1}] (\cdot, \tilde{\xi}_{n+1}) |X_{n+1}] (y, \tilde{\xi}_{n+1}) \right] 
= \int_{(L^2[0,t])^2} \text{A}(v_1, v_2, y + \tilde{\xi}_{n+1}, \tilde{\xi}_{n+1}) \exp \left\{ -\frac{1}{4\lambda} \|v_1 - v_2 - (\tilde{v}_1 - \tilde{v}_2)\|_2^2 \right\} \]
where the last equality follows from Theorem 2.2. By the Morera’s theorem and the dominated convergence theorem, we have the results. □

Theorem 4.2 Let $1 \leq p \leq \infty$. Then, under the assumptions and notations given as in Theorem 3.3, we have for $\lambda \in \mathbb{C}_+$ and $w_\phi$-a.e. $y \in C[0, t]$

\[
T_\lambda[(F_1 \ast F_2)_\lambda|X_n](\cdot, \bar{\zeta}_n)\mid X_n](y, \bar{\zeta}_n) = \left[ T_\lambda[F_1|X_n]\left( \frac{y}{\sqrt{2}}, \frac{\zeta_n + \xi_n}{\sqrt{2}} \right) \right] \left[ T_\lambda[F_2|X_n]\left( \frac{y}{\sqrt{2}}, \frac{\zeta_n - \xi_n}{\sqrt{2}} \right) \right]
\]

for $P_{X_n}$-a.e. $\bar{\zeta}_n, \bar{\zeta}_n \in \mathbb{R}^{n+1}$ and $T_\lambda^{(p)}[(F_1 \ast F_2)_\lambda|X_n](\cdot, \bar{\zeta}_n)\mid X_n](y, \bar{\zeta}_n)$ can be expressed by the right-hand side of the above equality replacing $\lambda$ by $q$.

Proof. For notational convenience, let $\bar{\xi}_n = (\xi_0, \xi_1, \cdots, \xi_n)$, $\bar{\zeta}_n = (\zeta_0, \zeta_1, \cdots, \zeta_n) \in \mathbb{R}^{n+1}$ and $\bar{\zeta}_{n+1} = (\zeta_0, \zeta_1, \cdots, \zeta_n, \zeta_{n+1})$ for $\zeta_{n+1} \in \mathbb{R}$. For $\lambda > 0$ and $P_{X_n}$-a.e. $\bar{\xi}_n, \bar{\zeta}_n \in \mathbb{R}^{n+1}$, we have by Theorems 1.3 and 3.3

\[
T_\lambda[(F_1 \ast F_2)_\lambda|X_n](\cdot, \bar{\xi}_n)\mid X_n](y, \bar{\zeta}_n) = \left[ \frac{\lambda}{2\pi(t-t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} \int_C \int_{(L_2[0,t])^2} \exp\left\{ \frac{i}{\sqrt{2}} \lambda^{-\frac{1}{2}} \left[ (v_1 + v_2, y + (\zeta_n + \xi_n) - \frac{1}{4\lambda}(t-t_n)[\bar{v}_1(t) - \bar{v}_2(t)]^2 \right. \right.
\]

\[
\left. + \frac{\lambda(\zeta_{n+1} - \zeta_n)^2}{2(t-t_n)} \left\} d\sigma_1(v_1) d\sigma_2(v_2) d\sigma_2(x) d\zeta_{n+1}.$

By (10), (13), Theorem 1.1 and the Fubini’s theorem, we have

\[
T_\lambda[(F_1 \ast F_2)_\lambda|X_n](\cdot, \bar{\xi}_n)\mid X_n](y, \bar{\zeta}_n)
and the dominated convergence theorem, we have the results.

where the last equality follows from Theorem 2.3. By the Morera’s theorem and the dominated convergence theorem, we have the results.

\[
= \left[ \frac{\lambda}{2\pi(t - t_n)} \right]^{\frac{1}{2}} \int_{(L_2[0,t])^2} \exp \left\{ \frac{i}{\sqrt{2}} \left[ (v_1 + v_2, y) + \sum_{j=1}^{n} \left[ (\bar{v}_1(t_j) + \bar{v}_2(t_j))(\zeta_j - \zeta_{j-1}) \right] - \frac{1}{4\lambda} \left[ (t - t_n)[\bar{v}_1(t) - \bar{v}_2(t)]^2 + \|v_1 - v_2 - (\bar{v}_1 - \bar{v}_2)\|_2^2 + \|v_1 + v_2 - (\bar{v}_1 + \bar{v}_2)\|_2^2 \right] \right\} d\zeta_n + d\sigma_1(v_1)d\sigma_2(v_2) \\
= \int_{(L_2[0,t])^2} \exp \left\{ \frac{i}{\sqrt{2}} \left[ (v_1 + v_2, y) + \sum_{j=1}^{n} \left[ (\bar{v}_1(t_j) + \bar{v}_2(t_j))(\zeta_j - \zeta_{j-1}) + [\bar{v}_1(t_j) - \bar{v}_2(t_j)](\xi_j - \xi_{j-1}) \right] - \frac{1}{4\lambda} \left[ (t - t_n)[\bar{v}_1(t) - \bar{v}_2(t)]^2 + [\bar{v}_1(t) + \bar{v}_2(t)]^2 \right] \right\} d\sigma_1(v_1)d\sigma_2(v_2) \\
= \int_{(L_2[0,t])^2} \exp \left\{ \frac{i}{\sqrt{2}} \left[ (v_1, y) + \sum_{j=1}^{n} \bar{v}_1(t_j) \left[ (\zeta_j - \zeta_{j-1}) + (\xi_j - \xi_{j-1}) \right] \right] - \frac{1}{2\lambda} \left[ (t - t_n)[\bar{v}_1(t)]^2 + \|v_1 - \bar{v}_1\|_2^2 \right] d\sigma_1(v_1) \right\} \int_{L_2[0,t]} \exp \left\{ \frac{i}{\sqrt{2}} \left[ (v_2, y) + \sum_{j=1}^{n} \bar{v}_2(t_j) \left[ (\zeta_j - \zeta_{j-1}) - (\xi_j - \xi_{j-1}) \right] \right] - \frac{1}{2\lambda} \left[ (t - t_n)[\bar{v}_2(t)]^2 + \|v_2 - \bar{v}_2\|_2^2 \right] d\sigma_2(v_2) \right\} \\
= T_\lambda[F_1|X_n]\left( \frac{y}{\sqrt{2}}, \frac{\bar{\zeta}_n + \bar{\xi}_n}{\sqrt{2}} \right) \left[ T_\lambda[F_2|X_n]\left( \frac{y}{\sqrt{2}}, \frac{\bar{\zeta}_n - \bar{\xi}_n}{\sqrt{2}} \right) \right] \\
\]

References


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