Int. Journal of Math. Analysis, Vol. 3, 2009, no. 9, 443 - 456

On the Geometry of the Second Order Tangent Bundle with the Diagonal Lift Metric

Mohammed Hamou DIDA[1], Fouzi HATHOUT[2] and Mustapha DJAA[3]

Geometry, Analysis, Control and Applications
Laboratory of Saïda University pb 138
"20000" Algeria

Abstract

In this paper we use the diagonal lift ^{D}g of Riemannian metric g of manifold M_n to the tangent bundle of order two T^2M_n of M_n introduced in [8], we associate to ^{D}g its Levi-Civita connection of T^2M_n and we calculate its curvature tensor, On T^2M_n there exist two an almost complex structure $F_1$and F_2 adapted with ^{D}g, the parallelism and integrability of which is given through the vanishing of curvature and we derives relations between the geometry properties of (M_n, g) and $(T^2M_n, {}^{D}g)$.

Mathematics Subject Classification: Primary 53A45; Secondary 53B25

Keywords: Tangent bundle of order 2, Riemannian metric, λ-lift, Diagonal lift, Levi-Civita connection, curvature tensor, f-structure, parallelism, Nijenhuis tensor, integrability, sectional curvature, scalar curvature

1 Introduction

The second order tangent bundle T^2M of smooth manifold M is smooth manifold of equivalent classes of curves c on M that agree up to their acceleration denoted $[c, x]_2$ or j^2c. So, that is a generalization of the tangent bundle TM.

Dodson and Radivoiovici ([3]) proved that T^2M of finite n-dimensional M becomes a vector bundle over M with structure group the general linear

[1]didamohammed@yahoo.com
[2]f.hathout@caramail.com
[3]djaamustapha@hotmail.com

group $GL(2n;\mathbb{R})$ and 3n-dimensional manifold if and only if the manifold M is endowed with additional structure: a linear connection ∇.

The λ-lift of tensor fildes and H-lift of given connection on manifold M to the tangent bundle of higher order T^rM $(\lambda, H = 1, ..r)$ where T^rM is the smooth manifold of equivalent classes of curves c on M that agree up to their r-velocity or a manifold of r-jet, is a generalization of vertical and horizontal lift of structure geometric to the tangent bundle TM.([11])

The connection ∇ defines a diffeomorphism S between the second order tangent bundle T^2M and the Whitney sum of two copies of the tangent bundles TM. S is a fibre diffeomorphism of locally trivial bundle but it is not an isomorphism of natural bundles. Next, using the vertical and horizontal lift $(X^V,\, X^H)$ (see [7],[9]) of vector fields $X \in C^\infty(M)$ we define by the λ-lift the adapted frame $\{X^0, X^1, X^2\}$, so a sequence of distributions $E^0,\ E^1$ and E^2 on T^2M such that $T(T^2M) = E^0 \oplus E^1 \oplus E^2$.

In the present paper, we equipped T^2M for Riemanian manifold (M, g) with diagonal lift metric ${}^Dg$ defined in ([3],[12]), and using the λ-lift of vector fields X to the tangent bundle T^2M, we calculate the Levi-Civita connection ${}^D\nabla$, the curvature tensor ${}^DR$, the sectional curvatures ${}^DK$ and the scalar curvature ${}^DS$.

We proof that their exist on T^2M two f-structure F_1 and F_2 adapted with ${}^Dg$ that his parallelism and integrability is lied with the geometry of $(T^2M, {}^D g)$.

At the end, we derives relations between the geometry of (M, g) and $(T^2M, {}^D g)$.

2 Second Tangent Bundle

Let M be an n-dimensional smooth differentiable manifold and $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$ a corresponding atlas. For each $x \in M$ we define a equivalence relation on $C_x = \{c : (-\varepsilon, \varepsilon) \to M \ / \ c \text{ is smooth and } c(0) = x, \varepsilon > 0\}$ by

$$c \approx_x h \Longleftrightarrow c'(0) = h'(0) \ and \ c''(0) = h''(0),$$

where by c' and c''we denote the first and the second, respectively, derivation of c :

$$\begin{aligned} c' &: (-\varepsilon, \varepsilon) \to TM \ ; \ t \to [dc(t)](1) \\ c'' &: (-\varepsilon, \varepsilon) \to T(TM) \ ; \ t \to [dc'(t)](1). \end{aligned}$$

Definition 1. *We define the second tangent space of M at the point x to be the quotient $T^2_xM = C_x/ \approx_x$and the second tangent bundle of M the union of all second tangent space: $T^2M = \underset{x\in M}{\cup} T^2_xM$. We denote $[c, x]_2$ the equivalence class of c with respect to $\approx_x$and $[c]_2$ an element of T^2M.*

More that, an additional structure on M, T^2M become a vector bundle: a linear connection.

Theorem 2. *If we assume that a linear connection ∇ is defined on the manifold M, then T^2M becomes a Banach vector bundle with structure group the general linear group $GL(2n;\mathbb{R})$.*

Proof. see [4]. □

Now, let ∇ be a linear connection on M. Let $\pi_2 : T^2M \to M$ be a projection defined by $\pi_2([c]) = c(0)$, if $(U, x^1, ..., x^n)$ is a chart on M, then we consider the induced chart $(\pi_2^{-1}(U), x^{i,\lambda})$ on T^2M defined by

$$x^{i,\lambda}([c]_2) = \frac{1}{\lambda!}\frac{d^\lambda}{dt^\lambda}(x^i \circ c)(0) \quad for\ i = 1, .., n\ and\ \lambda = 0, 1, 2.$$

Using the connection ∇ we can define the mapping

$$\begin{aligned} S &: T^2M \to TM + TM \\ S([c]_2) &= (\dot{c}(0), (\nabla_{\dot{c}}\dot{c})(0)) \end{aligned}$$

∇_c denotes the covariant derivation along c and $\dot{c}$ is the velocity vector field of c.

The mapping S is defined as

$$\nabla_{\dot{c}}\ \dot{c}^i = (\frac{d^2c^i}{dt^2} + p^i)\frac{\partial}{\partial x^i}$$

where p^i is a polynomial of degree two on $\frac{d^2c^i}{dt^2}$.

Thus using the induced charts on T^2M and $TM + TM$ we have

$$S(\{x^{i,\lambda}\}) = (\{\zeta^{i,\lambda}\}),$$

where $x^{i,0} = \zeta^{i,0}$ and for $\lambda = 1, 2$ we have $\zeta^{i,\lambda} = x^{i,\lambda} + p^i(\{x^{j,1}\}^{j=1,..,n})$.

This implies that S is a diffeomorphism.

3 λ-lift from M to T^2M

If f is a function on M, and for any vector fields X on M, we denote by f^V and f^C the vertical lift and complete lift respectively of f to TM defined by

$$f^V = f \circ \pi \quad ; \quad f^C(X) = X(f)^V.$$

Let be X a vector field on M. Then there is one and only one vector field X^V on TM called the vertical lift of X such that

$$X^V(f^V) = 0.$$

The connection ∇ defines the horizontal distribution H on TM such that

$$T(TM) = V \oplus H \quad \text{where } V = \ker d\pi$$

Since for every point z of TM

$$d_z\pi_{/H_z} \ : H_z \to T_{\pi(z)}M$$

is an isomorphism, then, if X a vector field on M, we can define

$$X^H(z) = (d_z\pi_{/H_z})^{-1}(X_{\pi(z)})$$

X^H is a vector field on TM called the horizontal lift of X to TM.

Consequently, $\{X^H, X^V\}$ is a 2n-frame which will be called the **adapted frame** to ∇ in TM.

Now, for any vector field X on M we shall consider three vectors fields X^0, X^I et X^{II} on T^2M defined by

$$\begin{array}{rcl} X^0 & = & S_*^{-1}(X^H + X^H) \\ X^I & = & S_*^{-1}(X^V + 0) \\ X^{II} & = & S_*^{-1}(0 + X^V) \end{array} \tag{1}$$

We defines a sequence of distributions E^0, E^1 and E^2 on T^2M such that

$$T(T^2M) = E^0 \oplus E^1 \oplus E^2.$$

If we put in T^2M

$$X^0 = (\tfrac{\partial}{\partial x_i})^0,\ X^1 = (\tfrac{\partial}{\partial x_i})^I\ ;\ X^2 = (\tfrac{\partial}{\partial x_i})^{II} \tag{2}$$

and by diffeomorphism S we easily obtain

$$X^0 = S_*^{-1}(X^H, X^H),\ X^1 = S_*^{-1}(X^V, 0),\ X^2 = S_*^{-1}(0, X^V). \tag{3}$$

We have,

Definition 3. *if X is a vector fields on U, X^λ $(\lambda = 1, 2, 3)$ is called the **λ-lift** of X to T^2M.*

Consequently, $\{X^0, X^1, X^2\}$ is a 3n-frame which will be called the **adapted frame** to ∇ in T^2M.

λ-lift where studied in ([11]) to the tangent bundle of higher order T^rM, and in the case of $r = 1$ ([6]) we have $X^1 = X^V$ and $X^0 = X^H$ but in ([2]) in the case of $r = 1$, $X^1$and X^0 coincide with X^V and X^C where X^C is the complet lift of X.

Let now M be a Riemannian manifold with nondegenerate metric g whose components is a coordinate neighborhood U are g_{ij}.

4 Diagonal Lift of Riemannian Metric to T^2M

For any tensor field g of type $(0,2)$ in M, there exist a unique tensor field ${}^Dg \in \mathfrak{T}_2^0(T^2M)$ witch for any vectors fields X, Y on M and any $i, j = 0,1,2$, we have

$$ {}^Dg(X^i, Y^j) = \delta^i_j \; g(X,Y) \circ \overset{2}{\pi} \tag{4} $$

and locally in T^2M we have

$$ {}^Dg = g_{ij}dx^0_i \otimes dx^0_j + g_{ij}dx^1_i \otimes dx^1_j + g_{ij}dx^2_i \otimes dx^2_j \tag{5} $$

thus from (4-1) and (4-2) ^{D}g has components of the form

$$ ({}^Dg_{\beta\alpha}) = \begin{pmatrix} g_{ij} & 0 & 0 \\ 0 & g_{ij} & 0 \\ 0 & 0 & g_{ij} \end{pmatrix} \tag{6} $$

with respect to the adapted frame $\{X^0, X^1, X^2\}$ in T^2M.

From (4-3) we follows that if g is a Riemannian metric in M, then ${}^Dg$ is a Riemannian metric in T^2M.

Definition 4. *The metric ${}^Dg$ is called the **diagonal lift** of the tensor field g to T^2M given by*

$$ \begin{cases} i) \; {}^Dg_\zeta(X^i, Y^i) = g_p(X,Y) \\ ii) \; {}^Dg_\zeta(X^i, Y^j) = 0 \qquad i,j = 0,1,2 \text{ and } i \neq j \end{cases} $$

for all $\zeta = (p, u, u') \in T^2M$.

The diagonal lift of tensor field of type (0,2) to second frame bundle was studies in ([9]) and in ([12]) he give a definition of diagonal lift of tensor field of type (0,2) at tangent bundle of higher order, it was generalization of diagonal lift studied by S.Sasaki in case of TM.([14])

5 Levi-Civita connection of ^{D}g

Taking account that the Levi-Civita connection ∇ is torsion free we shall need the following identities

$$ \begin{aligned} \left[X^0, Y^0\right] &= [X,Y]^0 - \sum_{k=1,2} (R(X,Y)u)^k \\ \left[X^0, Y^j\right] &= (\nabla_X Y)^j \\ \left[X^i, Y^j\right] &= 0 \qquad \forall i,j = 1,2. \end{aligned} \tag{7} $$

(see [10],[11])

And by KOSZULE formula, the Levi-Civita connection ${}^D\nabla$ of $(T^2M, {}^D g)$ is given as following

$$\begin{array}{lcl} 1/\ {}^D\nabla_{X^0}Y^0 & = & (\nabla_X Y)^0 - \frac{1}{2}\sum_{k=1,2}(R(X,Y)u)^k \\ 2/\ {}^D\nabla_{X^0}Y^1 & = & (\nabla_X Y)^1 + \frac{1}{2}(R(u,Y)X)^0 \\ 3/\ {}^D\nabla_{X^0}Y^2 & = & (\nabla_X Y)^2 + \frac{1}{2}(R(u,Y)X)^0 \\ 4/\ {}^D\nabla_{X^1}Y^0 & = & {}^D\nabla_{X^2}Y^0 = \frac{1}{2}(R(u,X)Y)^0 \\ 5/\ {}^D\nabla_{X^1}Y^1 & = & {}^D\nabla_{X^1}Y^2 = {}^D\nabla_{X^2}Y^1 = {}^D\nabla_{X^2}Y^2 = 0 \end{array} \tag{8}$$

for any vectors fields $X,\ Y \in C^\infty(M)$ and $\zeta = (p, u, u') \in T^2M$.

Having determined the Levi-Civita connection we are ready to calculate the Riemannian curvature tensor of T^2M. But first we state the following useful lemma.

Lemme 5. *Let $F : T^2M \to T^2M$ be a smooth bundle endomorphism of T^2M. Then*

$$\left\{ \begin{array}{l} i)\ ({}^D\nabla_{X^i}F(\eta)^0)_\zeta = F(X)^0_\zeta + \frac{1}{2}(R(u,X)F(\eta))^0_\zeta \\ ii)\ ({}^D\nabla_{X^i}F(\eta)^j)_\zeta = F(X)^j_\zeta \qquad i,j = 1,2 \\ ii)\ ({}^D\nabla_{X^0}F(\eta)^k)_\zeta = {}^D\nabla_{X^0}F(u)^k_\zeta \quad k = 0,1,2 \end{array} \right.$$

for any $\zeta = (p, u, u') \in T^2M$ and $X, \eta \in C^\infty(TM)$.

6 The Curvature Tensor

Let R be a curvature tensor of (M, g), and ${}^DR$ is curvature tensor of $(T^2M, {}^D g)$ equipped with the diagonal lift of g. Then the following formula hold

$$\begin{aligned} 1/\ {}^DR(X^0,Y^0)Z^0 = {} & (R(X,Y)Z)^0 + \tfrac{1}{2}(R(R(Y,Z)u,u)X - R(R(X,Z)u,u)Y \\ & -2R(R(X,Y)u,u)Z)^0 + \tfrac{1}{2}\textstyle\sum_{k=1,2}((\nabla_Z R)(X,Y)u)^k \end{aligned}$$

$$\begin{aligned} 2/\ {}^DR(X^0,Y^0)Z^i = {} & (R(X,Y)Z)^i + \tfrac{1}{2}((\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X)^0 \\ & -\tfrac{1}{4}\textstyle\sum_{k=1,2}(R(X,R(u,Z)Y)u - R(Y,R(u,Z)X)u)^k \end{aligned}$$

$$\begin{aligned} 3/\ {}^DR(X^1,Y^1)Z^0 = {} & (R(X,Y)Z)^0 + \tfrac{1}{4}(R(u,X)R(u,Y)Z - R(u,Y)R(u,X)Z)^0 \\ & -(g(u,X)R(u,Y)Z - g(u,Y)R(u,X)Z)^0 \end{aligned}$$

$$4/\ {}^DR(X^i,Y^2)Z^0 = (R(X,Y)Z)^0 + \tfrac{1}{4}(R(u,X)R(u,Y)Z - R(u,Y)R(u,X)Z)^0$$

$$5/\ {}^DR(X^i,Y^0)Z^0 = \tfrac{1}{2}(\nabla_Y R)(u,X)Z) - \tfrac{1}{4}\textstyle\sum_{k=1,2}(2R(Y,Z)X - R(Y,R(u,X)Z)u)^k$$

$$6/\ {}^DR(X^i,Y^0)Z^j = (\tfrac{1}{2}R(X,Z)Y + \tfrac{1}{4}R(u,X)R(u,Z)Y)^0$$

7/ ${}^D R(X^1, Y^2)Z^i = {}^D R(X^1, Y^1)Z^i = {}^D R(X^2, Y^2)Z^i = 0$
for any $\zeta = (p, u, u') \in T^2M$, $i, j = 1, 2$ and $X, Y, Z \in C^\infty(TM)$.

7 f-structure on T^2M

An f-structure on a manifold M is known to be a field of endomorphisms f acting on its tangent bundle and satisfying the condition $f^3 + f = 0$ (see [6]). The number $r = \dim Imf$ is constant at any point of M and called a rank of the f-structure. Recall that an f-structure on a (pseudo)Riemannian manifold (M, g) is called a metric f-structure, if $g(fX, X) = 0$, $X \in \mathfrak{T}^1_0(M)$. In the case the triple (M, g, f) is called a metric f-manifold. It is clear that the tensor field $\Omega(X, Y) = g(X, fY)$ is skew-symmetric, i.e. is a 2-form on M. Ω is called a fundamental form of a metric f-structure .

In [15] Vohra and Singh introduced the following terminology
1) $fAK-$manifold if and only if $d\Omega(FX, FY, FZ) = 0$, for any vector fields X, Y, Z on M.
2) $fH-$manifold if and only if F is partially integrable.
3) $fK-$manifold if and only if $\nabla_{FX}F = 0$, for any vector field X on M, ∇ being the Levi-Civita connection of g.

We denote N_F the Nijenhuis tensor of f-structure F defined by

$$N_F(\tilde{X}, \tilde{Y}) = F^2 \left[\tilde{X}, \tilde{Y}\right] + \left[F\tilde{X}, F\tilde{Y}\right] - F\left[F\tilde{X}, \tilde{Y}\right] - F\left[\tilde{X}, F\tilde{Y}\right],$$

for all $\tilde{X}, \tilde{Y} \in C^\infty(T^2M)$.

Moreover, they get

Proposition 6. *M is $fK-$manifold if and only if M is $fAK-$manifold and $fH-$manifold.*

For a linear connection ∇, we define two tensors fields $F_1$and F_2 of type (1,1) on T^2M by

$$F_1(X^0) = X^1 \ ; \ F_1(X^k) = -\delta^1_k \ X^0, \tag{9}$$
$$F_2(X^0) = X^2 \ ; \ F_2(X^k) = -\delta^2_k \ X^0 \quad for \ k = 1, 2 \tag{10}$$

for any vector field X on M.

Then the matrix representation of F_h $(h = 1, 2)$ is

$$F_h = \begin{pmatrix} 0 & \delta^j_i \ \delta^1_h & \delta^j_i \ \delta^2_h \\ -\delta^j_i \ \delta^1_h & 0 & 0 \\ -\delta^j_i \ \delta^2_h & 0 & 0 \end{pmatrix}$$

with respect to the adapted frame $\{X^0, X^1, X^2\}$. Therefore, F_h has constante rank $2n$ and ${F_h}^3 + F_h = 0$, which implies

Theorem 7. *For a linear connection ∇ on M, there exist an **f-structure** F_h $(h=1,2)$ defined by (9) and (10) on T^2M.*

Let $\mathfrak{i}_h = -{F_h}^2, \mathfrak{m}_h = {F_h}^2 + I$ $(h=1,2)$ be the projection operator of F_h $(h=1,2)$ and $\mathfrak{L}_h = Im\mathfrak{i}_h$, $\mathfrak{M}_h = Im\mathfrak{m}_h$ the complementary distributions associated to $\mathfrak{i}_h$ and $\mathfrak{m}_h$; they have dimension $2n$ and n respectively.

Since we have

$$\begin{aligned} \mathfrak{i}_h(X^0) &= X^0 \ ; \ \mathfrak{i}_h(X^k) = \delta_h^k \ X^h \\ \mathfrak{m}_h(X^0) &= 0 \ ; \ \mathfrak{m}_h(X^k) = (1-\delta_h^k) \ X^k \end{aligned}$$

for $k=1,2,\ h=1,2$ and any vector field X on M.

Then $\left\{X^0, X^h\right\}$ span $\mathfrak{L}_h$ and $\left\{X^{\delta_h^1+1}\right\}$ span $\mathfrak{M}_h$.

Proposition 8. *$\mathfrak{L}_h$ is completely integrable if and only if $R\equiv 0$ (i.e, ∇ is locally flat).*

Proof. A necessary and sufficient condition for the distribution $\mathfrak{L}_h$ to be integrable is that

$$N_F(\mathfrak{L}_h X, \mathfrak{L}_h Y) = 0.$$

and by formula (7) we deduce the proof. □

Now, we shall proof that the diagonal lift metric ${}^Dg$ is adapted to the f-structure $F_h(h=1,2)$ and we study here parallelism. For this, we recall some definitions.

Definition 9. *Suppose that there is given an f-structure F on manifold M. Then a Riemannian metric g on M is said to be **adapted** to F (or hor-Erhesmannian in ([5]) if:*
i) The distributions $\mathfrak{L}$ and $\mathfrak{M}$ canonically associated to F are orthogonal with respect to g.
ii) $g(X, FX) = 0$ for any vector field X on M.

Definition 10. *If there is given a connection ∇ on manifold N. A tensor field F of type (1,1) is said to be **parallel** if $\nabla F \equiv 0$.*

From the definitions (3), (9) and (10), we easily deduce

Proposition 11. *For any vector field $\tilde{X}$ on M*

$${}^Dg(\tilde{X}, F_h\tilde{X}) = 0 \ for \ h=1,2.$$

Proposition 12. *$\mathfrak{L}_h$ and $\mathfrak{M}_h$ are orthogonal with respect to ${}^Dg$.*

Proof. The result follows from definition (3) and taking into account that $\left\{X^0, X^h\right\}$ span $\mathfrak{L}_h$ and $\left\{X^{\delta_h^1+1}\right\}$span $\mathfrak{M}_h$. □

From proposition (8) and (11), we have

Theorem 13. *The diagonal lift metric ^Dg is adapted to the f-structure F_1 and F_2.*

Next, we have

Proposition 14. *The covariant differential ∇^D of F_1 and F_2 are given by*
1/ $^D\nabla\ F_1\left(X^0, Y^0\right) = \frac{1}{2}(R(X,Y)u - R(u,Y)X)^0$
2/ $^D\nabla F_1\left(X^1, Y^1\right) = -\frac{1}{2}(R\left(u, X\right)Y)^0$
3/ $^D\nabla F_1\left(X^2, Y^2\right) =^D \nabla F_1\left(X^1, Y^2\right) = 0$
4/ $^D\nabla F_1\left(X^0, Y^1\right) = -\frac{1}{2}(R(u,Y)X)^0 + \frac{1}{2}\sum_{k=1,2}(R(X,Y)u)^k$
5/ $^D\nabla F_1\left(X^0, Y^2\right) = -\frac{1}{2}(R\left(u, Y\right)X)^1$
and
1/ $^D\nabla\ F_2\left(X^0, Y^0\right) = (R\left(u, Y\right)X)^0$
2/ $^D\nabla F_2\left(X^1, Y^1\right) = 0$
3/ $^D\nabla F_2\left(X^2, Y^2\right) =^D \nabla F_2\left(X^1, Y^2\right) = -\frac{1}{2}(R\left(u, X\right)Y)^0$
4/ $^D\nabla F_2\left(X^0, Y^1\right) = -\frac{1}{2}(R\left(u, Y\right)X)^2$
5/ $^D\nabla F_2\left(X^0, Y^2\right) = \frac{1}{2}(R\left(X, Y\right)u)^1 - \frac{1}{2}(R\left(u, X\right)Y)^2$
for any vector field X and Y on M.

Thus, we deduce

Proposition 15. *The f-structures F_1 and F_2 is parallels if and only if $R \equiv 0$.*

The integrability of the f-structure F is given by

Definition 16. *The f-structure F is integrable if and only if $N_F = 0$.*

Taking account that $^D\nabla$ has free torsion and formulas (5-1), (5-2), we have

Proposition 17. $N_{F_1}\ (X^0, Y^0) = -(R(X,Y)u)^1$
$N_{F_1}(X^1, Y^1) = \sum_{k=1,2}(R(X,Y)u)^k$
$N_{F_1}(X^1, Y^1) = N_F(X^i, Y^j) = 0$
and
$N_{F2}\ (X^0, Y^0) = -(R(X,Y)u)^2$
$N_{F_2}(X^1, Y^1) = \sum_{k=1,2}(R(X,Y)u)^k$
$N_{F_2}(X^2, Y^2) = N_F(X^i, Y^j) = 0$
for all $X, Y \in C^\infty(TM)$ and $i \neq j \in \{0,1,2\}$.

Proposition 18. *F_1 and F_2 are partially integrable if and only if $R \equiv 0$ (i.e, ∇ is locally flat).*

Proof. It is immediately that is $N(\mathfrak{L}_h X, \mathfrak{L}_h Y) = 0$. □

Proposition 19. *The f-structure F_1 and F_2 are integrable if and only if $R \equiv 0$ (i.e, ∇ is locally flat).*

Then we have

Theorem 20. *The following assertions are equivalentes*
1/ ∇ is locally flat.
2/ $\mathfrak{L}_h$ is completely integrable.
3/ F_1 and F_2 are partially integrable.
4/ F_1 and F_2 are integrable.

The fondamental form on T^2M can be defined by $\Omega(X^i, Y^j) = {}^D g(F_h X^i, Y^j)$ for $i, j = 0, 1, 2$ and $h = 1, 2$.

Proposition 21. *We obtain in T^2U*
1) $\Omega_1(X^0, X^0) = \Omega_1(X^0, X^2) = \Omega_1(X^1, X^2) = \Omega_1(X^1, X^2) = \Omega_1(X^2, X^2) = 0$
$\Omega_1(X^0, X^1) = g(X, X)$
2) $\Omega_2(X^0, X^0) = \Omega_2(X^0, X^1) = \Omega_2(X^I, Y^I) = \Omega_2(X^I, X^2) = \Omega_2(X^2, X^2) = 0$
$\Omega_2(X^0, X^2) = g(X, X)$

Theorem 22. *1/ $(T^2M, F_i, ^D g)$ is always $fAK-$manifold.*
2/ $(T^2M, F_i, ^D g)$ is $fK-$manifold if and only if (M, g) is locally Euclidean.

Proof. 1) Follows directly from (11).
2) Follows taking account (1) theorem 20 and proposition 6. □

8 Geometric Consequences

Theorem 23. *Let (M, g) be a Riemannian manifold and the second order tangent bundle T^2M be equipped with ${}^D g$ the diagonal lift of g. Then $(T^2M, ^D g)$ is flat if and only if (M, g) is flat.*

Proof. It is a direct consequence of curvature tensor formula, if $R \equiv 0$ then ${}^D R \equiv 0$. We now assume that ${}^D R \equiv 0$ and calculate the Riemann curvature tensor for three 0-lift vector fields at $\zeta = (p, 0, 0)$ then

$$(R_p(X, Y)Z)^0 = R_\zeta(X^0, Y^0)Z^0 = 0$$

Therefore (M, g) is flat. □

For the sectional curvatures of $(T^2M, ^D g)$ we have

Proposition 24. *Let $\zeta \in T^2M$ and $X, Y \in TM$ be two orthonormal tangent vectors at p. Let ${}^DK(X^i, Y^j)$ denote the sectional curvature of the plane spanned by X^λ and Y^λ $(\lambda = 0, 1, 2)$, Then we have*

$$\begin{aligned} {}^DK(X^0, Y^0) &= K(X, Y) - \frac{3}{2}\,||R(X, Y)u||^2 \\ {}^DK(X^0, Y^i) &= \frac{1}{4}\,||R(u, Y)X||^2 \\ {}^DK(X^i, Y^j) &= 0 \end{aligned}$$

for i,j=1,2 , and ||cdot|| is the norm induced by g.

Using the well-known form of the curvature tensor for Riemannian manifolds of constant curvature we immediately get the following.

Corollary 25. *Let (M, g) be a Riemannian manifold of constant sectional curvature k. Then*

$$\begin{aligned} {}^DK(X^0, Y^0) &= k - \frac{3}{2}k^2(g(u, X) + g(u, Y)) \\ {}^DK(X^0, Y^i) &= \frac{1}{4}k^2 g(u, X)^2 \\ {}^DK(X^i, Y^j) &= 0 \end{aligned}$$

for any orthonormal vector fields $X^i, Y^j \in C^\infty(TM)$ and i,j=1,2

We can now compare the scalar curvatures on (M, g) and $(T^2M, {}^D g)$, then we have

Proposition 26. *Let S be the scalar curvature of g and ${}^DS$ be the scalar curvature of ${}^Dg$. Then the following equation holds*

$${}^DS = S - \tfrac{1}{2}\sum_{i,j=1}^{n} ||R(X_i, X_j)u||^2\,,$$

where $\{X_1, ..., X_n\}$ is a local orthonormal frame for T^2M.

Corollary 27. *If (M, g) has a constant sectional curvature $\mathcal{K}$, then*

$${}^DS = (n-1)\mathcal{K}\ (n - \mathcal{K}\,||u||^2).$$

Proof. we have

$$\begin{aligned}
{}^{D}S &= S - \tfrac{1}{2}\sum_{i,j=1}^{n} \|R(X_i, X_j)u\|^2 \\
&= S - \tfrac{1}{2}\mathcal{K}^2\sum_{i,j=1}^{n} \|g(X_j, u)X_i - g(X_i, u)X_j\|^2 \\
&= S - \tfrac{1}{2}\mathcal{K}^2\sum_{i,j=1}^{n} (g(X_j, u)^2 + g(X_i, u)^2 - 2g(X_i, u)g(X_j, u)g(X_i, X_j)) \\
&= n(n-1)\mathcal{K} - \mathcal{K}^2(n-1)\|u\|^2 = (n-1)\mathcal{K}(n - \mathcal{K}\|u\|^2).
\end{aligned}$$

□

Theorem 28. *Let (M, g) be a Riemannian manifold and the second order tangent bundle T^2M be equipped with ${}^{D}g$ the diagonal lift of g. Then $(T^2M, {}^{D}g)$ has constant scalar curvatures if and only if (M, g) is flat.*

Proof. The statement follows directly from proposition (26). □

Corollary 29. *Let (M, g) be a Riemannian manifold and the second order tangent bundle T^2M be equipped with ${}^{D}g$ the diagonal lift of g. Then $(T^2M, {}^{D}g)$ has constant scalar curvatures if and only if the scalar curvature is zero.*

Proof. The result is an immediate consequence of theorem (23) and theorem (28). □

Corollary 30. *Let (M, g) be a Riemannian manifold and the second order tangent bundle T^2M be equipped with ${}^{D}g$ the diagonal lift of g. Then $(T^2M, {}^{D}g)$ is Einstein if and only if $(T^2M, {}^{D}g)$ is flat.*

Proof. The statement is a direct consequence of theorem (28). □

Corollary 31. *Let (M, g) be a Riemannian manifold and the second order tangent bundle T^2M be equipped with ${}^{D}g$ the diagonal lift of g. Then $(T^2M, {}^{D}g)$ is locally symmetric if and only if $(T^2M, {}^{D}g)$ is flat.*

Proof. $(T^2M, {}^{D}g)$ is locally symmetric if

$$({}^{D}\nabla_{\tilde{X}}\ {}^{D}R)(\tilde{Y}, \tilde{X}) = 0, \text{ for all } \tilde{Y}, \tilde{X} \in C^\infty(TM).$$

□

Theorem 32. *Let (M, g) be a Riemannian manifold and the second order tangent bundle T^2M be equipped with ${}^{D}g$ the diagonal lift of g. Then the f-structure F_1 and F_2 are parallel and integrable if and only if (M, g) is flat.*

Proof. Its direct consequence from proposition 14 and 17. □

The diagonal lift ^{D}g of g give a **similar geometry** of Riemannian manifold and its second order tangent bundle equipped with ^{D}g.

References

[1] A.Morimoto, Prolongation of geometric structures. Lect. Notes Math. Inst. Nagoya 1969.

[2] A. Morimoto, Liftings of tensors fields and connections to tangent bundles of Higher order, Nogoya Math. Jour.,40 (1970), 99-120

[3] C.T.J.Dodson and M.S.Radivoiovici, Tangent and Frames Bundles of order two Analele stiintifice ale Univesitatii "AL. I Gusa", 28 (1982), 63-71.

[4] C.T.J.Dodson and G.N.Galanis, Second Order Tangent Bundle of infinite dimensional manifold. J. Geom. Phys. 52 (2004), 127-136.

[5] I. Vaisman, Connexion remarquables sur les varietes hor-ehresmanniennes. C.R.Acad. Sci. Paris, 273 (1971), 1253-1256.

[6] K. Yano and S. Ishihara,Tangent and cotangent Bundles, Marcel Dekker Inc., New York 1973.

[7] K. Yano and S. Ishihara, Horizontal lifts of tensors fields and connection to the tangent bendles. j. Math. Mech. 16 (1967), 1015-1030.

[8] K. Nomizu and S. Kobayashi, Foundation of Differential Geometry, vol I. Intersciense, New York-London 1963.

[9] L.A. Cordero, C.T.J. Dodson and M. de Leon. Differential Geometry of frame bundles. (Kluwer, Dordrecht, 1989)

[10] M. de Leon and E. Vazquez, On the geometry of the tangent bundle of order 2. An. Univ. Bucarestii34 (1985)40-48.

[11] M. Djaa and J. Gancarzewicz, The geometry of tangent bundles of order r. boletin Academia Galega de Ciencias, spain.Vol. IV, 147-165 (1985).

[12] M. Djaa, Prolongation des structures geometriques au fibre tangent d'ordre superieur et classification spectrale des operateurs de multiplications, PHD thesis oran university (1998) Algeria

[13] S. Gudmundsson, E. Kappos, On the Geometry of tangent bundles Expo. math. 20(2002), 1-41.

[14] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds. Tohoku Math. J., 10 (1958), 338-354.

[15] Vohra, V.C. and Singh, K.D. , Some Structure on a f-structure manifold.

Received: April 28, 2008