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Eigendistributions for the Invariant Differential Operators on the Affine Group

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Abstract

It is known that $\chi_\xi(x) = e^{-i<\xi,x>}$ are eigendistributions for any invariant differential operator on $\mathbb{R}^2$. In this paper, we show that if T is an eigendistribution of an invariant differential operator on $\mathbb{R}^2$, then an eigendistribution for an invariant differential operator on the Affine group G can be obtained. To this end, an existence theorem is put forward. Out of this theorem a fundamental solution of these invariant differential operators has been also obtained.

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1 Preliminary and results.

Let A be the affine group which consists of all matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \qquad x > 0,\ y \in \mathbb{R}. \tag{1.1}$$

Let $\mathbb{R}_+ = \{x \in \mathbb{R};\ x > 0\}$ be the multiplicative group of all positive real numbers. Let $G = \mathbb{R} \propto_\rho \mathbb{R}_+$ be the group of the semi-direct product of $\mathbb{R}$ and $\mathbb{R}_+$, via the group homomorphism $\rho : \mathbb{R}_+ \longrightarrow Aut(\mathbb{R})$ defined by:

$$\rho(x)(y) = x \cdot y \tag{1.2}$$

for any $x \in \mathbb{R}_+$ and $y \in \mathbb{R}$, where $Aut(\mathbb{R})$ is the group of all automorphisms of $\mathbb{R}$. the multiplication of two elements $X = (y, x)$ and $Y = (y', x')$ in G is given by

$$\begin{aligned} X \cdot Y &= (y, x) \cdot (y', x') \\ &= (y + xy', x \cdot x')1.3 \end{aligned} \tag{1.1}$$

The inverse of an element $X = (y, x)$ in G is:

$$X^{-1} = (y, x)^{-1} = (-x^{-1}y, x^{-1}) \tag{1.4}$$

In view of the group isomorphism $\Phi : G \longrightarrow A$ defined by:

$$\Phi(y, x) = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

we can identify the group A with the group G.

Let $C^\infty(G)$, $\mathcal{D}(G)$, $\mathcal{D}'(G)$, $\mathcal{E}'(G)$ respectively be the space of C^∞-functions, C^∞-functions with compact support, distributions and distributions with compact support. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra g of G; which is canonically isomorphic onto the algebra of all distributions on G supported by $\{(0, 1)\}$ the identity element of G. For any $u \in \mathcal{U}$, one can define a differential operator P_u on$G$ as follows:

$$\begin{aligned} P_u f(X) &= u * f(X) \\ &= \int_G f(Y^{-1}X)u(Y)\ dY 1.5 \end{aligned} \tag{1.2}$$

for any $f \in C^\infty(G)$, where $X = (y, x)$, $Y = (y', x')$ and $dY = dy' \frac{dx'}{x'}$ is the right Haar measure on G. The mapping $u \longmapsto P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all right invariant differential operators on$G$. For more details see [5,10].

Let $V = \mathbb{R} \times \mathbb{R}_+$ be the group of the direct product of $\mathbb{R}$ by $\mathbb{R}_+$. we denote by $S\ (V)$ the symmetrice algebra of V. For every $u \in\ S\ (V)$, we can associate a differential operator Q_u on V as follows:

$$\begin{aligned} Q_u f(X) &= u *_c f(X) \\ &= \int_V f(X - Y)u(Y)\ dY 1.6 \end{aligned} \tag{1.3}$$

for any $f \in C^\infty(V)$, $X \in V$ and $Y \in V$, where $*_c$ signifies the commutative convolution product on the group V and $dY = dy\frac{dx}{x}$ is the Lebesgue measure on V. The mapping $u \longmapsto Q_u$ is an algebra isomorphism of $S\ (V)$ onto the algebra of all invariant differential operators on V, which are nothing but the algebra of differential operators with constant coefficiants on V. Then there is a unique linear bijection $\tau : \mathcal{U} \longrightarrow\ S\ (V)$.

2 Invariant Functions

Let $L = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ be the group of the mixed product $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_+$, with multiplication

$$\begin{aligned} X \cdot Y &= (z, y, x)(z', y', x') \\ &= (z + xz', yy', xx')2.1 \end{aligned} \tag{2.1}$$

for all $(z, y, x) \in L$ and $(z', y', x') \in L$. The inverse of an element $X = (z, y, x)$ in L is given by:

$$\begin{aligned} X^{-1} &= (z, y, x)^{-1} \\ &= (-x^{-1}z, y^{-1}, x^{-1})2.2 \end{aligned} \tag{2.2}$$

In this case, we can identify G with the closed subgroup $\mathbb{R} \times \{1\} \propto_\rho \mathbb{R}_+$ and V with $\mathbb{R} \times \mathbb{R}_+ \times \{1\}$.

Definition 2.1: For every $f \in C^\infty(L)$, one can define a function $\widetilde{f} \in C^\infty(L)$ as follows:

$$\widetilde{f}(z, y, x) = f(yz, 1, yx) \tag{2.3}$$

for all $(z, y, x) \in L$.

Remark 2.2: The function $\widetilde{f}$ is invariant in the following sense:

$$\widetilde{f}(kz, yk^{-1}, xk) = \widetilde{f}(z, y, x) \tag{2.4}$$

for any $(z, y, x) \in L$ and $k \in \mathbb{R}_+$. The restriction $R\widetilde{f}$ of $\widetilde{f}$ on G (resp. on V) belongs to $\mathcal{S}(G)$(resp. $\mathcal{S}(V)$) if $f \in \mathcal{S}(L)$.

Proposition 2.3. For every function $F \in C^\infty(L)$ invariant in sense(2.4) and for every $u \in \mathcal{U}$,we have

$$u \ \star F((z, y); x) = u \ \star_c F((z, y); x) \tag{2.5}$$

for every $(z, y; x) \in L$, where $\star$ signifies the convolution product on G with respect the variables (z, x) and $\star_c$signifies the convolution product on V with respect the variables $(z, y\ )$.

Proof. In fact we have

$$\begin{aligned} P_uF(z,y,x) &= u * F(z,y,x) \\ &= \int_G F\left[(b,a)^{-1}(z,y,x)\right] u(b,a)db\frac{da}{a} \\ &= \int_G F\left[(-a^{-1}b,a^{-1})(z,y,x)\right] u(b,a)db\frac{da}{a} \\ &= \int_G F(-a^{-1}(z-b),y,xa^{-1})u(b,a)db\frac{da}{a} \\ &= \int_V F((z-b),ya^{-1},x)u(b,a)db\frac{da}{a} \\ &= u *_c F(z,y,x) = Q_uF(z,y,x) \end{aligned}$$

where P_u and Q_u are the invariant differential operators on G and V respectively

3 Fourier Transform and the Divisioin of Distributions.

The Schwartz space $\mathcal{S}(G)$ of G can be considered as the Schwartz space $\mathcal{S}(\mathbb{R}$ x $\mathbb{R}_+$). Let $\mathcal{S}'(G)$ be the space of all tempered distributions on G. The action ρ of the group $\mathbb{R}_+$on $\mathbb{R}$ defines a natural action ρ on the dual group $(\mathbb{R})^*$ of the group $\mathbb{R}$ ($(\mathbb{R})^* \simeq \mathbb{R}$), which is given by :

$$\rho\,(x)\,\xi = x.\xi \tag{3.1}$$

for any $\xi \in \mathbb{R}$ and $x \in \mathbb{R}_+$

Definition 3.1. *For every $f \in \mathcal{S}(G)$, one can define its Fourier transform $\mathcal{F}f$ by :*

$$\mathcal{F}f\ \ (\xi,\lambda) = \int_G f(y,x)\ e^{-\,i\,\xi\,y}\ x^{-i\,\lambda}\frac{dx}{x}\ dy \tag{3.2}$$

for any $\xi \in \mathbb{R}$, and $\lambda \in \mathbb{R}$. It is clear that $\mathcal{F}f \in \mathcal{S}(G)$ and the mapping $f \rightarrow \mathcal{F}f$ is isomorhism of the topological vector space $\mathcal{S}(G)$.

Proposition 3.2. For every $u \in D(G)$, and $F \in \mathcal{D}(L)$, we have

$$\int_{\mathbb{R}} \mathcal{F}(\overset{\vee}{u} * \widehat{f}\,)\,(\xi,\lambda,\ \mu\,)\ d\ \mu = \mathcal{F}(\widehat{f}\ \ )\,(\xi,\lambda,\ 1)\ \mathcal{F}(\ \overset{\vee}{u}\ )\ (\xi,\lambda) \tag{3.3}$$

for any $\xi \in \mathbb{R}$, and $\lambda \in \mathbb{R}$, where $\overset{\vee}{u}(y,x) = u((y,x)^{-1})$ for any $(y,x) \in G$

Proof. First, we have:

$$u \star F((z,y);x) = u \star_c F((z,y);x)$$

$$= \int_G F(z-b), y\ a^{-1},\ x)u(b,a)db\tfrac{da}{a}$$

secondly:

$$\int_{\mathbb{R}} \mathcal{F}(\overset{\vee}{u} * \widehat{f}\,)\ (\xi,\lambda,\ \mu\ )d\ \mu = \int_{\mathbb{R}} \mathcal{F}(\overset{\vee}{u} *_c \widehat{f}\,)\ (\xi,\lambda,\ \mu\ )\ d\ \mu$$

$$= \mathcal{F}(\widehat{f}\,)\ (\xi,\lambda,1)\mathcal{F}(\ \overset{\vee}{u}\ )\ (\xi,\lambda)$$

Theorem 3.3. Every invariant differential operator u on G which is not identically 0 has a tempered fundamental solution.

Proof. For each complex number s with positive real part, we can define a distribution $T^{\ S}$ on L by:

$$\langle T^{\ s},\ f\ \ \rangle = \int_{\mathbb{R}^3}\int\int \left[\left|\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)\right|^2\right]^s \mathcal{F}(f\ \ )(\xi,\lambda,\mu)d\xi d\ \lambda d\mu$$

for every $f \in \mathcal{S}(\ L\ )$. By Atiyah-Bernstein theorems [1] and [2], the function $s \to T^{\ s}$ has a meromorphic continuation in the whole complex plan, which is analytic at $s = 0$ and its value at this point is the Dirac measure on the group$L$. Now we can define another distribution $\widehat{T^{\ s}}$, as follows

$$\begin{aligned}\left\langle \widehat{T^{\,s}}, f\right\rangle &= \left\langle T^{\,s}, \widehat{f}\right\rangle \\ &= \int_{\mathbb{R}^3}\int\int \left[\left|\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)\right|^2\right]^s \mathcal{F}(f\ \ )(\xi,\lambda,\mu)d\ \xi d\ \lambda d\mu\end{aligned}$$

for every $f \in \mathcal{S}(L)$ and $s \in \mathbb{C}$, with Re(s) ≥ 0

Note that the distribution $\widehat{T^{\,s}}$is invariant in sense (2.4) and we have

$$\begin{aligned}
\left\langle \widehat{u * v *_c T^{\ s}}, f \right\rangle &= \left\langle u * v *_c T^{\ s}, \widehat{f} \right\rangle \\
&= \left\langle T^s, \overset{\vee}{v} *_c \overset{\vee}{u} * \widehat{f} \right\rangle \\
&= \int\limits_{\mathbb{R}^3}\int\int \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi, \lambda) \right|^2 \right]^s \mathcal{F}(\ \overset{\vee}{v} *_c \overset{\vee}{u} * \widehat{f}\ )(\xi, \lambda, \mu) d\ \xi d\ \lambda d\mu
\end{aligned}$$

here

$$v(z, y) = \overline{u(-z,\ y^{-1})}$$

and

$$\overset{\vee}{v} *_c f = \int\limits_G f((z-a, yb^{-1})\overset{\vee}{v}((a, b); c) dadbdc$$

is the commutative convolution product on G. By proposition 3.1, we get:

$$\left\langle \widehat{u * \widetilde{u} *_c T^{\ s}}, f \right\rangle = \int\limits_{\mathbb{R}^3}\int\int \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi, \lambda) \right|^2 \right]^{s+1} \mathcal{F}(\ \widehat{f}\ )(\xi, \lambda, \mu) d\ \xi d\ \lambda d\mu$$

hence

$$\widehat{u * v *_c T^{\ s}} = \widehat{T^{\ s+1}} \tag{3.4}$$

In view of invariance (2.4), the restriction of the distributions $\widehat{u * v *_c T^{\ s}} = \widehat{T^{\ s+1}}$ on the sub-group $\mathbb{R} \times \{0\} \underset{\rho}{\propto} \mathbb{R}_+ \simeq G$ are nothing but the distributions

$$u * v *_c T^{\ s} = T^{\ s+1} \tag{3.5}$$

i.e.

$$u * v *_c T^{\ s}(z, 1, x) = T^{s+1}(z, 1, x)$$

The distribution $T^{\ s}$ can be expanded a round $s = -1$, ([1],[2]) in the form

$$T^s = \sum_{j\,=\,-(2n+1)}^{\infty} \alpha_j (s+1)^j \tag{3.6}$$

where each α_j is a distribution on G. But $u * v *_c T^{\,s} = T^{\,s+1}$ can not have a pole at $s = -1$ (since $T^{\,0} = \delta_G$) and so we must have:

$$\begin{aligned} u * v \;\; *_c\, \alpha_j &= 0 \quad \text{for} \quad j < 0 \\ u * v \;\; *_c\, \alpha_0 &= \delta_G \end{aligned}$$

whence the theorem.

4 An Existence Theorem and Eigendistributions.

In the following, we state and prove the following existence theorem, which leads us to deduce the eigendistributions of an element $u \in \mathcal{D}(G)$

Theorem 4.1. Let $P_{\underset{u}{\vee}}$ and $Q_{\underset{u}{\vee}}$ be the invariant differential operators on G and V respectively, then the following conditions are equivalent:

$$(i)\, Q_{\underset{u}{\vee}}\, \mathcal{D}^{'}(V\;) = \mathcal{D}^{'}(V.)$$

$$(ii) P_{\underset{u}{\vee}}\; \mathcal{D}^{'}(G) = \mathcal{D}^{'}(G). \tag{4.1}$$

Proof: Let δ be the Dirac measure on $\mathcal{R}_+$, then $T \mapsto i\; T = T \otimes \delta$ is a one-to-one continuous mapping from $\mathcal{D}^{'}(B\;)$ into $\mathcal{D}^{'}(L)$. By equation (2.5), we have

$$\begin{aligned} < \widetilde{i(\breve{u} \star_c T)}, F > &= < i(\breve{u} \star_c T), \tilde{F} > \\ &= < (\breve{u} \star_c T) \otimes \delta, \tilde{F} > \\ &= < (\breve{u} \star_c (T \otimes \delta), \tilde{F} > \\ &= < T \otimes \delta, u \star_c \tilde{F} > \\ &= < T \otimes \delta, u \star \tilde{F} > \\ &= < iT, u \star \tilde{F} > \\ &= < \breve{u} \star iT, \tilde{F} > \\ &= < \widetilde{\breve{u} \star iT}, F > \end{aligned}$$

for any $F\; \epsilon \mathcal{D}(L)$and $T\; \epsilon\; \mathcal{D}^{'}(V)$. Thus

$$\widetyle{\breve{u} \star iT} = i(\widetilde{\breve{u} \star_c T}) \tag{4.2}$$

Note that $\breve{u}$ in the left hand side of equation (4.2)signifies $\breve{u}(g) = u(g^{-1})$ for any $g\epsilon G$, and $\breve{u}$ in the right hand side of this equation signifies $\breve{u}\ (g) = u(-g)$ for any $g\ \epsilon V$. Now, as well known in the theory of differential operators with constant coeficients, for every distribution S on V, there is a distribution T on V [12], such that

$$\breve{u} \star_c T = S \tag{4.3}$$

Consequently,

$$\begin{aligned} i(\widetilde{\breve{u} \star_c T}) &= \widetilde{\breve{u} \star iT} \\ &= \widetilde{\breve{u} \star (T \otimes \delta)} \\ &= \widetilde{iS} \\ &= \widetilde{S \otimes \delta} \end{aligned}$$

By the restriction on G, we get

$$\begin{aligned} (\breve{u} *_c T) \otimes \delta(z,1,x) &= \breve{u} \star (T \otimes \delta)(z,1,x) \\ &= S \otimes \delta(z,1,x) \end{aligned}$$

for any $((z,y);t)\epsilon G$.

Let $\mathcal{R}$ be any distribution on $\mathbb{R}$, then

$$\begin{aligned} \breve{u} \star (T \otimes \delta) \star_c \mathcal{R}(z,1,x) &= \breve{u} \star (T \otimes \mathcal{R})(z,1,x) \\ &= S \otimes \mathcal{R}(z,1,x) \end{aligned}$$

where:

$$\breve{u} \star (T \otimes \delta) \star_c \mathcal{R}(z,1,x) = \int_R \breve{u} \star (T \otimes \delta)(z,1,x)\mathcal{R}(xk^{-1})dk$$

Since $\mathcal{D}'(\mathbb{R}) \otimes \mathcal{D}'(\mathbb{R})$ is dense in $\mathcal{D}'(G)$, then for any distribution S' on G there is a distribution E on G such that

$$\breve{u} \star E(z,1,x) = S'(z,1,x) \tag{4.4}$$

for any $((z, y) \in G$. This proves that (i) implies (ii).

In order to prove that (ii) implies (i), consider the group $K = G \times \mathbb{R}_+$ and the mapping $F \mapsto \tilde{F}$, which is defined by

$$\tilde{F}((z, y\ ; x) = F((-yz\ , 1, x\ y) \tag{4.5}$$

for any $F \epsilon C^\infty(K)$ and $((z, y; x)\epsilon K$. Then it is easy to show that (ii) implies (i).

Corollary 4.2. If $T(z, y)$ is a fundamental solution for $\breve{u}\ \epsilon \mathcal{D}(V)$, then $iT(z, 1, x)$ is also a fundamental solution for $\breve{u}\ \epsilon \mathcal{U}$.

Theorem 4.3. If $T\ (z, y)$ is an eigendistribution of an element $u\ \epsilon S(V)$, then $iT(z, 1, x)$ is an eigendistribution of u considered as an invariant differential operator on G.

Proof: In fact we have:

$$\begin{aligned} \widetilde{\breve{u} \star iT}\ (z, y, x) &= i(\widetilde{\breve{u} \star_c T})(z, y, x) \\ &= \widetilde{\breve{u} \star_c iT}\ (z, y, x) \end{aligned}$$

taking the restriction on G, we get

$$\begin{aligned} \breve{u} \star iT\ (z, 1, x) &= i(\breve{u} \star_c T)(z, 1, x) \\ &= \breve{u} \star_c iT\ (z, 1, x) \end{aligned}$$

If T is an eigendistribution of $\breve{u}$, as an invariant differential operator on V, we obtain

$$\widetilde{\breve{u} \star iT} = \widetilde{\lambda iT} = \widetilde{\breve{u} \star_c iT}$$

Thus, we get

$$\breve{u} \star iT\ (z, 1, x) = \lambda iT\ (z, 1, x) \tag{4.6}$$

Convolving each side of equation(4.6)by a distribution $\mathcal{R}$ on $\mathbb{R}$ with respect to x, we have

$$\breve{u} \star (T \otimes \mathcal{R})(z, 1, x) = \lambda(T \otimes \mathcal{R})(z, 1, x)$$

equivalently; $(T \otimes \mathcal{R})\breve{} \star u\ (z, 1, x) = \lambda(T \otimes \mathcal{R})\breve{}(z, 1, x)$

Now, let T_ξ be the distribution on V defined by:

$$\begin{aligned} < T_\xi, f > &= < T_\xi(z,y),\ f\ (\ z,y) > \\ &= \int_V f\ (\ z,y\ ) e^{-i <\xi,\ (z,y)>} d\,X \end{aligned} \tag{4.7}$$

for any $f\ \epsilon \mathcal{D}(V)$, where $X = (z,y) \in V$, $\xi = (\lambda, \mu) \in\ V$ and
$< \xi, (z,y\ ) >= \lambda z + \mu\ ln\ y$,
then we have:

Corollary 4.4. Let $T_\xi(z,y)$ be the distributions on V defined by (4.7), then the distributions $T_\xi \otimes \chi_\alpha(z,1,x\ )$on G are eigendistributions for any invariant differential operator on G, where $\chi_\alpha(x)= x^{-i\ \alpha}$

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