Int. Journal of Math. Analysis, Vol. 3, 2009, no. 12, 563 - 570

Fixed Point Theorems for Occasionally Weakly Compatible Maps in Probabilistic Semi-Metric Space

Harish Chandra and Arvind Bhatt

Department of Mathematics and DST-CIMS
Banaras Hindu University, Varanasi, India 221005
harish_pathak2004@yahoo.com
arvindbhu_6june@rediffmail.com

Abstract. In this paper we prove common fixed point theorems for a pair of occasionally weakly compatible maps in probabilistic semi - metric space. These results are extensions of the results which we have obtained on symmetric spaces [3].

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fixed points, Compatible maps, Weakly comatible maps, Occasionally weakly compatible maps, PM - space

1. Introduction

Fixed point theory in probabilistic metric spaces can be considered as a part of Probabilistic Analysis, which is a one of the emerging areas of interdisciplinary mathematical research with many diverse applications. The theory of probabilistic metric spaces was introduced in **1942** by Menger [12] in connection with some measurements in Physics. Over the years, the theory has found several important applications in the investigation of physical quantities in quantum particle physics and string theory as studied by El Naschie [14] and [15]. The area of probabilistic metric spaces is also of fundamental importance in probabilistic functional analysis. The first effort in this direction was made by Sehgal [22], who in his doctoral dissertation, initiated the study of contraction mapping theorems in probabilistic metric spaces. Since than, Sehgal and Bharucha - Reid [1] obtained a generalization of Banach Contraction Principle on a complete Menger space which is an important step in the development of fixed point theorems in Mengar space.

Sessa [19] initiated the tradition of improving commutativity in fixed- point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [7] soon enlarged this concept to compatible maps. The notion of compatible mappings in a Mengar space has been introduced by Mishra [13]. After this, Jungck and Rhoades [8] gave the concept of weakly compatible maps. Recently, Jungck and Rhoades [9] introduced the concept of Occasionally weakly compatible maps. This concept is the most general among all the commutativity concepts.

In this paper we prove common fixed point theorems for two pairs and four pairs of occasionally weakly compatible of pair of self maps on probabilistic semi-metric spaces. These results are generalisations of the results obtained in the context of symmetric spaces [3].

We begin with the following basic definitions of concepts relating to probabilistic metric spaces for ready reference and also for the sake of completeness.

2. Basic Definitions

Definition 2.1. [20]. A real valued function f on the set of real numbers is called a distribution function if it is non-decreasing, left continuous with $\inf_{t\in R} f(t) = 0$ and $\sup_{t\in R} f(t) = 1$.

The Heaviside function H is a distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.2. [20]. Let X be a non - empty set and Δ denote the set of all distribution functions defined on X. An ordered pair (X, F) is called a probabilistic metric space if F is a mapping from $X \times X$ into Δ satisfying the following conditions:

(i) $F_{xy}(t) = H(t)$ if and only if x = y.
(ii) $F_{xy}(t) = F_{yx}(t)$.

(iii) $F_{xy}(0) = 0$.
(iv) If $F_{xy}(t_1) = 1$ and $F_{yz}(t_2) = 1$, then $F_{xz}(t_1 + t_2) = 1$ for all x, y, z in X and $t_1, t_2 \geq 0$.

If only (1), (2) and (3) hold, the ordered pair (X, F) is said to be a probabilistic semi - metric space. We note that every symmetric(semi-metric) space (X, d) can be realized as a probabilistic semi - metric space by taking $F : X \times X \longrightarrow \Delta$ defined by $F_{xy}(t) = H(t - d(x, y)$ for all x, y in X. So probabilistic semi - metric spaces provide a wider framework than that of the symmetric spaces and are better suited in many situations.

Definition 2.3. [20]. A t-norm is a function $T : [0, 1]\times[0, 1]\longrightarrow [0, 1]$ satisfying the following conditions:

(i) $T(a, 1) = a, T(0, 0) = 0$.
(ii) $T(a, b) = T(b, a)$.
(iii) $T(c, d) \geq T(a, b)$ for $c \geq a$, $d \geq b$.
(iv) $T(T(a, b), c) = T(a, T(b, c))$ for all a, b, c in $[0, 1]$.

Definition 2.4. [20]. A Menger probabilistic metric space (X, F, T) is an ordered triad, where T is a t - norm, and (X, F) is probabilistic metric space satisfying the following condition:

$$F_{xz}(t_1 + t_2) \geq T(F_{xy}(t_1), F_{yz}(t_2)) \text{ for all } x, y, z \text{ in } X \text{ and } t_1, t_2 \geq 0.$$

Definition 2.5. Let(X, F) be a probabilistic semi - metric space. The (ϵ, λ) - topology in (X, F) is generated by the family of neighborhoods

$$U = \{(U_v(\epsilon, \lambda)) : (v, \epsilon, \lambda \in X \times R^+ \times (0,\ 1))\},$$
$$\text{where } U_v(\epsilon, \lambda) = \{u : u \in X, F_{u,v}(\epsilon) > 1 - \lambda\}.$$

If a t - norm T is such that $\sup_{x<1} T(x, x) = 1$ then (X, F, T) is, with the (ϵ, λ) topology, a metrizable topological space.

Definition 2.6. [20]. Let (X, F) be a probabilistic semi - metric space. A sequence $\{x_n\}$ in $(X,\ F)$ is said to converge a point $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that

$$F_{x_n,x}(\epsilon) > 1 - \lambda \text{ for all } n \geq N(\epsilon, \lambda).$$

Definition 2.7. [20]. Let (X, F) be a probabilistic semi - metric space. A sequence $\{x_n\}$ in (X, F) is said to be a cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integar $N(\epsilon, \lambda)$ such that

$$F_{x_n,x_m}(\epsilon) > 1 - \lambda \text{ for all } n, m \geq N(\epsilon, \lambda).$$

Definition 2.8. [20]. A probabilistic semi - metric space (X, F) with continuous t- norm is said to be complete if every cauchy sequence in X converge to a point in X.

Definition 2.9. [25]. Two self mappings f and g of a probabilistic semi - metric space (X, F) are said to be weakly commuting if $F_{fgx,gfx}(t) \geq F_{fx,gx}(t)$ for each x in X and $t > 0$.

Definition 2.10. [13]. Two self mappings f and g of a probabilistic semi - metric space $(X, F,)$ are said to be compatible if $F_{fgx_n,gfx_n}(t) \longrightarrow 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$.

Definition 2.11. [9]. Let (X, F) be a probabilistic semi - metric space. f and g be self maps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case $w = fx = gx$ is called a point of coincidence of f and g.

The following concept [2] is a proper generalization of nontrivial weakly compatible maps which do have a coincidence point.

Definition 2.12. [8]. Two self-mappings f and g of a probabilistic semi - metric space (X, F) are said to be weakly compatible if they commute at their coincidence points, i.e., if $fx = gx$ for some x∈ X, then $fgx = gfx$.

It is easy to see that two compatible maps are weakly compatible but converse is not true

Definition 2.13. [9]. Two self mappings f and g of a probabilistic semi - metric space (X, F) are said to be occasionally weakly compatible(owc) iff there is a point x in X which is a coincidence point of $f$and g at which f and g commute.

Lemma 2.1. [9]. Let (X, F) be a probabilistic semi - metric space. f and g are self maps of X and let f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

Proof. Since f and g are owc, there exists a point $x \in X$ such that $fx = gx = w$ and $fgx = gfx$. Thus, $ffx = fgx = gfx$, which says that ffx is also a point of coincidence of f and g. Since the point of coincidence $w = fx$ is unique by hypothesis, $gfx = ffx = fx$, and $w = fx$ is a common fixed point of f and g.

Moreover, if z is any common fixed point of f and g, then $z = fz = gz = w$ by the uniqueness of the point of coincidence. □

3. Main Results

Let a function ϕ be defined by $\phi : [0,1] \longrightarrow [0,1]$ satisfying the condition $\phi(q) > q$, for all $q < 1$.

Theorem 3.1. Let (X, F) be a probabilistic semi - metric space. If f and g are Occasionally weakly compatible maps on X and

$$(1) \qquad F_{fx,fy}(t) \geq \phi\left[Min\left\{F_{gx,gy}(t), F_{gx,fy}(t), F_{gy,fx}(t), F_{gy,fy}(t)\right\}\right],$$

where for all $x, y \in X$ and $t > 0$. Then f and g have a unique common fixed point.

Proof. Since f and g are occasionally weakly compatible, there exists a point $u \in X$ such that $fu = gu, fgu = gfu$.We claim that fu is the unique common fixed point of f and g. We first assert that fu is a fixed point of f.
For, if $ffu \neq fu$.Then from equation (1), we get

$$\begin{aligned} F_{fu,ffu}(t) &\geq \phi[Min\{F_{gu,gfu}(t), F_{gu,ffu}(t), F_{gfu,fu}(t), F_{gfu,ffu}(t)\}] \\ &= \phi[Min\{F_{fu,ffu}(t), F_{fu,ffu}(t), F_{fu,ffu}(t), F_{gfu,gfu}(t)\}] \\ &= \phi[F_{fu,ffu}(t)] \\ &> F_{fu,ffu}(t). \end{aligned}$$

This is a contradiction. So $ffu = fu$ and $ffu = fgu = gfu = fu$. Hence fu is a common fixed point of f and g.

Now we prove uniqueness. Suppose that $u, v \in X$, such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then from equation (1),

$$F_{fu,fv}(t) \geq \phi[Min\{F_{gu,gv}(t), F_{gu,fv}(t), F_{gv,fu}(t), F_{gv,fv}(t)\}].$$

But as $fu = u$ and $fv = v$, we get

$$\begin{aligned} F_{u,v}(t) \geq& \phi[Min\{F_{u,v}(t), F_{u,v}(t), F_{v,u}(t), F_{v,v}(t)\}] \\ &= \phi(F_{u,v}(t)) > F_{u,v}(t). \end{aligned}$$

This is a contradiction. So $u = v$. Therefore the common fixed point of f and g is unique. □

Now we give an example which satisfies the conditions of above theorem.

Example 3.1. Let $X = [0,1]$ and $\phi : X \longrightarrow X$ be defined as

$$\phi(q) = \tfrac{1+q}{2}, \text{ then } \phi(q) > q \ , \ 0 \leq q < 1,$$

and $f(x) = \frac{1+2x}{3}$, $g(x) = \frac{1+4x}{5}$ f and g satisfy all the conditions with respect to the distribution defined

$$F_{x,y}(t) = \begin{cases} e^{\frac{-|x-y|}{t}}, & \text{if } t > 0 \\ 0, & \text{if } t = 0. \end{cases}$$

In this example f and g are Occasionally weakly compatible maps and f and g satisfy the equation (1). f and g have a unique common fixed point 1.

Theorem 3.2. Let (X, F) be a probabilistic semi - metric space. If f and g are Occasionally weakly compatible maps in X and

$$(2) \qquad \begin{aligned} F_{fx,fy}(t) \geq F_{gx,gy}\left(\frac{t}{a}\right) &+ Min\left\{F_{fx,gx}\left(\frac{t}{b}\right) F_{fy,gy}\left(\frac{t}{b}\right)\right\} \\ &+ Min\left\{F_{gx,gy}\left(\frac{t}{c}\right), F_{gx,fx}\left(\frac{t}{c}\right), F_{gy,fy}\left(\frac{t}{c}\right)\right\}, \end{aligned}$$

for all $x, y \in X$ with $f(x) \neq fy$ and $t > 0$ where $0 < a < 1$, $0 < b < 1$ and $0 < c < 1$. Then f and g have a unique common fixed point.

Proof. We claim that f and g have a unique point of coincidence $w = fx = gx$. If possible, suppose there is another point of coincidence $fy = gy = w'$ and $w' \neq w$. Then $F_{fx,fy}(t) \neq 1 \; \forall \; t > 0$. So from equation (3), we get

$$(3) \qquad \begin{aligned} F_{fx,fy}(t) \geq F_{gx,gy}\left(\frac{t}{a}\right) &+ Min\left\{F_{fx,gx}\left(\frac{t}{b}\right) F_{fy,gy}\left(\frac{t}{b}\right)\right\} \\ &+ Min\left\{F_{gx,gy}\left(\frac{t}{c}\right), F_{gx,fx}\left(\frac{t}{c}\right), F_{gy,fy}\left(\frac{t}{c}\right)\right\}, \end{aligned}$$

This is a contradiction. Which implies that $F_{fx,fy}(t) = 1$. Hence we get $fx = fy$. Therefore fx is unique. Now from Lemma (2.1) it follows that f and g have a unique common fixed point.

□

Theorem 3.3. Let (X, F) be a probabilistic semi - metric space. Suppose that f, g, S, T are self maps of X and the pairs $\{f, S\}$ and $\{g, T\}$ are Occasionally weakly compatible. If

$$(4) \qquad F_{fx,gy}(t) > Min\left\{F_{Sx,Ty}(t), F_{Sx,fx}(t), F_{Ty,gy}(t), F_{Sx,gy}(t), F_{Ty,fx}(t)\right\},$$

for all $x, y \in X$ with $fx \neq gy$ and $t > 0$. Then f, g, S and T have a unique common fixed point.

Proof. By hypothesis there exists points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. Suppose that $F_{fx,gy}(t) \neq 1$ for all $t > 0$.

Then from equation (4)

$$F_{fx,gy}(t) > Min\,\{F_{fx,gy}(t), F_{fx,fx}(t), F_{gy,gy}(t), F_{fx,gy}(t), F_{gy,fx}(t)\}$$
$$> F_{fx,gy}(t).$$

This is a contradiction. Hence $F_{fx,\,gy}(t) = 1$ for all $t > 0$. This implies that fx = gy. So $fx = Sx = gy = Ty$. Moreover, if there is another point z such that $fz = Sz$, then, using (4) it follows that $fz = Sz = gy = Ty$, or $fx = fz$ and $w = fx = Sx$ is the unique point of coincidence of f and S. Then by Lemma (2.1) it follows that w is the unique common fixed point of f and S. By symmetry there is a unique point $z \in X$ such that $z = gz = Tz$. Suppose that $w \neq z$. Using (4) $F_{w,z}(t) = F_{fw,gz}(t) > F_{fw,gz}(t)$. This implies that $F_{w,z}(t) > F_{w,z}(t)$. This is a contradiction. Therefore $w = z$ and w is a unique point of concidence of f, g, S, T. By lemma (2.1) w is the unique common fixed point of f, g, S, T. □

References

[1] A.T.Bharucha and Reid, Fixed point theorems in Probabilistic analysis, Bull.Amer. Math. Soc, **8**2(1976), 641-657.

[2] M.A.Al-Thagafi and Naseer Shahzad, Generalized I-nonexpansive selfmaps and invarient approximations, **2**4(5)(2008), 867-876.

[3] H.Chandra and A.Bhatt, Common fixed point theorems for weakly compatible maps in symmetric spaces, Communicated.

[4] O.Hadzic and E.Pap, Fixed point theorems in PM Spaces, Kluwer Academic Publishers, 2001.

[5] K.Jha and R.P.Pant, Common fixed point theorems by altering distances, Tamkang Math. J, **3**5(2004), 109-116.

[6] G.Jungck, Commuting mappings and fixed points, Amer. Math. Monthly , **7**3(1976), 261-263.

[7] G.Jungck, Compatible mappings and common fixed points, Int. J. Math. & Math. Sci, **9**(1986), 771-779.

[8] G.Jungck, Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East J. Math. Sci, **4**(1996), 19-215.

[9] G.Jungck and B.E.Rhoades, Fixed point Theorems for occasionally weakly Compatible Mappings, Fixed point theory ,**7**(2006), 286-296.

[10] R.Kannan, Some results on fixed points, Bull. Calcutta Math. Soc **6**0(1968), 71-76.

[11] S.Kumar and R.Chugh, Common fixed point theorem using minimal commutativity and reciprocal continuity conditions in metric space. Sci. Math. Japonica , **5**6(2002), 197-203.

[12] K.Menger, Statistical metrices, Nat. Acad.Sci.USA **2**8(1942), 535-537.

[13] S.N.Mishra, Common fixed points of compatible mappings in PM - spaces, Math. Japon ,(1991), 283-289.

[14] MS.EL Naschie, On the uncertainty of Cantorian geometry and two - slit experiment, Chaos Solitons and Fractals, (1998), 517-29.

[15] MS.EL Naschie, On the verifications of heterotic string theory and ϵ^{∞}, Chaos, Solitons and Fractals ,**2**32(2000), 397-407.

[16] S. Padaliya and R.P.Pant, Common fixed point theorems for R-weakly commuting mappings of type (A), Soochow J. Math, **3**1(2005), 155-163.

[17] R.P.Pant and V.Pant, Common fixed points under strict contractive conditions, J.Math.Anal.Appl.Soc, (2000), 327-332.

[18] R.P.Pant, Common fixed points of four maps, Bull. Calcutta Math.Soc, **9**0(1998), 281-286.

[19] S.Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math., **3**2(1982), 149-153.

[20] B.Schweizer and A.Skalar, Statistical metric spaces, North Holland Amsterdam, (1983).

[21] Schweizer and A.Skalar, Probabilistic metric spaces, Pacific J. of Math **1**0(1960), 313-324.

[22] V.M.Sehgal, Some fixed point theorems in functional analysis and probability. Ph.D. Dissertation, Wayne State Univ. Michigan, (1966).

[23] V.M.Sehgal, A.T.Bharucha and Reid, Fixed points of contraction mappings on Probabilistic metric spaces, Math. Systems Theory, **0**6(1972), 97-102.

[24] S.P.Singh and B.E.Meade, On common fixed point theorems, Bull. Austral. Math. Soc, **1**6(1977), 49-53.

[25] S.L.Singh, B.D.Pant and R.Talwar, Fixed points of weakly commuting mappings on Mengar spaces, Jnanabha **2**3(1993), 115-122.

Received: August, 2008