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Regularization Inertial Proximal Point Algorithm for Convex Feasibility Problems in Banach Spaces

Nguyen Buong

Vietnamse Academy of Science and Technology
Institute of Information Technology
18, Hoang Quoc Viet, q. Cau Giay, Ha Noi, Vietnam
nbuong@ioit.ac.vn

Truong Minh Tuyen

Department of Mathematics
Thai Nguyen University, Vietnam
tm.tuyentm@gmail.com

Abstract

The purpose of this paper is to give a regularization variant of the inertial proximal point algorithm not only in an infinite-dimensional Banach space E, which is an uniformly smooth and uniformly convex, but also in its finite-dimensional approximations for finding a point in the nonempty intersection $\cap_{i=1}^{N} C_i$, where $N \geq 1$ is an integer and each C_i is assumed to be the fixed point set of a nonexpansive mapping $T_i : E \rightarrow E$, without the weak sequential continuous property for a normalized duality maping of E.

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1. Introduction

Let E be an uniformly smooth and uniformly convex Banach space (having the approximation property [20]) with its dual space E^* that is strictly convex. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. We write $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$.

Definition 1.1. A mapping j from E onto E^* satisfying the condition

$$j(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \quad \text{and} \quad \|f\| = \|x\|\}$$

is called the normalized duality mapping of E.

The modulus of smoothness of E is the function

$$\rho_E : [0, \infty) \to [0, \infty)$$

defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \le 1, \|y\| \le \tau\right\}.$$

Definition 1.2. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau\to 0}(\rho_E(\tau)/\tau) = 0.$$

Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces, $W^p_m, 1 < p < \infty$, are uniformly smooth.

Definition 1.3. A Banach space E is said to be

(i) uniformly convex, if for any $\varepsilon, 0 < \varepsilon \le 2$, the inequalities $\|x\| \le 1, \|y\| \le 1$, and $\|x-y\| \ge \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) \ge 0$ such that $\|(x+y)/2\| \le 1-\delta$.

(ii) and strictly convex, if for $x, y \in S_E$ with $x \ne y$, then

$$\|(1 - \lambda)x + \lambda y\| < 1 \quad \forall \lambda \in (0, 1)$$

where we use S_E to denote the unit sphere $S_E = \{x \in E : \|x\| = 1\}$.

Note that the modulus of convexity of E is the function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| : \|x\| = 1, \|y\| = 1, \|x - y\| = \varepsilon\}.$$

E is uniformly convex if and only if

$$\delta_E(\varepsilon) > 0 \quad \forall \varepsilon > 0,$$

and E is strictly convex if and only if the normalized duality mapping j of E is strictly monotone, i.e.,

$$\langle x - y, j(x) - j(y)\rangle \ge 0$$

and the symbol "=" is achieved if and only if $x = y$.

It is wellknown [1] that when E is the space of type L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces, $W^p_m, 1 < p < \infty$, for $0 < \varepsilon \le 2$ we have

$$\begin{aligned}\delta_E(\varepsilon) &\ge 16^{-1}(p - 1)\varepsilon^2, \text{for} \quad 1 < p \le 2,\\ \delta_E(\varepsilon) &\ge p^{-1}(p - 1)(\varepsilon/2)^p, \text{for} \quad p \ge 2.\end{aligned}$$

It is also well known [23] that if E^* is strictly convex then j is single-valued. The important properties of j are $j(-x) = -j(x)$ and uniformly continuous on bounded domain.

Definition 1.4. A mapping A from E to E is said to be Lipschitz continuous with the Lipschitz constant $L > 0$, if

$$\|A(x) - A(y)\| \leq L\|x - y\|.$$

When $L = 1$, A is called a nonexpansive mapping.

Definition 1.5. A mapping A from E to E is said to be accretive, if

$$\langle A(x) - A(y), j(x - y)\rangle \geq 0 \quad \forall x, y \in D(A),$$

where $D(A)$ denotes the domain of A, m-accretive, if $R(A + \lambda I) = E$ for $\lambda > 0$ where $R(A)$ and I denote the range of A and the identity mapping of E, respectively. When E is a Hilbert space H, A is said to be monotone and maximal monotone, respectively.

Definition 1.6. A mapping T in E is called λ-strictly pseudocontractive in the terminology of Browder and Petryshyn [5], if for all $x, y \in D(T)$, there exists $\lambda > 0$ such that

$$\langle T(x) - T(y), j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|x - y - (T(x) - T(y))\|^2. \tag{1.1}$$

Clearly, (1.1) can be written in the form

$$\langle (I - T)(x) - (I - T)(y), j(x - y)\rangle \geq \lambda\|(I - T)(x) - (I - T)(y)\|^2. \tag{1.2}$$

In Hilbert space H, (1.1) (and hence (1.2)) is equivalent to the inequality

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|(I - T)(x) - (I - T)(y)\|^2, \quad k = 1 - \lambda.$$

Clearly, when $k = 0$, i.e. $\lambda = 1$, T is nonexpansive.

Definition 1.7. A mapping A in E is called Fréchet differentiable at a point $x \in D(A)$, if

$$A(x + h) - A(x) = B(x)h + o(\|h\|) \quad \forall x + h \in D(A),$$

where $B(x)$ is a linear bounded mapping from E to E denoted by $A'(x)$.

Definition 1.8. Let K be a nonempty closed convex subset of E. Then for each $x \in K$, the set $I_K(x)$ defined by

$$I_K(x) = \{y \in E : y + \lambda(z - x), z \in K, \lambda \geq 0\}$$

is called a *inward set*. A mapping $S : E \to K$ is said to satisfy the weakly inward condition, if $Sx \in \overline{I_K(x)}$ (the closure of $I_K(x)$) for each $x \in K$.

Definition 1.9. Let $P : E \to K$ be a mapping. P is said to be
(i) *sunny,* if for each $x \in K$ and $t \in [0, 1]$, we have

$$P(tx + (1 - t)Px) = Px;$$

(ii) *a retraction of E onto K,* if $Px = x$ for all $x \in K$;
(iii) *a sunny nonexpansive retraction,* if P is sunny, nonexpansive retraction of E onto K;
(iv) K is said to be a *sunny nonexpansive retract* of E, if there exists a sunny nonexpansive retraction of E.

We are concerned with the following convex feasibility problem:

$$\text{finding an } x_* \in C := \cap_{i=1}^N C_i, \tag{1.3}$$

where $N \geq 1$ is an integer and each C_i is assumed to be the fixed point set $F(T_i)$ of a nonexpansive mapping $T_i : E \to E, i = 1, 2, ..., N$.

For solving (1.3), the iteration method

$$x_{n+1} = P(\alpha_{n+1} f(x_n) + (1 - \alpha_{n+1})T_{n+1}x_n), \tag{1.4}$$

where $x_0 \in E$ is any given initial data, $f(x) : K \to K$ is a contractive mapping, $T_n = T_{n(\mathrm{mod})N}$, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and P is a sunny nonexpansive retraction of E onto K, is given in [10].

Theorem 1.1 [10] *Let E be a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let K be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E and P is a sunny nonexpansive retaction from E onto K. Let $f : K \to K$ be a given Banach contraction mapping with a contractive constant $0 < \beta < 1$, and let $T_i : E \to E, i = 1, 2, ..., N$, be nonexpansive mappings satisfying the conditions:*

(i) $\cap_{i=1}^N (F(T_i) \cap K) \neq \emptyset$;

(ii) $\cap_{i=1}^N F(T_i) = F(T_1 T_N ... T_2) = ... = F(T_N T_{N-1} ... T_1) = F(S)$, where $S = T_N T_{N-1} ... T_1$;

(iii) The mapping $S : K \to E$ satisfies the weakly inward condition;

For any $x_0 \in K$, let $\{x_n\}$ be the sequence defined by (1.4). If the following conditions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{i=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{i=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \alpha_n/\alpha_{n+1} = 1$,

then the sequence $\{x_n\}$ converges strongly to a point $p \in \cap_{i=1}^N (F(T_i) \cap K)$ which is the unique solution of the following variational inequality:

$$\langle p - f(p), j(p - u)\rangle \leq 0, \quad \forall u \in \cap_{i=1}^N (F(T_i) \cap K).$$

If E is a Hilbert space and $f(x) = u$ (a given point in K), then (1.4) is equivalent to

$$x_{n+1} = \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n,$$

which was introduced and studied by Bauschke [4] in 1996.

If $N = 1$, E either is a uniformly smooth Banach space or a reflexive Banach space with a weakly sequentially continuous duality mapping j and K is a nonempty closed convex subset of E, $T : K \to K$ is a nonexpansive mapping, and $f : K \to K$ is a contractive mapping, then (1.4) is equivalent to the following sequence:

$$x_{n+1} = \alpha_{n+1}f(x_n) + (1 - \alpha_{n+1})Tx_n \tag{1.5}$$

which is studied by Xu [22]. Algorithm (1.5) is an extention of the first introduced and studied by Moudafi [12] in the setting of Hilbert space.

It is easy to see that if T is a λ-strictly pseudocontractive mapping (and hence a nonexpansive mapping), then the mapping $A := I - T$ is accretive, and Lipschitz continuous with Lipschitz constant $L = 1/\lambda$. Therefore, the problem of finding a fixed point of a λ-strictly pseudocontractive mapping or a nonexpansive mapping is equivalent to finding a zero of the following operator equation

$$A(x) = 0, \tag{1.6}$$

involving the accretive mapping A.

When A is m-accretive in a Hilbert space H, i.e. A is maximal monotone, Rockafellar [16] considered the the iteration method

$$c_nA(x_{n+1}) + x_{n+1} = x_n, \quad x_0 \in H, \tag{1.7}$$

where $c_n > c_0 > 0$, which is called the proximal point algorithm, and posed an open question whether (or not) the proximal algorithm (1.7) always converges strongly. This question was resolved in the negative by Güler [11]. To obtain the strong convergence Solodov and Svaiter [19] have combined the proximal point algorithm with simple projection step onto intersection of two halfspaces containing solution set. Further, Attouch and Alvarez [3] considered an extension of the proximal point algorithm (1.7) in the form

$$c_nA(u_{n+1}) + u_{n+1} - u_n = \gamma_n(u_n - u_{n-1}), u_0, u_1 \in H, \tag{1.8}$$

which is called an inertial proximal point algorithm, where $\{c_n\}$ and $\{\gamma_n\}$ are two sequences of positive numbers. Note that this algorithm was proposed by Alvarez in [2] in the context of convex minimization. Then, Moudafi [15] applied this algorithm for variational inequalities, Moudafi and Elisabeth [14] studied the algorithm by using enlargement of a maximal monotone operator, and Moudafi and Oliny [13] considered convergence of a spliting inertial proximal method. The main result in these papers is the weak convergence of the algorithm in Hilbert spaces.

Ryazantseva [17] extended the proximal point algorithm (1.7) for the case that A is a m-accretive mapping in a properly Banach space E and proved the weak convergence the sequence of iterations $\{x_n\}$ of (1.7) to a solution of (1.6) which is assumed to be unique. Then, to obtain the strong convergence for algorithm (1.7), Ryazantseva [18] combined the proximal algorithm with the regularization, named regularization proximal algorithm, in the form

$$c_n(A(x_{n+1}) + \alpha_n x_{n+1}) + x_{n+1} = x_n, \quad x_0 \in E. \tag{1.9}$$

Under some conditions on c_n and α_n, the strong converegence of $\{x_n\}$ of (1.9) is guaranteed only when j is weak sequential continuous and strong continuous, and the sequence $\{x_n\}$ is bounded. The last requirement will be satisfied, when the set of solutions for (1.6) is bounded. Without the bounded condition, recently the strong convergence of a regularization variant of the proximal point algorithm is established in [24] for equation (1.6) involving a maximal monotone mapping A in Hilbert space.

For finding approximative solutions of (1.3) in a general case $N > 1$, one of the authors [9] considered an operator version of the Tikhonov regularization method in the form

$$\sum_{i=1}^{N} A_i(x) + \alpha_n x = 0, A_i = I - T_i, \tag{1.10}$$

depending on the positive regularization parameter α_n that tends to zero as $n \to +\infty$. Equation (1.10) is the simple form of the following equation

$$\sum_{i=1}^{N} \alpha_n^{\mu_i} A_i(x) + \alpha_n x = 0, 0 \le \mu_i < 1. \tag{1.11}$$

If $\mu_1 = 0$ and $\mu_i < \mu_{i+1}, i = 1, 2, ..., N-1$, then algorithm (1.11) is investigated in [7] and [8] for finding a common solution for a system of potential monotone hemi-continuous mappings $A_i : E \to E^*$ and a common fixed point of a finite family of strictly pseudocontractive mappings in Hilbert spaces, respectively. In the case that $\mu_i = 0, i = 1, 2, ..., N$, and without conditions (i)-(iii) and the weak sequential continuous property of the normalized duality mapping j of E in theorem 1.1, we have the following result.

Theorem 1.2. [9] *(i) For each $\alpha_n > 0$, problem (1.10) has a unique solution x_n.*
(ii) If the sequence $\{\alpha_n\}$ is chosen such that

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} = 0,$$

for any positive integer p, then

$$\lim_{n\to+\infty} x_n = x_* \in C.$$

Moreover,

$$\|x_{\alpha_n} - x_{\alpha_{n+p}}\| \le \|y\| \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} \quad y \in C. \tag{1.12}$$

Further, the author [9] considered also the finite dimensional approximations of (1.10) by the sequence of the following problems

$$\sum_{i=1}^{N} A_i^n(z) + \alpha_n z = 0, z \in E_n, \tag{1.13}$$

where $A_i^n = P_n A_i P^n, P_n$ is a linear projection of E into its subspace E_n such that

$$E_n \subset E_{n+1}, \quad P_n x \to x, n \to +\infty, \forall x \in E.$$

Let

$$\gamma_n(y) = \|(I - P_n)y\|$$

where $y \in C$.

Theorem 1.3. [9] *(i) For each $\alpha_n > 0$, equation (1.13) has a unique solution z_n.*
(ii) If $\gamma_n(y) = o(\alpha_n)$ for each $y \in C$, $T_i, i = 1, ..., N$, are Fréchet differentiable with

$$\|T_i'(x) - T_i'(y)\| \le L_i\|x - y\|, L_i > 0, \tag{1.14}$$

and the sequence $\{\alpha_n\}$ is chosen such that

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n} = 0,$$

for any positive integer p, then

$$\lim_{n\to+\infty} z_n = z_* \in C.$$

Moreover,

$$\|z_{\alpha_n} - z_{\alpha_{n+p}}\| \le R \frac{|\alpha_n - \alpha_{n+p}|}{\alpha_n}, \tag{1.15}$$

where R is a positive constant.

In this paper, we present another method for solving problem (1.3). That is the inertial proximal point algorithm considered in combination with regularization not only in E, but also in the finite dimensional approximations of E.

2. Main results

We formulate the following fact needed in the proof of our results.

Lemma 2.1 [21] *Let* $\{a_n\}, \{b_n\}, \{c_n\}$ *be the sequences of positive numbers satisfying the conditions*

(i) $a_{n+1} \leq (1-b_n)a_n + c_n, b_n < 1,$

(ii) $\sum_{n=0}^{\infty} b_n = +\infty, \quad \lim_{n\to+\infty} c_n/b_n = 0.$

Then, $\lim_{n\to+\infty} a_n = 0.$

Now, consider a sequence $\{u_n\} \subset E$ defined by

$$\tilde{c}_n(\sum_{i=1}^{N} A_i(u_{n+1}) + \alpha_n u_{n+1}) + u_{n+1} - u_n = \gamma_n(u_n - u_{n-1}), u_0, u_1 \in E, \quad (2.1)$$

Since the mapping $\sum_{i=1}^{N} A_i$ is Lipschitz continuous and accretive on E, it is m-accretive [6]. Therefore, the element u_{n+1} in (2.1) is uniquely defined.

Theorem 2.2. *Assume that* E *is an uniformly convex and uniformly smooth Banach space with the strictly convex* E^**, and the parameters* $\tilde{c}_n, \gamma_n$ *and* α_n *are chosen such that*

(i) $0 < c_0 < \tilde{c}_n < C_0, 0 \leq \gamma_n < \gamma_0, \alpha_n \searrow 0,$

(ii) $\sum_{n=1}^{\infty} b_n = +\infty, b_n = \alpha_n\tilde{c}_n/(1+\alpha_n\tilde{c}_n),$
$\sum_{n=1}^{\infty} \gamma_n b_n^{-1}\|u_n - u_{n-1}\| < +\infty,$

(iii) $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n b_n} = 0.$

Then, the sequence $\{u_n\}$ *defined by (2.1) converges strongly to an element in* C.

Proof. We rewrite equations (2.1) and (1.10) in their equivalent forms

$$d_n(\sum_{i=1}^{N} A_i(u_{n+1}) + u_{n+1} = \beta_n u_n + \beta_n\gamma_n(u_n - u_{n-1}),$$
$$d_n\sum_{i=1}^{N} A_i(x_n) + x_n = \beta_n x_n,$$
$$d_n = \beta_n\tilde{c}_n, \beta_n = 1/(1+\alpha_n\tilde{c}_n).$$

After subtracting the second equality from the first one and multiplying by $j(u_{n+1} - x_n)$ we get

$$d_n\langle \sum_{i=1}^{N}(A_i(u_{n+1}) - A_i(x_n)), j(u_{n+1} - x_n)\rangle + \langle u_{n+1} - x_n, j(u_{n+1} - x_n)\rangle$$
$$= \beta_n\langle u_n - x_n, j(u_{n+1} - x_n)\rangle + \beta_n\gamma_n\langle u_n - u_{n-1}, j(u_{n+1} - x_n)\rangle.$$

Again, by virtue of the property of A_i and j it is not difficult to verify the following inequality

$$\|u_{n+1} - x_n\| \le \beta_n \|u_n - x_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\|.$$

Consequently, on the base of (1.12) we have

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &\le \|u_{n+1} - x_n\| + \|x_{n+1} - x_n\| \\ &\le \beta_n \|u_n - x_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\| + 2\|y\| \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} \\ &\le (1 - b_n)\|u_n - x_n\| + c_n,\end{aligned}$$

where $c_n = \beta_n \gamma_n \|u_n - u_{n-1}\| + 2\|y\|(\alpha_n - \alpha_{n+1})/\alpha_n$. Since the serie in (ii) is convergent, then $\gamma_n b_n^{-1} \|u_n - u_{n-1}\| \to 0$, as $n \to +\infty$. Lemma 2.1 guarantees that $\|u_{n+1} - x_{n+1}\| \to 0$ as $n \to +\infty$.

On the other hand, from

$$\lim_{n\to+\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n b_n} = 0,$$

it follows

$$\lim_{n\to+\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} = 0.$$

Hence,

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) > 0 : \forall n > N(\varepsilon) \Longrightarrow \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} < \frac{\varepsilon}{p},$$

for any fixed integer $p > 0$.

Thus,

$$\begin{aligned}0 < \frac{\alpha_n - \alpha_{n+p}}{\alpha_n} &= \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + \frac{\alpha_{n+1} - \alpha_{n+2}}{\alpha_n} + ... + \frac{\alpha_{n+p-1} - \alpha_{n+p}}{\alpha_n} \\ &\le \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + \frac{\alpha_{n+1} - \alpha_{n+2}}{\alpha_{n+1}} + ... + \frac{\alpha_{n+p-1} - \alpha_{n+p}}{\alpha_{n+p-1}} < \varepsilon.\end{aligned}$$

It means that $\lim_{n\to+\infty}(\alpha_n - \alpha_{n+p})/\alpha_n = 0$ for any fixed positive integer p. Theorem 1.2 permits us to conclude that $\lim_{n\to\infty} x_n = x_* \in C$. So,

$$\lim_{n\to\infty} u_n = x_* \in C.$$

Theorem is proved.

Consider a finite-dimensional variant of (2.1) defined by

$$\begin{gathered}\tilde{c}_n \Big(\sum_{i=1}^{N} A_i^n (v_{n+1}) + \alpha_n v_{n+1} \Big) + v_{n+1} - v_n = \gamma_n (v_n - v_{n-1}), \\ v_0, v_1 \in E_1.\end{gathered} \tag{2.2}$$

Note that for two elements $v_{n-1}, v_n \in E_n$, the solution v_{n+1} of (2.2) exists in the same reason as for u_{n+1} and also belongs to E_n. Since $E_n \subset E_{n+1}$, then v_n, v_{n+1} and the solution v_{n+2} also belong to E_{n+2}. So, the sequence $\{v_n\}$ is well defined.

Theorem 2.3. *Assume that E is an uniformly convex and uniformly smooth Banach space with the strictly convex E^*, $\gamma_n(y) = o(\alpha_n)$ for each $y \in C$, $T_i, i = 1, ..., N$, are Fréchet differentiable with (1.14), and the parameters $\tilde{c}_n$ and α_n are chosen such that*

(i) $0 < c_0 < \tilde{c}_n < C_0, \alpha_n \searrow 0, \gamma_n(y) = o(\alpha_n)$,

(ii) $\sum_{n=1}^{\infty} b_n = +\infty, b_n = \alpha_n\tilde{c}_n/(1+\alpha_n\tilde{c}_n)$,

(iii) $\lim_{n\to\infty} \frac{\alpha_n-\alpha_{n+1}}{\alpha_n b_n} = 0$.

Then, the sequence $\{v_n\}$ defined by (2.2) converges to an element in C.

Proof. We rewrite equations (2.2) and (1.13) in their equivalent forms

$$d_n \sum_{i=1}^{N} A_i^n(v_{n+1}) + v_{n+1} = \beta_n v_n,$$

$$d_n \sum_{i=1}^{N} A_i^n(z_n) + z_n = \beta_n z_n.$$

After subtracting the second equality from the first one and multiplying by $j(u_{n+1} - x_n)$ we get

$$\begin{aligned} d_n\langle\sum_{i=1}^{N}(A_i^n(v_{n+1}) - A_i^n(z_n)), j^n(v_{n+1} - z_n)\rangle + \langle v_{n+1} - z_n, j^n(v_{n+1} - z_n)\rangle \\ = \beta_n\langle v_n - z_n, j(z_{n+1} - z_n)\rangle. \end{aligned}$$

Again, by virtue of the property of A_i and j we obtain the following inequality

$$\|v_{n+1} - z_n\| \le \beta_n\|v_n - z_n\|.$$

Consequently, from (1.15) we have the following estimate

$$\begin{aligned} \|v_{n+1} - z_{n+1}\| &\le \|v_{n+1} - z_n\| + \|v_{n+1} - z_n\| \\ &\le \beta_n\|v_n - z_n\| + R\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} \\ &\le (1 - b_n)\|v_n - z_n\| + c_n, \end{aligned}$$

where $c_n = R(\alpha_n - \alpha_{n+1})/\alpha_n$. Lemma 2.1 guarantees that $\|v_{n+1} - z_{n+1}\| \to 0$ as $n \to +\infty$. From

$$\lim_{n\to+\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n b_n} = 0,$$

it implies that

$$\lim_{n\to+\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} = 0.$$

And, by the similar argument as in the proof of theorem 2.2, we have

$$\lim_{n\to+\infty} \frac{\alpha_n - \alpha_{n+p}}{\alpha_n b_n} = 0$$

for each positive integer p. Applying theorem 1.3, we obtain that $\lim_{n\to+\infty} z_n = z_*$. So, $\lim_{n\to+\infty} v_n = z_*$. Theorem is proved.
Remark The sequences $\alpha_k = \alpha_0(1+k)^{-\alpha}$ and

$$b_k = b_0(1+k)^{-b}\frac{1}{1+\|z_k - z_{k-1}\|},$$

where α_0, b_0 are some positive constants and

$$0 < \alpha < \frac{1}{2}, b > \alpha + 1,$$

satisfy all conditions in the theorem.

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